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FERMILAB-PUB-25-0944

arXiv:2512.16527

This manuscript has been authored by Fermi Forward Discovery Group, LLC under Contract No. 89243024CSC000002 with the U.S. Department of Energy, Office of Science, Office of High Energy Physics.

Delafield-1220-2025

The Nontrivial Vacuum Structure of an Extended $t\bar{t}$ BEH (Higgs) Bound state

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In a recent reformulation of top-quark condensation for the Brout-Englert-Higgs boson, we introduced an extended internal wave-function, $\phi(r)$. We show how this leads to a *manifestly* Lorentz invariant formalism, where the absence of "relative time" is a gauge invariance of the bilocal field theory. This dictates a novel and nontrivial Lorentz invariant vacuum structure for the BEH boson, the relativistic generalization of a condensed matter state such as a BCS or Bose-Einstein condensate.

I. INTRODUCTION

Beginning with Schrödinger, any two-body bound state can be described in a semiclassical limit as a bilocal field, $H(x, y)$ [1, 2]. For the ground state ansatz this factorizes in barycentric (center-of-mass system) coordinates as:

$$H(x^\mu, y^\mu) \sim H(X^\mu)\phi(r^\mu) \tag{1}$$

In the case of a Brout-Englert-Higgs (BEH) boson the constituents are massless chiral top and anti-top quarks located at space-time coordinates $(x^\mu, y^\mu) = X^\mu \pm r^\mu$. $H(X^\mu)$ can be viewed as the standard model (SM) BEH isodoublet with electroweak charges and $\phi(r^\mu)$ is a complex scalar which is electroweak neutral [3, 4]. $H(X^\mu)$ describes the center-of mass motion of the BEH boson, and $\phi(r^\mu)$ is then the internal wave-function.

A theory of a BEH boson composed of $\bar{t}t$, known as "top condensation," was proposed in the 1990's [5–8], deploying the Nambu-Jona-Lasinio (NJL) model [9] with renormalization group (RG) improvements [6]. The NJL model, however, is pointlike, lacking $\phi(r^\mu)$, which leads to difficulties when there is a large hierarchy between the composite scale, M_0, and the electroweak scale $|\mu|$ (the symmetric phase mass of the BEH boson, $|\mu| = 88$ GeV). The inclusion of the internal wave-function yields significant improvement in the predictions of the low energy parameters. In the limit $|\mu| \ll M_0$ there is significant wave-function spreading of $\phi(\vec{r})$ and the resulting dilution effects dominate the low energy effective theory. This brings its predictions into concordance with experiment, virtually eliminating fine-tuning, and predicting the new mass scale of the binding interaction, $M_0 \sim 6$ TeV, ref.([3],[4]) summarized in Appendix A.

We emphasize that the theory is *manifestly* Lorentz invariant. The challenge is, however, that the internal wave-function, $\phi(r^\mu)$, introduces *a priori* unwanted dependence upon "relative time." This is $r^0 = (x^0 - y^0)/2$ in the rest frame of the bound state, but boosted relative time would occur in any frame. Due to the single time parameter of Hamiltonian based quantum mechanics, the relative time is unphysical and must be removed in a manner consistent with Lorentz invariance. This issue is avoided in the NJL model due to the pointlike interaction, but the relative time problem is well known to arise in any bound state with an extended interaction, [10]. It can be seen as arising in the free field limit by kinematics (below).

Relative time implies an "arrow of relative time," a timelike, unit, 4-vector ω_μ, associated with $\phi(r^\mu)$. The relative time is then τ, where $r^\mu = \omega^\mu \tau$. The absence of relative time can then be viewed as a gauge symmetry of the internal wave-function $\phi(r^\mu)$, where a "gauge transformation" is $r^\mu \to r^\mu + \omega^\mu \tau$ with τ acting as a gauge parameter. Given an ω^μ we can pass to a manifestly "gauge invariant" $\phi(r^\mu)$ field, the analogue of a"Stueckelberg" field, where the symmetry is built in:

$$\phi(r^\mu) \to \phi_\omega(r^\mu) \equiv \phi\left(\omega^\mu(\omega_\nu r^\nu) - r^\mu\right) \qquad \omega^2 \equiv \omega_\mu\omega^\mu = 1. \tag{2}$$

The issue then becomes: "what defines ω_μ?"

