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Quantum Aspects of Natural Top Quark Condensation

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In top quark condensation the Brout-Englert-Higgs (BEH) boson is a $\bar{t}t$ bound state. With a UV completion of a single coloron exchange interaction, a recent semiclassical treatment, [1, 2], gave a novel theory of the BEH boson as an extended object with composite scale $M_0 \sim 6$ TeV. Presently we obtain the semiclassical theory as an effective action, using the source/Legendre transformation techniques of Jackiw, *et al.*, [3, 4], and fermion loop effects in the large-N_c limit by deploying an auxiliary field to implement the sum of leading fermion loop diagrams. The theory remains natural at the loop level, with fine tuning at the level of a few %, and the effective coupling of the 4-fermion interaction, $\overline{g}_0^2$, is significantly enhanced by quantum loops over the fundamental coloron coupling, g_0^2. Hence a relatively weaker "topcolor" theory can produce critical coupling in the effective BEH bound state theory.

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I. INTRODUCTION

Recently we revisited the idea of "top quark condensation," i.e., that the Brout-Englert-Higgs (BEH) boson is composed of top + anti-top quarks [1, 2]. Our new approach describes a bound state, consisting of a pair of relativisitic chiral fermions, arising from a single particle exchange of a perturbative massive gluon-like object with mass M_0 and coupling g_0 (dubbed a "coloron"). A semiclassical analysis, involving "bilocal fields," leads to a low-mass composite BEH boson with an extended internal wave-function, $\phi(r)$ [1, 2]. $\phi(r)$ satisfies a Schrödinger-Klein-Gordon (SKG) equation in a Yukawa potential with coupling g_0^2. Close to its critical coupling, $g_0^2 \sim g_c^2 \approx 8\pi^2/N_{color}$ the internal wave-function spreads, thus diluting $\phi(0)$. The BEH boson is then an extended object with a small eigenvalue, μ^2. We identify $\mu^2 = -(88)^2$ GeV^2, the BEH mass in the symmetric phase of the standard model (SM). The composite scale, the mass of the $SU(3)$ octet colorons, is then found to be $M_0 \sim 6$ TeV. Near critical coupling the low energy theory is approximatly scale invariant, in analogy to a second order phase transition in condensed matter physics. This effective scale invariance leads to the spreading of the wave-function $\phi(r)$, and is a universal phenomenon in any bound state system near its critical coupling, where the eigenvalue of the Hamiltonian is small compared to the internal parameters of the theory. Hence scale symmetry is realized dynamically as the custodial symmetry of the small mass BEH bound state.

Remarkably, the static observables, such as the induced top–quark BEH–Yukawa coupling, g_Y, and quartic coupling, λ, depend upon $\phi(0) \sim \sqrt{|\mu|/M_0}$ and become concordant with experiment [1]. Moreover, due to a subtlety of the dilution effect, the theory requires fine–tuning at the level of $\sim |\phi(0)|^2 \sim |\mu|/M_0$, rather than the usual $|\mu|^2/M_0^2$. Hence, identifying $\mu \sim 88$ GeV^2 we require fine tuning only at the level of $\sim$ few %. The colorons may be accessible to the LHC, where they would be pair produced and decay to $\bar{t}t$ pairs (also $\bar{b}b$ in extended topcolor models [5, 6]) in boosted jets. This is significantly different than old results obtained in the 1990's [7]−[11] which were based upon the renormalization group (RG) improved Nambu–Jona-Lasinio model (NJL, [12]), but did not contain an internal wave-function $\phi(r)$.

Presently we show how the semiclassical theory emerges from the underlying quantum field theory of the third generation quarks with the single coloron exchange interaction. We utilize the formalism developed in the early papers, mainly of Jackiw, *et al.*, [3][4], which yield the effective semiclassical action. We also introduce auxiliary fields that facilitate the sum of the leading fermion loop diagrams of order $(\hbar g_0^2 N_c)^n$. The approach can be considered a decomposition of a top quark field into semiclassical + quantum degrees of freedom, $\sim \psi + \sqrt{\hbar}\omega$. Integrating out the quantum fermionic fluctuations, ω, in the presence of fermionic source fields, $\xi_{L,R}$, yields a semiclassical effective "source" action. The action for the semiclassical fields, $\psi_{L,R}(x)$, and a composite bilocal Brout-Englert-Higgs field, $H^i(x, y)$, is then generated upon performing a Legendre transformation to remove the sources. The resulting effective action then allows treatment of all semiclassical phenomena in terms of the c-number, chiral spinor fields, $\psi_{L,R}(x)$, and formation of the bound state $H^i(x, y)$, reproducing our earlier work (Reference [1] should be considered a companion to the present paper).

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Mainly, we find that the sum of ω-loops with coloron coupling, g_0, generates a renormalized coupling for the 4-fermion interaction, $\overline{g}_0^2$, where:

$$\overline{g}_0^2 = g_0^2 \left(1 - \frac{g_0^2 N_c}{8\pi^2}\right)^{-1} \tag{1}$$

Essentially, the fermion loops act as in the NJL model, which enhances the binding of the 0^+ channel ground state, leading to a form of "critical amplification." The underlying coloron production, scattering and decay amplitudes are therefore governed by g_0^2, while the binding, which forms the BEH boson, is governed by the enhanced $\overline{g}_0^2 > g_0^2$. This implies that the $\overline{g}_0^2$ can be supercritical, while the underlying coloron coupling g_0^2 remains smaller and subcritical. Therefore the effective potential coupling in the 0^+ channel, $\overline{g}_0^2$, can be super-critical, to produce the low mass BEH bound state and the spontaneous symmetry breaking, in a weaker underlying topcolor model.

The induced top-quark BEH-Yukawa coupling $\overline{g}_Y \propto \overline{g}_0^2$ is also renormalized. Using the experimental value as an input, $\overline{g}_Y \approx 1$, we still obtain the composite scale (the coloron mass) $M_0 \approx 6$ TeV. We also observe how the BEH quartic coupling, $\lambda \propto \overline{g}_0^8 \propto \overline{g}_Y^4$, is generated by the underlying ω-loops. In our present scheme we find numerically that this is remarkably close to the SM result $\lambda \sim 0.25$, which represents a significant improvement over the old NJL model based top condensation that yielded $\lambda \sim 1$.

This also resolves putative issues that arise in the treatment of ref.[1]. One might be tempted to, e.g., loop the Yukawa interaction to argue that there is a problematic large correction to the BEH mass $\propto -\overline{g}_Y^2 N_c M_0^2/8\pi^2$. This would be an additive correction to μ^2 and one might conclude that the effective theory is therefore "unnatural." *This would, however, be incorrect.* This particular contribution is now already generated by integrating out the underlying ω_i and it would be double counting to do a fermion loop calculation in the effective semiclassical theory. Moreover, in the underlying theory this appears, not as a mass term, but as a multiplicative and enhancing correction to the effective single coloron exchange interaction and the internal wave-function back-reacts to it. It is, in fact, the source of the critical amplification of $\overline{g}_0^2$.

We find the Jackiw, *et al.*, techniques to be powerful, and work particularly well with the auxiliary field fermion loop sum technique introduced here.

II. EFFECTIVE SEMICLASSICAL ACTION

A. Sources and Legendre Transformation

We presently derive the effective top quark condensation action for a semiclassical chiral fermion fields $\psi_L^i(x)$ and $\psi_R(x)$, from the underlying quantum field theory, by integrating out third generation, color triplet, quark fields, $(\omega_L^i(x), \omega_R(y))$, where:[1]

$$\omega_L^i = \frac{(1-\gamma^5)}{2}\begin{pmatrix} t \\ b \end{pmatrix}, \qquad \omega_R = \frac{(1-\gamma^5)}{2} t, \tag{2}$$

where i is an electroweak $SU(2)$ index. We will use the methods of Jackiw, [3] introducing chiral source fields, Ξ_R^i and Ξ_L, then performing a Legendre transformation to obtain the effective action for the semiclassical fields $\psi_{L,R}(x)$. We then follow with the method of Cornwall, Jackiw and Tomboulis, (CJT [4]), *in the semiclassical theory* and introduce the two-body bound state wave-function source $K^i(x, y)$, together with semiclassical fields. A Legendre transformation then yields the full theory with effective fields $\psi_{L,R}$ and bilocal BEH boson $H^i(x, y)$. We will deploy a bilocal auxiliary field, $B(x, y)$, to expedite the sum of fermion loops in a large N_c approximation.

Our starting point is the fundamental action at the quantum field theory level of the topcolor model with the single coloron exchange interaction, suitably Fierz rearranged (as given in eq.(28) of ref.[1]):

$$S_{TC} = \int d^4x \bigg([\overline{\omega}_L^i(x)\; i\not{D}\, \omega_{iL}(x)] + [\overline{\omega}_R(x)\; i\not{D}\, \omega_R(x)] + g_0^2 \int d^4x\; d^4y\; [\overline{\omega}_L^i(x)\omega_R(y)] D_F(x-y) [\overline{\omega}_R(y)\omega_{iL}(x)] \tag{3}$$

where [...] denotes color singlet contraction of indices. The coloron propagator in Feynman gauge is:

$$iD_{\mu\nu}(x-y) = g_{\mu\nu} D_F(x-y); \qquad D_F(x-y) = -\int \frac{1}{q^2 - M_0^2} e^{iq(x-y)} \frac{d^4q}{(2\pi)^4}. \tag{4}$$

---

[1] Note that the b_R quark must also participate in topcolor in an anomaly-free scheme, which can be implemented as in *e.g.*, [5].

The interaction can be factorized by introducing a bilocal, isodoublet, auxiliary field $B^i(x,y)$ of mass dimension 3:

$$S_{TC} = \int d^4x \Big([\overline{\omega}^i_L(x)\ i\not{D}\,\omega_{iL}(x)] + [\overline{\omega}_R(x)\ i\not{D}\,\omega_R(x)] \Big)$$
$$+ \int d^4x\ d^4y\ \Big(g_0 D_F(x-y) \left(B^{i\dagger}(x,y)[\overline{\omega}_R(y)\omega_{iL}(x)] + [\overline{\omega}^i_L(x)\omega_R(y)]B_i(x,y) \right) - [B^{i\dagger}(x,y) D_F(x-y) B_i(x,y)] \Big). \quad (5)$$

The "equation of motion" of B_i is then:

$$B_i(x,y) = g_0[\overline{\omega}_R(y)\omega_{iL}(x)]; \qquad B^{i\dagger}(x,y) = g_0[\overline{\omega}^i_L(x)\omega_R(y)]. \quad (6)$$

Substituting the solution of eq.(6) back into eq.(5) yields eq.(3). This is analogous to the treatment of the local NJL model (e.g., as in eqs.(4) and (6) in ref.[1]), but differs presently by the bilocal factor, arising from a bilocal coloron exchange amplitude, of $D_F(x-y)$. Unlike the NJL model, we do not treat B_i as a composite field and we will integrate it out to sum fermion loops.

We can define the fermion kinetic terms and B_i interaction terms of eq.(5) as a matrix $K(x,y)$:

$$S_{TC} = \int d^4x\ d^4y\ \Big([\overline{\Omega}(x) K(x,y) \Omega(y)] - [B^{i\dagger}(x,y) D_F(x-y) B_i(x,y)] \Big), \quad (7)$$

where,

$$\Omega \equiv \begin{pmatrix} \omega_{iL} \\ \omega_R \end{pmatrix}, \qquad \overline{\Omega} \equiv \begin{pmatrix} \overline{\omega}^i_L & \overline{\omega}_R \end{pmatrix}, \qquad K(x,y) = \begin{pmatrix} S_F^{-1}(x,y) & g_0 D_F(x-y) B^\dagger(x,y) \\ g_0 D_F(x-y) B(x,y) & S_F^{-1}(x,y) \end{pmatrix}. \quad (8)$$

Here $S_F^{-1}(x,y)$ is the formal inverse of the Feynman propagator, $S_F(x,y) = (i\not{D})^{-1}\delta^4(x-y)$. Note that K is Dirac Hermitian, $\overline{K} = \gamma_0 K^\dagger \gamma_0 = K$.

We follow Jackiw [3] and add chiral source terms, $\Xi(x)$, for the fermion fields,

$$S_{TC}(\Xi,\Omega) = \int d^4x\ d^4y\ \Big([\overline{\Omega}(x) K(x,y) \Omega(y)] - [B^{i\dagger}(x,y) D_F(x-y) B_i(x,y)] \Big) + \int d^4x \Big(\overline{\Xi}(x)\Omega(x) + \overline{\Omega}(x)\Xi(x) \Big),$$
$$\text{where:} \qquad \Xi \equiv \begin{pmatrix} \xi_{iR} \\ \xi_L \end{pmatrix}, \qquad \overline{\Xi} \equiv \begin{pmatrix} \overline{\xi}^i_R & \overline{\xi}_L \end{pmatrix}. \quad (9)$$

The effective action, W, is defined by the path integral:

$$W = -i\hbar \ln \int D\Omega D\overline{\Omega}\ \exp\left(\frac{i}{\hbar} S'(\Xi,\Omega) \right) \quad (10)$$

To evaluate W we begin by shifting the integration variables to cancel cross-terms,

$$\Omega(x) \to \Omega(x) - \int d^4y\ K^{-1}(x,y)\Xi(y); \qquad \overline{\Omega}(x) \to \overline{\Omega}(x) - \int d^4y\ \overline{\Xi}(y) K^{-1}(y,x), \quad (11)$$

and the shifted action becomes:

$$S_{TC} = \int d^4x\ d^4y\ \Big([\overline{\Omega}(x) K(x,y) \Omega(y)] - \overline{\Xi}(x) K^{-1}(x,y) \Xi(y) - [B^{i\dagger}(x,y) D_F(x-y) B_i(x,y)] \Big) \quad (12)$$

The path integral is now $W = W(\Omega) + W(\Xi) + W(B)$ where,

$$W(\Omega) = -i\hbar \ln \int D\Omega D\overline{\Omega}\ \exp\left(\frac{i}{\hbar} \int d^4x\ d^4y\ \Big([\overline{\Omega}(x) K(x,y) \Omega(y)] \Big) \right),$$
$$W(\Xi) = -\int d^4x\ d^4y\ \overline{\Xi}(x) K^{-1}(x,y) \Xi(y), \qquad W(B) = -\int d^4x\ d^4y\ \Big([B^{i\dagger}(x,y) D_F(x-y) B_i(x,y)] \Big). \quad (13)$$

We then define the effective semiclassical field, Ψ:

$$\Psi \equiv \begin{pmatrix} \psi_{iL} \\ \psi_R \end{pmatrix}, \qquad \overline{\Psi} \equiv \begin{pmatrix} \overline{\psi}^i_L & \overline{\psi}_R \end{pmatrix} \quad (14)$$

where,

$$\overline{\Psi}(x)=\frac{\delta W(\Xi)}{\delta\Xi(x)}=-\int d^4y\ \overline{\Xi}(y)K^{-1}(y,x);\qquad \Psi(x)=\frac{\delta W(\Xi)}{\delta\overline{\Xi}(x)}=-\int d^4y\ K^{-1}(x,y)\Xi(y);\tag{15}$$

Hence, inverting these relations,

$$\Xi(x)=-\int d^4y\ K(x,y)\Psi(y);\qquad \overline{\Xi}(x)=-\int d^4y\ \overline{\Psi}(y)K(y,x),\tag{16}$$

Ξ should be regarded as a functional of Ψ by eqs.16 .

We then perform a Legendre transformation to obtain the effective action, $S(\Psi,B)$, swapping Ξ for the semiclassical field, ψ, retaining the auxiliary field B:

$$\begin{aligned}S(\Psi,B)&=W(\Omega)+W(B)+W(\Xi)-\int d^4x\ \left(\frac{\delta W(\Xi)}{\delta\Xi(x)}\Xi(x)+\overline{\Xi}(x)\frac{\delta W(Xi)}{\delta\overline{\Xi}}\right)\\&=W(\Omega)+W(B)-\int d^4x\ d^4y\ \overline{\Xi}(x)K^{-1}(x,y)\Xi(y)-\int d^4x\ (\overline{\Psi}(x)\Xi(x)+\overline{\Xi}(x)\Psi(x))\\&=W(\Omega)+\int d^4x\ d^4y\ \Big(\overline{\Psi}(x)K(x,y)\Psi(y)-[B^{i\dagger}(x,y)D_F(x-y)B_i(x,y)]\Big).\end{aligned}\tag{17}$$

We obtain the normal-sign kinetic terms for the semiclassical fields $\psi_{L,R}$. From eq.(8) we have:

$$\begin{aligned}S(\Psi,B)=W(\Omega)&+\int d^4x\Big([\overline{\psi}_L(x)i\not{D}\,\psi_L(x)]+[\overline{\psi}_R(x)\ i\not{D}\,\psi_R(x)]\Big)-\int d^4x\ d^4y\ \Big([B^{i\dagger}(x,y)D_F(x-y)B_i(x,y)]\Big)\\&+\int d^4x\ d^4y\ \Big(g_0D_F(x-y)(B^{i\dagger}(x,y)[\overline{\psi}_R(y)\psi_{iL}(x)]+[\overline{\psi}^i_L(x)\psi_R(y)]B_i(x,y)\Big)\end{aligned}\tag{18}$$

The "equation of motion" of B_i is now:

$$B_i(x,y)=g_0[\overline{\Psi}_R(y)\Psi_{iL}(x)]+\frac{\delta W(\Omega)}{\delta B^{\dagger i}(x,y)},\qquad B^{i\dagger}(x,y)=g_0[\overline{\Psi}^i_L(x)\Psi_R(y)]+\frac{\delta W(\Omega)}{\delta B_i(x,y)}.\tag{19}$$

If we ignore the $W(\Omega)$ terms, and insert these back into the new action we recover the original classical action of eq.(3), but now given in terms of the semiclassical fields Ψ:

$$S_{TC}=\int d^4x\Big([\overline{\psi}_L(x)i\not{D}\,\psi_L(x)]+[\overline{\psi}_R(x)\ i\not{D}\,\psi_R(x)]+g_0^2\int d^4x\ d^4y\ [\overline{\psi}^i_L(x)\psi_R(y)]D_F(x-y)[\overline{\psi}_R(y)\psi_{iL}(x)].\tag{20}$$

The quantum loop effects are contained in $W(\Omega)$, which we calculate in Section III. We have obtained the semiclassical action for the c-number fields $(\psi_{iL}(x),\psi_R(y))$.

B. Full Effective Action for Bilocal BEH boson via Cornwall-Jackiw-Tomboulis [4]

With the semiclassical theory we now include a source for a bilocal field which will lead to the semiclassical BEH boson field, $H_i(x,y)$. Note that, since we have already integrated out the underlying quantum fields, ω, the bound state $H_i(x,y)$ that we obtain presently *is composed only of the semiclassical modes* $\psi_{L,R}$. We can operate at the semiclassical level, following the formalism Cornwall, Jackiw, and Tomboulis, (CJT) [4]. This is distinct from the NJL model, or the usual applications of CJT, where the bound state is composed in loops of a sum over all of the ω quantum fields.