For simple two-body bound states, where $H(X) \sim \exp(iP_\mu X^\mu)$, then ω^μ can be identified with the normalized 4-momentum, $\omega^\mu = P^\mu/\sqrt{|P^2|}$ (e.g. via Lagrange multipliers). Hence, a two-body spherically symmetrical bound state becomes:

$$H(X)\phi_\omega(r^\mu) = \exp(iP_\mu X^\mu)\; \phi\left(\sqrt{((P_\mu r^\mu)^2/P^2) - r_\mu r^\mu}\right) \tag{3}$$

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$\phi(r^\mu)$ then reverts to $\phi(\vec{r})$ in the rest frame where it can be treated as a solution to a static Schrödinger-Klein-Gordon (SKG) equation [3].

However, there remains the challenging question when $\mu^2 < 0$ and spontaneous symmetry breaking occurs: "How can we have a Lorentz invariant vacuum state with $P^\mu = 0$, but nonzero ω_μ?" That is, "what determines ω^μ in the vacuum?" Any constraint that locks P^μ to ω^μ in the two-body case ceases to exist in the vacuum and ω^μ would seem to become arbitrary. While one possibility is that $\phi(r^\mu) \rightarrow$(constant) in the vacuum, this does not lead to a consistent result with the internal dynamics of $\phi(r)$.

It would seem nonsensical to assert that the vacuum is defined by a condensate that occurred in a particular Lorentz frame with a particular ω_μ. If, for example, ω_μ is somehow associated with the local cosmic rest frame we would obtain induced Lorentz violating effects in the electroweak physics, e.g., vacuum Cerenkov radiation for all particles that receive mass from the BEH boson, top-quarks to electrons and neutrinos [11]. We would also have potentially large radiatively induced Lorentz non-invariant corrections to electrodynamics [12] (we have estimated these effects in [3] and while they are suppressed as $\sim |\mu|^2/M_0^2$, they appear nonetheless to be problematic). We would prefer a starting point in which the vacuum is manifestly Lorentz invariant and contains no preferred ω^μ. Nonetheless, we require ω_μ to define the solutions, $\phi_\omega(r^\mu)$, to implement the relative time invariance of our theory.

Therefore, we propose the following solution to the vacuum problem: *the vacuum is a Lorentz invariant sum over all frames of the individual solutions $\phi_\omega(r^\mu)$ in each frame.* We will introduce a Lorentz invariant integral over ω^μ, leading to a novel internal wave-function for the vacuum state, $\Phi(r^\mu)$, with (unrenormalized) $H'(X)$.

$$H(x, y) = H'(X)\Phi(r) \qquad \text{where,} \qquad \Phi(r) = \mathcal{N} \int d^4\omega\ \delta(\omega^2 - 1)\ \phi_\omega(r^\mu) \tag{4}$$

With this definition of the vacuum, $\Phi(r)$ is a collective state, similar to a condensed matter system, such as a BCS superconductor, [14], or Bose-Einstein condensate (BEC) in, e.g., the Gross-Pitaevskii model, or Ginzburg-Landau model of superconductivity [15]. However, our new vacuum differs from these because it is Lorentz invariant.

For example, in a BEC, created in a laboratory, one coaxes a large number, N, of bosons into a state with common 3-momentum, such as $\vec{P} = 0$. Hence two-body scattering states are defined by the relative momenta, $\vec{Q}_i$, and would have bilocal wave-functions, $\sim a_i e^{i\vec{Q}_i\cdot\vec{r}}$. The Hamiltonian kinetic term for the pairs is then $\sim \sum_i |a_i|^2 Q_i^2 \propto N$ while the short-distance, two-body, attractive scattering potential is $\sim -M \sum_i \sum_j a_i^\dagger a_j \propto N^2$. In the large N limit the Hamiltonian is then dominated by the two-body scattering effects and is diagonalized by a coherent state that is a sum over all $\vec{Q}_i$ as , $\Phi \sim \sum_i a_i e^{i\vec{Q}_i\cdot\vec{r}}/\sqrt{N}$. This dominance of the interaction is a key feature of condensates. It occurs in the BCS state of a superconductor (analogously the Ginzburg-Landau wave-function), that is a sum (or product in kets) over all Cooper pairs with $\vec{Q}_i$ lying on the Fermi surface in momentum space, [14, 15]. These states exist in a preferred, $\vec{P} = 0$ frame, the rest frame of the system.

The vacuum correlation function we propose presently, $\Phi(r)$, is a sum over *constituent wave-function solutions of the SKG equation*, each with $P^\mu = 0$ but nontrivial $\phi_\omega(r^\mu)$, each specified by a different Lorentz frame, ω_μ. The ω_μ are timelike and span the timelike hyperboloid in momentum space. The $\phi_\omega(r^\mu)$, in a sense, are the relativistic generalization of "Cooper pairs" [14]. While Cooper pairs span the Fermi surface in a superconductor, the $\Phi \sim \int_\omega \phi_\omega$ span the future timelike hyperboloid in the 4-vectors, ω. It follows that the invariant integral over ω_μ yields a Lorentz invariant $\Phi(r)$.