We add an additional source term for a complex bilocal field, $Q_i(x,y)$, that couples only *to a subset of the semiclassical (to-be-bound) modes*, $[\overline{\psi}_R(x)\psi_{iL}(y)]_b$, in eqs.(18), while the factorization field, B_i, couples to all fermion pairs, including to-be-free fermion pairs, $[\overline{\psi}_R(x)\psi_{iL}(y)]_f$:

$$\begin{aligned}S=W(\Omega)&+\int_x\Big([\overline{\psi}_Li\not{D}\,\psi_L]+[\overline{\psi}_R\ i\not{D}\,\psi_R]\Big)\\&+\int_{xy}\ \Big(-[B^{i\dagger}D_FB_i]+\big(g_0(B^{i\dagger}+Q^{i\dagger})D_F[\overline{\psi}_R\psi_{iL}]_b+g_0B^{i\dagger}D_F[\overline{\psi}_R\psi_{iL}]_f+h.c.\big)\Big)\end{aligned}\tag{21}$$

Here we abbreviate integrals as:

$$\int_{u...v} = \int d^4u..d^4v \tag{22}$$

(and we will suppress writing field arguments where obvious, e.g., $B^{i\dagger}(x,y)D_F(x,y)B_i(x,y) \to B^{i\dagger}D_F B_i$, etc.).

We then have:

$$\frac{\delta S}{\delta Q^{i\dagger}} = g_0 D_F[\overline{\psi}_R \psi_{iL}]_b \equiv g_0\sqrt{N_c}M_0^2 D_F H_i; \qquad \frac{\delta S}{\delta Q_i} = g_0 D_F[\overline{\psi}^i_L \psi_R]_b \equiv g_0\sqrt{N_c}M_0^2 D_F H^{i\dagger}; \tag{23}$$

where we anticipate the normalization of H_i, $[\overline{\psi}_L(x)\psi_R(y)] \to [\overline{\psi}_L(x)\psi_R(y)]_f + M_0^2\sqrt{N_c}H^i(x,y)$, from ref. [1].[2]

We perform the Legendre transformation, following [4], and using eq.(23):

$$S' = S - \int_{xy}\left(K^\dagger \frac{\delta S}{\delta K^\dagger} + K\frac{\delta S}{\delta K}\right) \tag{25}$$

hence,

$$S' = W(\Omega) + \int_x \Big([\overline{\psi}_L i\not{D}\,\psi_L] + [\overline{\psi}_R\; i\not{D}\,\psi_R]\Big) + \int_{xy}\Big(-[B^{i\dagger}D_F B_i] + g_0(B^{i\dagger}D_F C_i + h.c.)\Big) \tag{26}$$

where we define,

$$C_i(x,y) = \sqrt{N_c}M_0^2 H_i(x,y) + [\overline{\psi}_R(x)\psi_{iL}(y)]_f \tag{27}$$

If we ignore the quantum corrections (ω-loops), the "equation of motion" of B_i becomes:

$$B_i(x,y) = g_0 C_i(x,y), \qquad B^{i\dagger}(x,y) = g_0 C^{i\dagger}(x,y) \tag{28}$$

Substituting B_i into eq.(26) we thus obtain the semiclassical result,

$$\begin{aligned} S'(\Psi,H) = &\int_x \Big([\overline{\psi}_L(x) i\not{D}\,\psi_L(x)] + [\overline{\psi}_R(x)\; i\not{D}\,\psi_R(x)]\Big) \\ &+ g_0^2\int_{xy}\big(N_c M_0^4 H^\dagger(x,y)D_F(x-y)H(x,y)\big) + \sqrt{N_c}M_0^2(H^{i\dagger}(x,y)D_F(x-y)[\overline{\psi}_R(y)\psi_{iL}(x)]_f + h.c.) \\ &+ g_0^2\int_{xy}[\overline{\psi}^i_L(x)\psi_R(y)]_f D_F(x-y)[\overline{\psi}_R(y)\psi_{iL}(x)]_f \end{aligned} \tag{29}$$

We have therefore obtained results identical to the starting point of our previous theory, eq.(54), in ref.[1]. Note the BCS-like enhancement factor of N_c in the $H^\dagger D_F H$ interaction term, where N_c is the analogue of the number of Cooper pairs in a BCS state [13].

C. Kinetic Terms and Factorization Ansatz

The "to-be-bound" fermion kinetic terms remain in the action S':

$$S_{KT} = \int_x \Big([\overline{\psi}^i_L(x) i\not{D}\,\psi_{iL}(x)]_b + [\overline{\psi}_R(x)\; i\not{D}\,\psi_R(x)]_b\Big) \tag{30}$$

---

[2] Only the color singlets bind, hence with color indices (a,b),

$$\overline{\psi}^a_R(x)\psi_{ibL}(y) \to \overline{\psi}^a_R(x)\psi_{ibL}(y)_f + M_0^2\frac{\delta^a_b}{\sqrt{N_c}}H_i(x,y). \tag{24}$$

which yields the $\sqrt{N_c}$ factors in eq.(23). Note that in [1] we used $M^2 = \epsilon M_0^2$ as an arbitrary normalizing mass scale for H, but we then showed that $\epsilon \to 1$ in the critical coupling limit. We are presently abbreviating the discussion and assume $\epsilon = 1$ at the outset.

These have to be replaced by the kinetic term of the bound state field $H_i(x,y)$. From eq.(23), in the free field limit (small g_0), the kinetic terms S_{KT} imply the equations of motion. Technically with $H \equiv H_i \frac{\tau^i}{2}$ these are best written as commutators,

$$[D_y, [D_y, H(x,y)]] = 0 \qquad [D_x^\dagger, [D_x^\dagger, H(x,y)]] = 0 \tag{31}$$

Omitting complications from the gauge fields these follow from the square of the Dirac equation, e.g. $[\overline{\psi}_L(y)[\partial\!\!\!/ , [\partial\!\!\!/ , \psi_{iR}(x)]]_b = \partial_x^2[\overline{\psi}_L(y)\psi_{iR}(x)]]_b = 0$, etc. A free field Lorentz invariant action yields the equations of motion by variation:

$$S_{KT} \to M_0^4 \int d^4x\, d^4y \left(Z|D_R^\dagger H(x,y)|^2 + Z|D_L H(x,y)|^2 \right) \tag{32}$$

where the covariant derivatives are as defined in the NJL model:

$$D_{L\mu} = \frac{\partial}{\partial y_\mu} - ig_2 W_\mu^A(y)\frac{\tau^A}{2} - ig_1 W_\mu^0(y)\frac{Y_L}{2}; \qquad D_{R\mu}^\dagger = \frac{\partial}{\partial x_\mu} + ig_1 W_\mu^0(x)\frac{Y_R}{2}, \tag{33}$$

where W_μ^A (W_μ^0) denotes the $SU(2)$ $U(1)_Y$ gauge fields and the weak hypercharges are $Y_L = 1/3$, $Y_{tR} = 4/3$, (and the presently unused $Y_{bR} = -2/3$), and we will require $Y_H = -1$ for the BEH boson of the SM; the electric charges are as usual: $Q = I_3 + \frac{Y}{2}$. Note that D_L ($D_R^\dagger$) acts at coordinate y (x), and $D_R^\dagger$ acts on $\overline{t}_R$, hence the sign flip in the gauge field terms (the derivative $D^\dagger$ acts in the forward direction as we have written the kinetic term eq.32).

We now pass to barycentric coordinates,[3]

$$X^\mu = \frac{x^\mu + y^\mu}{2}, \quad r^\mu = \frac{x^\mu - y^\mu}{2}, \quad \partial_x = \frac{1}{2}(\partial_X + \partial_r), \quad \partial_y = \frac{1}{2}(\partial_X - \partial_r) \quad \text{Jacobian:} \quad \left|\frac{\partial(X,\rho)}{\partial(x,y)}\right| \equiv J^{-1} = 2^{-4}. \tag{34}$$

To work in the rest frame as in [1] we follow Yukawa, [15], and for the ground state we consider a factorized ansatz for $H_i(X,r)$:

$$H_i(x,y) = H_i(X+r, X-r) \equiv \hat{H}_i(X,r) \qquad \text{hence,} \qquad \sqrt{J/2}\, \hat{H}_i(X,r) = H_i(X)\phi(r). \tag{35}$$

We can introduce Wilson lines *as an approximation* to "pull-back" the gauge couplings from (x,y) to the center X. This is done by field redefinitions (as discussed in [1]):

$$H_i(x,y) \to W_R^\dagger(X,x) W_L(X,y)\, \sqrt{2/J}\; H_i(X)\phi(r), \tag{36}$$

where $H_i(X)$ will have mass-dimension 1, and and ϕ will be dimensionless. Note that $H_i(X)$, is distinguished from $H_i(x,y)$ by having dependence only upon X, and will become the effective BEH boson. The Wilson lines are not necessary. In the pure gauge limit these are simply gauge transformations on the H_i field. We introduce the Wilson lines as a conceptual tool to consolidate the electroweak charges at X, though the gauge charges are effectively consolidated at X in the low energy approximation. Moreover, the Wilson lines are renormalizable and are not sources (charges) for gauge fields. However, they do radiate, acting as currents.

With the pull-back localization of the BEH electroweak charges, the W and Z masses are generated in the usual way under spontaneous symmetry breaking.[4] We have the Lorentz invariant kinetic term action:

$$\int d^4X d^4r\; Z M_0^4 \left(|D_H H(X)|^2 |\phi(r)|^2 + |H(X)|^2 |\partial_{r^\mu}\phi(r)|^2 \right), \tag{37}$$

where the covariant derivative is:

$$D_{H\mu} = \frac{\partial}{\partial X^\mu} - ig_2 W_\mu^A(X)\frac{\tau^A}{2} - ig_1 W_\mu^0(X)\frac{Y_H}{2}. \tag{38}$$

The internal wave-function $\phi(r)$ is now a complex scalar that carries no gauge charges, and $Y_H = -Y_{tR} + Y_L = -1$, apropos the BEH field. (see Appendix A of [1] for the algebra of the Wilson line pullback).

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[3] Note: the Jacobian was misquoted in the preprint version of [1] but corrected in the published version.

[4] We have neglected the gluon Wilson line that extends from $-\vec{r}$ to $\vec{r}$, but of course, produces no color charge of the BEH boson.

D. Removing Relative Time

Consider the kinematics of a pair of massless particles of 4-momenta p_1 and p_2. We have $p_1^2 = p_2^2 = 0$. and two-body plane waves, $H(x,y) \sim \exp(ip_1 x + ip_2 y)$. We pass to the total momentum $P = (p_1 + p_2)$ and relative momentum $Q = (p_1 - p_2)$, and the plane waves become $\exp(iPX + iQr)$. Note that $P_\mu Q^\mu = p_1^2 - p_2^2 = 0$. This implies in the rest (barycentric) frame, $P = (P_0, 0)$ and $Q = (0, \vec{Q})$, and the free equation of motion, $P_0^2 - \vec{Q}^{\,2} = 0$. Hence, in the rest frame the dependence upon $\vec{X}$ (associated with $\vec{P}$) and, in particular the relative time, r^0 (associated with Q^0), drops out. If the particles are constituents of a bound state then this is the rest frame of the composite particle. In any other frame there is always a boosted $r^{0\prime}$ that is unphysical.

Bilocal fields have spurious dependence upon r^0 in the rest frame (or its boosted $r^{0\prime}$ in any other frame). This is an unphysical and unwanted degree of freedom. It is analogous to the situation in a gauge theory where $\partial_\mu A^\mu$ is unphysical.[5] To use the bilocal formalism, $r^{0\prime}$ must be projected out with Lorentz invariant constraints, analogous to gauge fixing. We have discussed this in greater detail in the present action formalism elsewhere [1, 2].

In the action of eq.(37) we included a normalization factor Z as in [1]. This is natural in mapping from the fermionic kinetic terms to the $\hat{H}_i$ kinetic terms, but it serves the role of allowing us to remove relative time. In the barycentric (center-of-mass) frame we assume an ansatz for $\phi(r)$ ($H_i(X)$) that has no dependence upon the relative time r^0 ($\vec{X}$). The field $H_i(X)$ will have the conventional volume normalization of a mass-dimension 1, complex field. The field $\phi(\vec{r})$ however, will describe the bound state and must be a compact (normalizable) field.

A canonical normalization of the $H_i(X)$ kinetic term in any frame (which leads to a normalized Noether current, such as $iH^\dagger \overleftrightarrow{\partial}_X H$) dictates a Lorentz invariant normalization constraint on $\phi(r)$:

$$1 = \int d^4r \; Z M_0^4 \; |\phi(r)|^2. \tag{39}$$

Since the relative time disappears kinematically in the rest frame, then $\phi(r) \to \phi(\vec{r})$ becomes a static field that has no dependence upon r^0. We then define the compact normalization for $\phi(\vec{r})$ in the rest frame as:

$$1 = M_0^3 \int d^3r \; |\phi(\vec{r})|^2, \qquad \text{which implies,} \qquad \int Z M_0 \; dr^0 = 1 \tag{40}$$

These conditions can be specified in a manifestly covariant form using the unit vector $\omega_\mu = P_\mu/\sqrt{P^2}$, and e.g., $r^0 = r_\mu \omega^\mu$, etc., as in Appendix C of [1]. Note that $\phi(\vec{r})$ is dimensionless with eq.(40).

This can be viewed as a "prenormalization" to define the free bilocal field and it's Noether charges [2], before turning on the interaction. The factor Z *is not absorbed into the fields* after turning on the interaction since we don't want the coupling constant, g_0^2, to be renormalized at the classical level. Indeed, Z can be viewed as an operator, $Z \sim \delta(M_0 r^0)$. We remark that all field theories, even local ones, are implicitly defined with a "canonical" prenormalization that defines the semiclassical action after which we turn on the interaction. Then, at loop level we are compelled to renormalize the fields to preserve the canonicality, which in turn renormalizes the coupling constants.

We then turn on the interaction with coupling constant g_0 of eq.(29), and the action, becomes,

$$S = \int d^4X \left(|D_X H(X)|^2 + |H(X)|^2 M_0^3 \int d^3r \; \left(-|\partial_{\vec{r}}\phi(\vec{r})|^2\right) + |H(X)|^2 \int dr^0 d^3r \left(M_0^4 \; 2g_0^2 N_c D_F(2r^\mu)|\phi(\vec{r})\,|^2\right)\right) \tag{41}$$

Note that there is no Z factor in the interaction term, and $|\partial_r \phi|^2 = |\partial_{r^0}\phi|^2 - |\partial_{\vec{r}}\phi|^2 \to -|\partial_{\vec{r}}\phi|^2$. We therefore integrate over r^0 in the interaction term :

$$\int dr^0 D_F(2r) = -\int dr^0 \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - M_0^2} e^{2iq_\mu r^\mu} = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3} \frac{1}{\vec{q}^{\,2} + M_0^2} e^{2iq_\mu r^\mu} = -\frac{1}{2} V_0(2|\vec{r}|) = \frac{e^{-2M_0|\vec{r}|}}{16\pi|\vec{r}|}. \tag{42}$$

The $\vec{q}$ momentum integral yields the familiar Yukawa potential (where $2|\vec{r}|$ is the separation of the particles). The action then becomes,

$$S = \int d^4X \left(|D_X H(X)|^2 + |H(X)|^2 M_0^3 \int d^3r \; \left(-|\partial_{\vec{r}}\phi(\vec{r})|^2 + g_0^2 N_c M_0 \frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}|\phi(\vec{r})|^2\right)\right). \tag{43}$$

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[5] In a gauge theory, such as QED, one may think this is relevant only to define the path integral. However this is an issue in classical electrodynamics, as well, and leads to the "transverse current" in general gauges [17]

While not "manifestly' so, this is "implicitly" Lorentz invariant as discussed in Appendix C of [1]. Note that, in the limit of suppressing the $\vec{q}^{\,2}$ in the denominators of the integrands of eq.(42), we obtain the large M_0^2 limit of the potential (using $J = 2^4$, and $\delta^3(\vec{r}) = (4\pi r^2)^{-1}\delta(r)$):

$$V_0(2r) \to -\frac{1}{M_0^2}\delta^3(2\vec{r}) = -\frac{1}{8M_0^2}\delta^3(\vec{r}) \qquad D_F(2r) \to \frac{1}{M_0^2}\delta^4(2r^\mu) = \frac{1}{JM_0^2}\delta^4(r^\mu) \tag{44}$$

This recovers the NJL model potential.

We emphasize that we can often use the NJL limit of the *potential*, but the bilocal field remains. For example, in the Dirac δ-function potential of $1+1$ quantum mechanics, the wave-function $\phi(r)$ is extended in space even though the potential is pointlike [1]. In $1+3$ dimensions the Dirac δ-function potential is actually ambiguous, and depends upon the limiting procedure used to define it. This is the analogue of what we are facing presently, but the Yukawa potential affords a well defined limiting procedure and unambiguous solution.

Including the Yukawa interaction and a loop generated quartic term we obtain the effective action for the composite BEH field,

$$\begin{aligned} S = &\int d^4X \left(|D_H H(X)|^2 + |H(X)|^2 \; M_0^3 \int d^3r \left(-|\partial_{\vec{r}}\phi(r)|^2 + g_0^2 N_c M_0 \frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}|\phi(\vec{r})|^2 \right) \right. \\ &\left. -\frac{\lambda}{2}(H^\dagger H)^2 - g_Y \left([\overline{\psi}_{iL}(X) t_R(X)]_f H^i(X) + h.c. \right) \right). \end{aligned} \tag{45}$$

The internal field $\phi(r)$ is "nested" within the action for a conventional pointlike BEH boson, $H(X)$. The static ϕ field has a Hamiltonian:

$$= M_0^3 \int d^3r \left(|\partial_{\vec{r}}\phi(r)|^2 - g_0^2 N_c M_0 \frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}|\phi(\vec{r})|^2 \right). \tag{46}$$

Extremalization of the Hamiltonian $\mathcal{M}$ yields the Schrödinger-Klein-Gordon (SKG) equation for $\phi(\vec{r})$ with the eigenvalue μ^2:

$$-\nabla^2\phi - g_0^2 N_c M_0 \frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}\phi(r) = \mu^2\phi. \tag{47}$$