Though the component fields, $\phi_\omega(r)$, satisfy a nontrivial *integro-differential equation* with dependence upon ω^μ, there is no requirement on the relative time in the vacuum (any Lagrange multipliers that lock ω^μ to a P^μ are now absent). Upon integrating over ω^μ the time component of r^μ returns in $\Phi(r^\mu)$, but the arrow of relative time, ω^μ, is removed. The BEH boson observed at the LHC, $h(X, r)$ and the Nambu–Goldstone phases that become longitudinal $W^\pm$ and Z^0 emerge as an "excitons" of this collective state, e.g., as $h(X, r) \propto h(X)\Phi(r^\mu)$. While dependence upon r^0 is now a feature of the vacuum wave-function, $\Phi(r)$, analogous to a vacuum two-point function, or a propagator, this makes no reference to a particular ω_μ.

The presence of $\Phi(r)$ is unobservable in the kinetic terms of the BEH boson, gauge bosons and top quarks, which conform exactly to the SM. We do, however, have access to its structure in the Yukawa and quartic interactions through an expansion in powers of r^μ, which corresponds to an expansion in inverse powers of M_0. This leads to a series of higher dimension operators, generating novel, Lorentz invariant, effects that are small. In the present paper we will only briefly describe how these effects can be extracted from the Yukawa coupling in the Section IV (see also [3]). We presently ignore the complications of the introduction of all flavors of quarks and leptons, in particular the b_R quark. We expect these to be perturbative and follow the earlier papers on extended technicolor [13] and "topcolor" from the 1990's [7]. In this paper we will mainly discuss the nontrivial vacuum.

This theory of the vacuum emerges from the underlying theory of two-body bound states with $H'(X)\Phi(r) = H(X)\widetilde{\Phi}(r)$ where $\widetilde{\Phi}(r) \propto \Phi(r)$ is the *classical average* over all $\phi_\omega(r^\mu)$ and $\langle H \rangle = v_{weak}$. In particular we have,

$$\widetilde{\Phi}(0) = \phi_\omega(0) \qquad \text{for any } \omega_\mu. \tag{5}$$

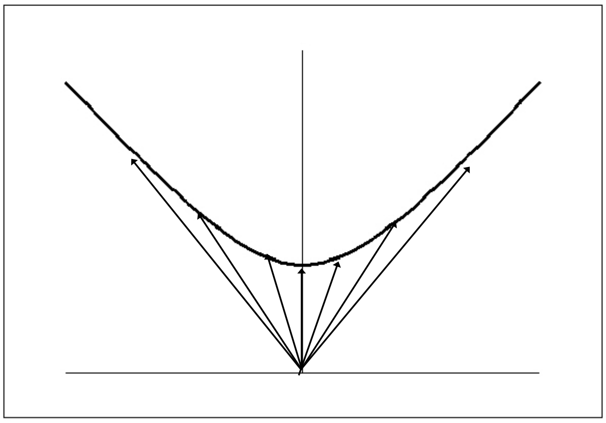

FIG. 1: Vacuum wave-function, $\Phi(r)$, spanning the timelike hyperboloid in 4-vectors ω_μ by integrating over "internal wave-functions," $\phi_\omega(r)$, to form the Lorentz invariant $\Phi(r) = \mathcal{N}\int d^4\omega\; \delta(\omega^2-1)\; \phi_\omega(r^\mu)$.

This relationship establishes that the predictions of the full theory M_0, λ, etc., with the full vacuum structure $\Phi(r)$ included, agree with the earlier results from the simple two-body bound state studies in [3], and are valid to order $|\mu|^2/M_0^2$. In the broken phase we then have:

$$H'(X)\Phi(r) = H(X)\widetilde{\Phi}(r^\mu) \to \exp(i\pi^a(X)\tau^a/2v_{weak})\begin{pmatrix} v_{weak} + \frac{1}{\sqrt{2}}h(X) \\ 0 \end{pmatrix}\widetilde{\Phi}(r^\mu) \tag{6}$$

where $h(X)$ is the "Higgs field" of the standard model. The $H'(X)\Phi(r)$ action will involve renormalized parameters but will be seen to be completely consistent with the two-body $H(X)\widetilde{\Phi}$ action, classically averaging over ϕ_ω. Here the constant zero 4-momentum VEV, v_{weak}, is carried by $H(X)$ and determined in the usual way by the minimum of the sombrero potential.