We find that the SKG equation has a critical coupling, $g_c^2 N_c/8\pi^2 = 1.06940$, for which $\mu^2 = 0$. Remarkably, this is very close to the quantum NJL critical coupling $g_{NJLc}^2 N_c/8\pi^2 = 1.00$ (see Section IV. (B) of [1]). When $g_0^2 > g_c^2$ the eigenvalue μ^2 becomes negative, $\to -|\mu|^2$. In such a solution the action for $H(X)$ then becomes the familiar,

$$S = \int d^4X \left(|D_H H(X)|^2 + |\mu|^2|H(X)|^2 - \frac{\lambda}{2}(H^\dagger H)^2 - g_Y \left([\overline{\psi}_L(X) t_R(X)]_f H(X) + h.c. \right) \right), \tag{48}$$

with the "sombrero potential":

$$-|\mu|^2|H(X)|^2 + \frac{\lambda}{2}(H^\dagger H)^2. \tag{49}$$

As discussed in detail in [1], the solution of the Schrödinger-Klein-Gordon SKG equation for $\phi(r)$ indeed extends to large distances, $\phi(r) \sim e^{-|\mu|r}/r$ where $|\mu| < M_0$ near critical coupling. This dilutes the value of $\phi(0) \sim \sqrt{|\mu|/M_0}$. We find that the Yukawa coupling $g_Y \propto \phi(0)$ and $\lambda \propto g_Y^4 \propto |\phi(0)|^4$. Moreover, inputting the known value of the Lagrangian mass of the BEH boson in the symmetric phase, which is $-|\mu|^2 = -(88)^2$ GeV^2, we find that the scale M_0 is $M_0 \approx 6$ TeV. The quartic coupling, λ, will be determined at loop level below and we find remarkable agreement with the SM as discussed below. The degree of fine-tuning of the theory is also suppressed by $\phi(0)$ in a subtle way. Rather than the naive result one would expect from the NJL model, $\delta g_0^2/g_c^2 \sim |\mu|^2/M_0^2 \sim 10^{-4}$, we now obtain a linear relation: $\delta g_0^2/g_c^2 \sim |\mu|/M_0 \sim 1\%$. To obtain these results we need the precise relationship between g_Y and $\phi(0)$ which follows from the induced Yukawa coupling of the bound state to free fermions.

E. The Induced Bound State Yukawa Interaction

We see in eq.(29) that the Yukawa interaction of the bound state with the free scattering state fermions is now induced from the $\sqrt{N_c}$ term, S':

$$g_0^2\sqrt{2N_c J} M_0^2 \int d^4X d^4r \; [\overline{\psi}_L^i(X+r)\psi_R(X-r)]_f D_F(2r) \; H_i(X)\phi(\vec{r}) + h.c.. \tag{50}$$

Consider the pointlike limit of the potential of eq.(44), $D_F(2r) \to (JM_0^2)^{-1}\delta^4(r)$:

$$\to g_0^2\sqrt{2N_cJ}\int d^4X\;[\overline{\psi}^i_L(X)\psi_R(X)]_f H_i(X)\phi(0)+h.c.. \tag{51}$$

We therefore see that the induced Yukawa coupling to the field $H(x)$ in the pointlike limit *of the potential* (which should be a reasonable low energy approximation) is:

$$g_Y = g_0^2\sqrt{2N_c/J}\;\phi(0) \tag{52}$$

Here the behavior of $\phi(0)$ is a suppression of g_Y that is a power-law, $\sim \sqrt{|\mu|/M_0}$ for small μ, near the critical coupling. This power-law behavior is significantly different than the slow RG (logarithmc) evolution in the old NJL-based top condensation model, which is why we now obtain $M_0 \sim 6$ TeV and reduced fine-tuning $\sim$ few % (rather than $M_0 \sim 10^{15}$ GeV and extreme fine-tuning $\sim 10^{-26}$).

III. QUANTUM FERMION LOOP CORRECTIONS TO THE SEMICLASSICAL THEORY

We recall the results for the fermion loops that arise in the pointlike NJL model with Yukawa interaction:

$$\int_x \left(g_0[\overline{\psi}_{iL}(x)\psi_R(x)]H^i(x) + h.c.\right). \tag{53}$$

This leads to the loop-induced terms in an effective action:

$$S = \int d^4x \left(ZDH^\dagger DH + M^2H^\dagger(x)H(x) - \frac{\lambda}{2}(H^\dagger H)^2\right). \tag{54}$$

where,

$$Z = \frac{g_0^2N_c}{8\pi^2}\ln(M_0/m), \qquad M^2 = \frac{g_0^2N_c}{8\pi^2}M_0^2, \qquad \lambda = \frac{g_0^4N_c}{4\pi^2}\ln(M_0/m). \tag{55}$$

The sign of the loop-induced M^2 term in the action is positive, representing an attractive negative term (tachyonic mass) in the effective potential, (while the quartic term is negative in the action, hence, repulsive in the potential). In the NJL model with bare mass M_0^2, the resulting mass of pointlike bound state is $\mu^2 = Z^{-1}(M_0^2 - M^2)$ and this defines the NJL critical coupling, $1 = g_0^2N_c/8\pi^2$, hence M^2 is the analogue of "binding energy" and causes the bound state to form.

This is a hint that the quantum loop effects will enhance the binding of the interaction in the semiclassical bilocal theory. However, it is important to allow the internal wave-function to respond to the attractive M^2 term, which now becomes part of the potential. Moreover, the NJL limit is a δ-function approximation, and we expect in the bilocal theory a softer contribution to the short distance potential than the mass M^2. Indeed, in the the loop calculation the NJL $M_0^2\delta^4(2r)$ cut-off is indistinguishable from the softer $D_F(2r)$ in the bilocal theory.

Hence, in the present analysis we will use the NJL limit in obtaining approximate bilocal loop integral results, but judiciously replacing $\delta^4(x-y) \to M_0^2D_F(x-y)$. Physically, the bilocal theory has a true momentum space cutoff of loop integrals $\sim M_0^2$, due to $D_F(q^2)$, while in the NJL theory this cut-off is implemented in loop integrals "by hand." The replacement of the $\delta^4(x-y)$ by $M_0^2D_F(x-y)$ approximately implements the true physics of the bilocal theory, allowing us to obtain a potential that, in turn, allows the "back-reaction" of the solution. We are invoking the "indistinguishability" of the $\delta^4(x-y)$ from the bilocal $M_0^2D_F(x-y)$, but maintaining the bilocality of the present theory.

By extending the auxiliary field equation to include the additional terms of $W(\Omega)$ we will obtain the quantum loop correction to the semiclassical action. These are the analogues of the terms in eq.(54) for the bilocal field. To compute $W(\Omega)$ we must integrate out Ω. This yields single-particle irreducible (1PI) diagrams leading to quadratic and quartic terms in B:

$$\begin{aligned}\frac{i}{\hbar}W(\Omega) = &\int_{xyx'y'}[B^\dagger(x,y)\mathcal{F}(x,y,x',y')B(x',y')],\\ &+\int_{x\ldots z'}[B^\dagger(x,y)B(x',y')]\mathcal{G}(x,y,x',y',w,z,w',z')[B^\dagger(w,z)B(w',z')] + ...,\end{aligned} \tag{56}$$

This largely parallels the loop corrections of the NJL model. This discussion is somewhat technical so we give the final results of the loop calculations in eqs.(7174).

A. Main Loop

Note the first term of eq.(56) contains the 4-point function $\mathcal{F}(x,y,x',y')$, given by:

$$\frac{i}{\hbar}W_{\mathcal{F}}(\Omega) = \int_{xyx'y'} [B^\dagger(x,y)\mathcal{F}(x,y,x',y')B(x',y')] =$$
$$-\frac{1}{2}g_0^2 N_c \int_{x\ldots y'} [B^\dagger(x,y)D_F(x-y)\,\mathrm{Tr}(S_F(x-x')S_F(y'-y))D_F(x'-y')B(x',y')] \tag{57}$$

where $S_F(x)$ is a fermionic Feynman propagator (we use conventions of [14]):

$$S_F(x) = \int \frac{d^4\ell}{(2\pi)^4}\frac{i\not\ell}{\ell^2+i\epsilon}e^{i\ell\cdot x} \qquad \text{and} \qquad \mathrm{Tr}\left(\frac{(1-\gamma^5)}{2}\not\ell\,\not\ell'\right) = 2\ell\cdot\ell' \tag{58}$$

In the loop integral we make the NJL approximation of taking a pointlike limit of the integral, defined by:

$$D_F(x-y) = -\int \frac{d^4q}{(2\pi)^4}\frac{e^{iq(x-y)}}{q^2-M_0^2} \to \int \frac{d^4q}{(2\pi)^4}\frac{e^{iq(x-y)}}{M_0^2} = \frac{1}{M_0^2}\delta^4(x-y) \tag{59}$$

then,

$$\mathcal{F}(x,y,x',y') \to -2g_0^2N_c\frac{\delta^4(x-y)\delta^4(x'-y')}{M_0^4}\int \frac{d^4\ell}{(2\pi)^4}\frac{d^4\ell'}{(2\pi)^4}\frac{\ell\cdot\ell' e^{i(\ell-\ell')(x-x')}}{\ell^2\ell'^2}. \tag{60}$$

It is useful to go to barycentric coordinates:

$$X = \frac{1}{2}(x+y), \quad r = \frac{1}{2}(x-y), \quad J = \left|\frac{\partial(x,y)}{\partial(X,r)}\right| = 2^4, \tag{61}$$

and define:

$$\widetilde{B}(X,r) \equiv B(X-r, X+r), \tag{62}$$

We will also define new integration variables, $\overline{X} = (X+X')/2$ and $R = (X-X')$, which have unit Jacobian.

Hence we have,

$$\int_{xyx'y'} [B^\dagger(x,y)\mathcal{F}(x,y,x',y')B(x',y')]$$
$$\to -g_0^2N_cJ^2\int_{XrX'r'}[\widetilde{B}^\dagger(X,r)\frac{\delta^4(2r)\delta^4(2r')}{M_0^4}\int\frac{d^4\ell}{(2\pi)^4}\frac{d^4\ell'}{(2\pi)^4}\frac{2\ell\cdot\ell' e^{2i(\ell-\ell')(X+r-X'-r')}}{\ell^2\ell'^2}\widetilde{B}(X',r')]$$
$$= -\frac{g_0^2N_c}{M_0^4}\int_{XX'}[\widetilde{B}^\dagger(X,0)\int\frac{d^4\ell}{(2\pi)^4}\frac{d^4\ell'}{(2\pi)^4}\frac{2\ell\cdot\ell'}{\ell^2\ell'^2}e^{i(\ell-\ell')(X-X')}\widetilde{B}(X',0)] \tag{63}$$

Note the cancellation of $J^2 = 2^8$ with the $(1/2^4)^2$ coming from integrating $\delta^4(2r)\delta^4(2r')$. Then $\widetilde{B}^\dagger(X,0)$ ($\widetilde{B}(X',0)$) is reduced to a pointlike field with dependence only upon X (X') where $r,r'\to 0$.

An outgoing (incoming) state is, $\widetilde{B}^\dagger(X,0)\sim\widetilde{B}_0^\dagger e^{iP(\overline{X}+R/2)}$, ($\widetilde{B}(X',0)\sim\widetilde{B}_0e^{-iP(\overline{X}-R/2)}$). Thus, integrating over $(\overline{X},R)$ we obtain,

$$-\frac{g_0^2N_c}{M_0^4}\int_{XR}e^{i(\ell-\ell'+P)(R)}[\widetilde{B}_0^{\dagger\prime}\widetilde{B}_0]\int\frac{d^4\ell}{(2\pi)^4}\frac{2\ell\cdot(\ell')}{\ell^2(\ell')^2} = -\frac{2g_0^2N_c}{M_0^4}\int_X[\widetilde{B}_0^{\dagger\prime}\widetilde{B}_0]\int\frac{d^4\ell}{(2\pi)^4}\frac{\ell\cdot(\ell+P)}{\ell^2(\ell+P)^2}$$
$$= \frac{ig_0^2N_c}{8\pi^2M_0^4}\int_{\overline{X}}[\widetilde{B}_0^\dagger\widetilde{B}_0]\left(M_0^2+P^2\ln\left(\frac{M_0}{m}\right)\right) \tag{64}$$

Here we use a Wick rotation and Euclidean momentum UV cut-off, $\ell^2 < M_0^2$.

The P^2 term corresponds to a loop-generated kinetic term for the auxiliary field B_i, (see eq.(83)). In the framework of the NJL model and original top condensation theory, [9], this corresponds the formation of a physical state, i.e., a resonance. We are interested presently in small P^2 so we will treat the $P^2\approx 0$ limit of the loop and return to this issue below.

The remaining M_0^2 term of eq.(64) is just the quadratically divergent two-point loop of the NJL model in disguise, as in eq.(55). We multiply by $-i\hbar$ to obtain the two-point loop contribution to the action $W_{\mathcal{F}}(\Omega)$,

$$W_{\mathcal{F}}(\Omega) = \frac{g_0^2 N_c}{8\pi^2 M_0^2}\int d^4X\; [\widetilde{B}^\dagger(X,0)\widetilde{B}(X,0)] \to \frac{g_0^2 N_c J}{8\pi^2 M_0^2}\int d^4X d^4r\; \widetilde{B}^\dagger(X,r)\delta^4(2r)\widetilde{B}(X,r) \tag{65}$$

While in the pointlike NJL model this term was interpreted as the quadratic negative term in the potential, $-g^2 N_c M_0^2/8\pi^2$, we see that, in the bilocal theory due to the implicit $\delta^4(2r)$, *this is a potential and not a mass term.* This presently acts as an attractive $\delta^4(2r)$ potential, and the wave-function $\phi(r)$ will react to it as such. However, as stated above, the $\delta^4(2r)$ is indistinguishable from the tree-level potential via: $\delta^4(2r) \to M_0^2 D_F(x-y)$ Therefore, using this replacement of the pointlike δ-function we obtain in the action ($P^2 \approx 0$):

$$W_{\mathcal{F}}(\Omega) = \int_{x\ldots y'} B^\dagger(x,y)\mathcal{F}(x,y,x',y')B(x',y') \approx \frac{g_0^2 N_c}{8\pi^2}\int d^4x\; d^4y\; B^\dagger(x,y)D_F(x-y)B(x,y) \tag{66}$$

where we have returned to the original (x,y) coordinates, and we have a two-body interaction. This is loop level, $\mathcal{O}\hbar$, and upon integrating out the auxiliary field, B, this will yield an enhancement of the semiclassical potential.

This result answers the potential criticism of [1], mentioned in the Introduction: If one computes the loop correction to the BEH mass by naively looping the Yukawa interaction in the semiclassical theory, then one would obtain the result of eq.(64), which appears to be a large correction to the BEH mass $\propto M_0^2$. Of course, this would be falsely double counting, since the only occurrence of the loop is in the underlying theory. Moreover, as emphasized above, *this is not a mass term for the bilocal field since it is* $\propto$ $\delta^4(2r)$ *and, rather, it is a short-distance (attractive) correction to the potential for* $\phi(r)$. The $\phi(r)$ field will adjust accordingly as the solution to the SKG equation, multiplicatively modifying the eigenvalue μ^2. Indeed, due to effective large distance scale symmetry near critical coupling, this must be interpreted as a radiative (multiplicative) correction to the effective coupling in the semiclassical potential. The loop integral leads to an attractive interaction which represents an enhancement of the effective coupling. We elaborate this further below.

B. Quartic Coupling

We also obtain the induced quartic interaction, which appears as an 8-point function in the effective semiclassical theory. This can be likewise evaluated in the NJL approximation. The loop integral involves $D_F^4(2r)\,\mathrm{Tr}\,(S_F)^4$ and we replace three of the D_F factors with $\delta^4(2r)/M_0^2$ which leaves one two-point integral of order g_0^4. The result can be inferred from the NJL calculation of the λ term in eq.(55) and is consistent with the detailed calculation:

$$\begin{aligned} W_{\mathcal{G}}(\Omega) &= \int_{x\ldots z'} B^\dagger(x,y)B(x',y')\mathcal{G}(x,y,x',y',w,z,w',z')B^\dagger(w,z)B(w',z') \\ &\approx \frac{g_0^4\hat{\lambda}}{2M_0^8}\left(\int_{xy}(B^\dagger(x,y)B(x,y))^2\delta^4(x-y)\right) \end{aligned} \tag{67}$$

where we define:

$$\hat{\lambda} = \frac{N_c}{4\pi^2}\ln\left(\frac{M_0}{m}\right). \tag{68}$$

We develop this further below.

C. Full Renormalized Action

The resulting action with the principal quantum effects considered here is then:

$$S' = W(\Omega) + \int_x\left([\overline{\psi}_L i\not{D}\,\psi_L] + [\overline{\psi}_R\, i\not{D}\,\psi_R]\right) + \int_{xy}\left(-[B^{i\dagger}D_F B_i] + g_0(B^{i\dagger}D_F C_i + h.c.)\right) \tag{69}$$

where C_i is defined in eq.(27) and:

$$\begin{aligned} W(\Omega) &= \int_{xyx'y'}[B^\dagger(x,y)\mathcal{F}(x,y,x',y')B(x',y')], + \int_{x\ldots z'}[B^\dagger(x,y)B(x',y')]\mathcal{G}(x,...,z')[B^\dagger(w,z)B(w',z')] + ..., \\ &= \frac{g_0^2 N_c}{8\pi^2}\int_{xy} B^\dagger(x,y)D_F(x-y)B(x,y) + \frac{g_0^4\hat{\lambda}}{2M_0^8}\int_{xy}(B^\dagger(x,y)B(x,y))^2\delta^4(x-y) \end{aligned} \tag{70}$$

Hence S' can be can be rewritten as:

$$S' = \int_x \left([\overline{\psi}_R i\not{D}\, \psi_R]_f + [\overline{\psi}_L i\not{D}\, \psi_L]_f \right) + \int_{xy} \left(g_0 B^{i\dagger}(x,y) D_F(x-y) C_i + h.c. \right)$$
$$- \int_{xy} [B^\dagger(x,y) D_F(x-y) B(x,y)] \left(1 - \frac{g_0^2 N_c}{8\pi^2}\right) + \frac{g_0^4 \hat{\lambda}}{2M_0^8} \int_{xy} [B^\dagger(x,y) B(x,y)]^2 \delta^4(x-y) \tag{71}$$

We now rescale B_i and g_0:

$$B_i' = \sqrt{R} B_i(x,y) \qquad \overline{g}_0 = R^{-1/2} g_0 \qquad \text{where,} \qquad R = \left(1 - \frac{g_0^2 N_c}{8\pi^2}\right). \tag{72}$$

We therefore have, in the 0^+ binding channel:

$$\overline{g}_0^2 = \frac{g_0^2}{1 - \frac{N_c g_0^2}{8\pi^2}} \tag{73}$$

The rescaling of g_0 sums the tower of fermion loops $\propto (g_0^2 N_c/8\pi^2)^n$ which is seen by expanding R^{-1}. The renormalized coupling $\overline{g}_0$ applies to the binding action, and not to the interaction of free fermions with the colorons (e.g., for the decay width of a coloron we would use the unrenormalized g_0). On energy scales $\ll M_0$, the enhanced $\overline{g}_0$ would apply in the 4-fermion interaction of free fermions, as is indicated by our effective action in eq.(77) below. However, for energy scales $\gg M_0$ the theory reverts to a pure topcolor gauge theory with coupling g_0 and the binding effects disappear. We see that $\overline{g}_0 > g_0$, so the underlying topcolor theory is weaker than the effective binding interaction. This is a "critical amplification" effect in the binding theory.

We then have,

$$S' = \int_x \left([\overline{\psi}_R i\not{D}\, \psi_R]_f + [\overline{\psi}_L i\not{D}\, \psi_L]_f \right) + \int_{xy} \left(\overline{g}_0 B_i'^{\dagger}(x,y) D_F(x-y) [\overline{\psi}_R(y) \psi_L^i(x)]_f + h.c. \right)$$
$$- \int_{xy} [B'^\dagger(x,y) D_F(x-y) B'(x,y)] + \frac{\overline{g}_0^4 \hat{\lambda}}{2M_0^6} \int_{xy} D_F(x-y) [B'^\dagger(x,y) B'(x,y)]^2 + ..., \tag{74}$$

where in the last term we use $\delta^4(x-y) \approx M_0^2 D_F(x-y)$.

The "equation of motion" of the auxiliary field, B', using eq.28, then takes the form,

$$B_i'(x,y) = \overline{g}_0 C_i(x,y) + \overline{g}_0^4 \hat{\lambda} M_0^{-6} B_i'(x,y) (B_j'^{\dagger}(x,y) B'^j(x,y)) + ..., \tag{75}$$

where we defined C_i in eq.(27). We then solve eq.(75) perturbatively in $\hat{\lambda}$ (where $\hat{\lambda}$ is defined in eq.(68):

$$B_i'(x,y) \approx \overline{g}_0 C_i + \overline{g}_0^4 \hat{\lambda} M_0^{-6} C_i (C^{\dagger j}(x) C_j(x)) + ... \tag{76}$$

Substituting into eq.(74) the action is then:

$$W(\Psi, H) = \int_x \left([\overline{\psi}_L(x) i\not{D}\, \psi_L(x)] + [\overline{\psi}_R(x)\; i\not{D}\, \psi_R(x)] \right)$$
$$+ \overline{g}_0^2 \int_{xy} \Big(N_c M_0^4 H^\dagger(x,y) D_F(x-y) H(x,y) + \sqrt{N_c} M_0^2 (H^{i\dagger}(x,y) D_F(x-y) [\overline{\psi}_R(y) \psi_{iL}(x)]_f + h.c.)$$
$$+ [\overline{\psi}_L^i(x) \psi_R(y)]_f D_F(x-y) [\overline{\psi}_R(y) \psi_{iL}(x)]_f \Big) + \frac{1}{2} \overline{g}_0^8 N_c \hat{\lambda} \int_{xy} \delta^4(x-y) [H^{\dagger i}(x,y) H_i(x,y)]^2 + ... \tag{77}$$

Passing to barycentric coordinates, $H^i(x,y) \to \sqrt{2/J}\; H^i(X)\phi(r)$, and, $M_0^2 D_F(2r) \to \delta^4(2r)$, we see that the Yukawa interaction becomes:

$$\overline{g}_0^2 \int_{xy} \sqrt{N_c} M_0^2 (H^{i\dagger}(x,y) D_F(x-y) [\overline{\psi}_R(y) \psi_{iL}(x)]_f + h.c.) \to \overline{g}_Y \int d^4X \left(H^{i\dagger}(X) [\overline{\psi}_R(X) \psi_{iL}(X)]_f + h.c. \right) \tag{78}$$

Therefore, we now have a renormalized Yukawa coupling:

$$\overline{g}_Y = \overline{g}_0^2 \sqrt{2N_c/J}\; \phi(0) \tag{79}$$

The physical top-Yukawa coupling is now given by $\overline{g}_Y$ and this will be set to its experimental value of unity as an input to define the theory and obtain M_0.

Furthermore, we have for the quartic term (where $\hat{\lambda}$ is defined in eq.(68):

$$\frac{1}{2}\overline{g}_0^8 N_c\hat{\lambda}\int_{xy}\delta^4(x-y)[H^{\dagger i}(x,y)H_i(x,y)]^2 \longrightarrow \frac{1}{2}\hat{\lambda}[\overline{g}_0^8 JJ^{-1}(2/J)^2N_c^2|\phi(0)|^4]\int_X (H^{i\dagger}(X)H_i(X))^2$$
$$\equiv \frac{\lambda}{2}\int_X\biggl(H^\dagger(X)H(X)\biggr)^2 \qquad \text{where, using eq.(79),} \qquad \lambda = \frac{\overline{g}_Y^4 N_c}{4\pi^2}\ln\biggl(\frac{M_0}{m}\biggr) \tag{80}$$

We can therefore write the leading terms in the full theory:

$$W(\Psi,H) = \int_x\biggl([\overline{\psi}_L(x)i\not{D}\,\psi_L(x)]_f + [\overline{\psi}_R(x)\;i\not{D}\,\psi_R(x)]_f\biggr)$$
$$+\int_X\biggl(DH^\dagger DH + |H(X)|^2\int_r\biggl(-|\nabla_{\vec{r}}\phi(\vec{r})|^2 + \bigl(\overline{g}_0^2 D_F(2r)|\phi(r)|^2\bigr)\biggr)\biggr)$$
$$+\int_{xy}\;\overline{g}_Y(H^{i\dagger}(X)[\overline{\psi}_R(X)\psi_{iL}(X)]_f + h.c.) + \;\overline{g}_0^2[\overline{\psi}_L^i(x)\psi_R(y)]_f D_F(x-y)[\overline{\psi}_R(y)\psi_{iL}(x)]_f$$
$$+\frac{\lambda}{2}\int_X (H^\dagger(X)H(X))^2 + ... \tag{81}$$

and with a solution to the SKG equation for $\phi(r)$ with eigenvalue μ^2 we have

$$W(\Psi,H) = \int_x\biggl([\overline{\psi}_L(x)i\not{D}\,\psi_L(x)]_f + [\overline{\psi}_R(x)\;i\not{D}\,\psi_R(x)]_f\biggr)$$
$$+\int_X\biggl(DH^\dagger DH - \mu^2|H(X)|^2\biggr)$$
$$+\,\overline{g}_0^2\int_{xy}\;[\overline{\psi}_L^i(x)\psi_R(y)]_f D_F(x-y)[\overline{\psi}_R(y)\psi_{iL}(x)]_f$$
$$+\,\overline{g}_Y(H^{i\dagger}(X)[\overline{\psi}_R(X)\psi_{iL}(X)]_f + h.c.) + \frac{\lambda}{2}\int_X (H^\dagger(X)H(X))^2 + ... \tag{82}$$

While we have extrapolated the loops to zero P^2, we note that eq.(64) contains a P^2 term. If we interpret this in the Wilsonian description of the NJL model, as in [1, 9], then eq.(64) implies that the theory is creating a loop level resonance of the B field with an action (ignoring gauge fields):

$$\sim Z'\partial_X B'^{i\dagger}\partial_X B'_i - M_0^2 B'^{i\dagger}B'_i \qquad \text{where,} \qquad Z' = \frac{\overline{g}_0^2 N_c}{8\pi^2}\ln\left(\frac{M_0}{m}\right) \tag{83}$$

where $m \sim |P|$. The on-shell resonance mass near critical coupling is therefore $M^2_{resonance} = M_0^2/\ln\left(\frac{M_0}{m}\right)$. (Here we have reverted to interpreting $M_0^2 D_F \to \delta^4(2r)$). Then eq.(76) is satisfied as $P^2 \ll M_0^2$. The formation and exchange of this resonance below threshold $P^2 \sim 0$ generates an attractive interaction and can be viewed as leading to the enhancement of the effective coupling $\overline{g}_0^2$ as $P^2 \to 0$.

IV. DISCUSSION AND CONCLUSIONS

A. Physical Consequences of the Improved Yukawa Coupling

The renormalized topcolor coupling in the 0^+ binding channel satisfies:

$$\overline{g}_0^2 = \frac{g_0^2}{1-\frac{N_c g_0^2}{8\pi^2}} \tag{84}$$

where g_0^2 is the underlying topcolor coupling. This represents "critical amplification" of the underlying coupling g_0^2 and all of the induced couplings, g_Y and λ (etc.). It is $\overline{g}_0^2$ that controls the formation of the semiclassical bound state. This amplification effect is not unexpected, since the quantum fermion loops are essentially NJL model effects, and these are strongly binding. We see that this can be understood as the formation of a resonance in the 0^+ as an NJL bound state, enhancing the effective coupling $\overline{g}_0^2$ as $P^2 \to 0$.

The bound state is critical (has zero mass μ^2) when $\overline{g}_0^2 = \overline{g}_{0c}^2$ where we determine numerically [1]:

$$\frac{\overline{g}_{0c}^2 N_c}{8\pi^2} = 1.06940. \qquad \text{cf. NJL model: } \frac{\overline{g}_{0c}^2 N_c}{8\pi^2} = 1 \tag{85}$$

This is the criticality condition of the Schrödinger equation with a Yukawa potential as translated into the SKG equation (note that we've often approximated $g_{0c}^2 \approx 8\pi^2/N_c$ using the NJL critical value in [1]).

Solutions to the SKG equation lead to the eigenvalue μ^2 which can be identified with the SM symmetric phase, $\mu^2 = -(88)^2$ GeV^2, which is negative (tachyonic) for supercritical $\overline{g}_0^2 > \overline{g}_{0c}^2$. The solutions are discussed in [1]. For example, if we assume a large distance solution of the form (the "skeletal solution" of Section IV.E of [1]), then

$$\phi(r) \approx \frac{ce^{-|\mu|r}}{r}; \qquad \text{then we find,} \qquad \phi(0) = \frac{2\sqrt{2}}{\pi^{3/2}}\sqrt{\frac{|\mu|}{M_0}} = 0.50795\sqrt{\frac{|\mu|}{M_0}} \tag{86}$$

We can then fit the renormalized Yukawa coupling to experiment,

$$\overline{g}_Y = 1 = \overline{g}_{0c}^2\sqrt{\frac{2N_c}{J}}\ \phi(0) = 17.236\ \phi(0) \approx 8.7548\sqrt{\frac{|\mu|}{M_0}}, \tag{87}$$

we then predict $M_0 = 6.745$ TeV.[6] Likewise, using an approximate numerical solution to the SKG equation yields $M_0 = 5.23$ TeV., (this is sensitive to a fitting function, and likely underestimates M_0). Conservatively we would estimate M_0 to lie in a range of order $5 < M_0 < 7$ TeV.

It should be emphasized that the collider physics predictions for properties of colorons will depend mainly upon g_0^2, *not* $\overline{g}_0^2$. The colorons form a QCD color octet, but with the exception of couplings to $\bar{t}t$ would not be singly produced and would be pair produced in gluon fusion processes, which is sensitive to the QCD coupling and M_0. Such coloron pairs lie near the energy limit of the LHC and will decay to pairs of $\bar{t}t$, and we also expect $\bar{b}b$ (topcolor must be extended to include the b_R-quark and a Z' to insure the $\bar{b}b$ channel is subcritical [5]). Coloron decay widths are controlled by g_0^2. If $\overline{g}_0^2$ is approximately critical, $N_c\overline{g}_0^2/8\pi^2 \approx (1.0694)$ then from eq.(84) we have,

$$\frac{g_0^2 N_c}{8\pi^2} \approx \frac{1.0694}{1 + 1.0694} = 0.51677 \tag{88}$$

and the underlying topcolor theory is subcritical, and $\alpha_{topcolor} = \frac{g_0^2}{4\pi} \approx 1.0823$ is marginally perturbative.

B. Physical Consequences of the Improved Quartic Coupling

For the quartic coupling we obtain above:

$$\lambda \approx \frac{\overline{g}_Y^4 N_c}{4\pi^2} \ln\left(\frac{M_0}{\mu}\right). \tag{89}$$

The log evolution is just that of a single fermion loop and matches the result for the pointlike NJL case, with $g_{top} = \overline{g}_Y$. Not surprisingly when we take the pointlike potential limit the loop result of the bilocal theory confirms a pointlike NJL calculation (see discussion of Section 4 of ref.([2]).

Experimentally, in the SM using the value of $m_{BEH} \approx 125$ GeV and $v_{weak} \approx 175$ GeV, we find $\lambda \approx 0.25$. In the old pointlike NJL top condensation model the quartic coupling was determined by the RG with "compositeness boundary conditions," where we obtained (running-down from the Landau pole at M_0 to $|\mu|$), the result $\lambda \sim 1$. This is too large and leads to predicted $m_{BEH} \sim 260$ GeV. Indeed, the quartic coupling is generally problematic for NJL based theories of a composite BEH boson.

[6] This was computed in [1] where we quoted 5.9 TeV, which resulted from using the NJL critical value $N_c g_{0c}^2/8\pi^2 = 1$ rather than the true $N_c g_{0c}^2/8\pi^2 = 1.0694$. However, we think these are within the uncertainty of the leading order calculations for the SKG solutions, and does not take RG running effects, e.g., logarithms, into account.

However, in the present bilocal scheme owing to suppression of $\overline{g}_Y \propto \phi(0)$, the quartic coupling is also suppressed $\propto |\phi(0)|^4$ and is now generated in RG running, from a value of zero at $M_0 = 6$ TeV, down to $|\mu| \sim 88$ GeV, using the induced $\overline{g}_Y$. Keeping only the $\overline{g}_Y^4$ contribution we would obtain numerically from eq.(89):

$$\lambda \approx (\overline{g}_Y^4)\frac{N_c}{4\pi^2}\ln\left(\frac{M_0}{|\mu|}\right) \approx 0.321 \quad \text{(cf., 0.25 experiment.);} \tag{90}$$

This result is significantly better than the old NJL top condensation model. However, prefactor at one loop level should reflect the full RG running of λ, (see, e.g., [16]), which would yield at leading log approximation:

$$\lambda \approx (\overline{g}_Y^4 - \overline{g}_Y^2\lambda - \lambda^2)\frac{N_c}{4\pi^2}\ln\left(\frac{M_0}{|\mu|}\right) \approx 0.230 \quad \text{(cf., 0.25 experiment.);} \tag{91}$$

This includes the λ^2 term which arises from loops with propagating internal BEH boson. For the present composite model this not likely to be correct, due to the extended size of the BEH wave-function. The terms of order $\overline{g}_Y^2\lambda$ are leg-renormalizations, due to the top quark loop and presumably can be retained. This would imply a result:

$$\lambda \approx (\overline{g}_Y^4 - \overline{g}_Y^2\lambda)\frac{N_c}{4\pi^2}\ln\left(\frac{M_0}{\mu}\right) \approx 0.243 \quad \text{(cf., 0.25 experiment.);} \tag{92}$$

Note that since we only know M_0 to precision of order $\sim 6 \pm 1$ TeV, we expect the λ prediction is uncertain by $\sim \pm 0.01$.

C. Summary

We have used the techniques of Jackiw, *et al.*, [3, 4] to obtain formally the effective action for the semiclassical third generation fields ψ_L, ψ_R in a single coloron exchange interaction field theory of topcolor. This describes the Brout-Englert-Higgs boson as a composite object with constituents that are the semiclassical fields composed of t and b quarks.

This marks a departure from the old RG improved NJL model which builds a composite state by integrating out all of the fermions, in analogy to confinement in QCD. For us, the bound state is softer, and more Hydrogenic, containing a subset of the low momentum modes in the semiclassical theory. This gives us the internal wave-function $\phi(r)$. Tuning the theory within a few percent of the critical coupling yields a hierarchy that defines a low mass BEH bound state. We have given the central results in the companion paper [1] and addressed the most significant quantum effects presently. Here we have introduced an auxiliary field, B_i, which consolidates the main large-N_c fermion loop contributions. Upon computing these, we integrate out B_i to obtain the full quantum corrected action.

Our most significant new result is that the topcolor coupling, g_0^2 is subcritical while the effective coupling in the binding potential $\overline{g}_0^2$ is enhanced and can be slightly supercritical:

$$\overline{g}_0^2 = g_0^2\left(1 - \frac{N_c g_0^2}{8\pi^2}\right)^{-1} \tag{93}$$

This in turn renormalizes the Yukawa coupling and quartic coupling. Hence the value of $\overline{g}_0^2$ is "critically amplified" above a weaker value of the underlying topcolor coupling g_0^2.

Inputting the experimental result for the renormalized Yukawa coupling, $\overline{g}_Y = 1$ with the symmetric phase BEH mass, $\mu^2 = -(88)^2$ GeV^2, yields the central prediction obtained for $M_0 \approx 6$ TeV as discussed in [1]. Hence the major results of [1] still hold, but with g_0^2 now replaced by $\overline{g}_0^2$. Due to the linear relationship between $\overline{g}_0^2$ and $|\mu|/M_0$, a consequence of the dilution effect of $\phi(0)$ as discussed in [1], we see the degree of fine-tuning of the hierarchy is of order $\sim |\mu|/M_0 \sim 1.4\%$.

We explicitly demonstrate that the BEH boson quartic coupling, λ, arises from loops $\sim N_c\overline{g}_Y^4 \ln(M/\mu)/4\pi^2 \propto (\phi(0))^4$. The result for the quartic coupling is $\lambda \sim 0.24$, compared to $\lambda \approx 0.25$ experimentally. The λ prediction is in astonishingly good agreement with experiment and we believe makes the natural top condensation idea compelling.

The "colorons" which mediate the binding interaction form a QCD color octet. They have a global $SU(3)$ symmetry and a conserved $SU(3)$ current, and must therefore be pair-produced. They will decay in the minimal model to $\bar{t}t$, but in more realistic topcolor models they can also decay to $\bar{b}b$ ([5][6][11] and references therein). They may be accessible to the LHC in the multi-TeV range [18–20], favoring the third generation in its couplings. It should be easy to obtain a lower bound on M_0 from single gluon or gluon fusion production of pairs of the colorons. The colorons will produce excesses in 4-top, $t\bar{t}t\bar{t}$, events. In an extended model we would also expect $t\bar{t}b\bar{b}$ and $b\bar{b}b\bar{b}$ anomalies to emerge. The more general topcolor models offer many possibilities within the arena of heavy quark flavor physics.

This theory, if confirmed, solves the "naturalness problem" of the BEH boson in the Standard Model and opens up a vista of new physics with new gauge interactions. Many avenues for further theoretical development exist. Notably, a revisitation of the Topcolor Z' [5], resonances, and new flavor physics associated with $\bar{b}b$ would be of interest. There is much to do to further develop these models in the context of the present dynamics.

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