The effective action of the $H'(X)\Phi(r)$ bilocal field is completely consistent with the simpler two-body $H(X)\phi_\omega(r)$ action. The main prediction is the existence of the new binding interaction at the scale $M_0 \approx 6$ TeV and the emergence of an octet of colorons coupled most strongly to third generation quarks. This interaction is only partially strong (approximately half critical) since the critical behavior occurs only in the binding channel where loop effects reinforce the binding interaction [4].

II. ABSENCE OF RELATIVE TIME AS A GAUGE INVARIANCE

We begin with the notion that the internal wave-function of two-body bound state, $\phi(r^\mu)$, must not depend upon "relative time," i.e., is independent of r^0 in the barycentric frame, or correspondingly r'^0 in any frame. This can be seen in the free field limit by kinematics if we consider a pair of equal-mass particles of 4-momenta p_1 and p_2, $p_1^2 = p_2^2 = m^2$, and a bilocal wave-function, $H(x,y) \sim \exp(ip_1x + ip_2y)$. We pass to the total momentum $P = (p_1 + p_2)$ and relative momentum $Q = (p_1 - p_2)$, and the plane waves become $\exp(iPX + iQr)$, where we define "barycentric coordinates," $x^\mu = X^\mu + r^\mu$, $y^\mu = X^\mu - r^\mu$. Note that $P_\mu Q^\mu = p_1^2 - p_2^2 = 0$. Therefore, in the center-of-mass frame, in which $P = (P^0, \vec{0})$ and $Q = (0, \vec{q})$, we see that $Q^0 = 0$. This implies there is no dependence in the bilocal state on r^0 through $\exp(iQ_0r^0) = 1$, and likewise no dependence upon a boosted relative time r'^0 in any other frame. If the particles are constituents of a bound state, then the "relative time" must decouple from the dynamics.

Given an arbitrary Lorentz invariant function, $\phi(r^\mu)$, in any frame there will generally be dependence upon a relative time, $\sim \tau$. This is analogous to the gauge dependent components of a vector potential, and it is an artifact of using the bilocal field description.[1] As described in the Introduction, the relative time, τ, can be written in any given frame as $r^\mu = \omega^\mu\tau$, and implicitly requires the timelike, 4-vector, ω^μ, the "arrow of relative time." Hence, eliminating dependence upon τ, requires a Lorentz invariant constraint, such as $\omega_\mu\partial^\mu\phi(r) = 0$.

[1] I am grateful to Bill Bardeen for some discussions that inspired this perspective.

In the symmetric phase of the standard model (SM), (or for any typical two-body bound state) the BEH boson contains such a vector, i.e., the 4-momentum P^μ carried by $H(X^\mu) \sim \exp(iP_\mu X^\mu)$. We can therefore bootstrap ω^μ to P^μ through a constraint relation $\omega^\mu \propto P^\mu$. For example, we can do this semiclassically by introducing Lagrange multipliers into the action, such as:

$$W = M_0^4 \int d^4X d^4r \; \lambda'(H^\dagger D_\mu H \phi^\dagger \partial_\mu \phi) \tag{7}$$

Hence we demand $\delta W/\delta\lambda' = 0$, which imposes the kinematic constraint $P_\mu Q^\mu = 0$. However, the deeper question remains: "what happens in the vacuum where $P_\mu = 0$ and ω^μ becomes unconstrained?"

Consider the bilocal field in barycentric coordinates:

$$H(x,y) \to H(X^\mu)\phi_\omega(r^\mu) \tag{8}$$

We will presently focus upon the fields, $\phi_\omega(r^\mu)$, as defined in eq.(2) which are invariant under the gauge transformation $r^\mu \to \omega^\mu\tau$ and thus have no dependence upon τ, though an implicit dependence upon ω^μ remains. Indeed, $\phi_\omega(r^\mu)$ is then the analogue of a "Stueckelberg" field, e.g., a gauge field such as $B_\mu = A_\mu - \partial_\mu\chi$ which is invariant under $A_\mu \to A_\mu + \partial_\mu\tau$ and $\chi \to \chi + \tau$. Note that eq.(2) satisfies the constraint equation $0 = \omega^\mu\partial_\mu\phi(r^\mu)$. In the following, $\phi(r^\mu)$ will refer to an arbitrary Lorentz invariant scalar, while $\phi_\omega(r^\mu)$ is of the (Stueckelberg) form of eq.2) with the relative time projected out.

Consider the action S_ϕ for the internal field $\phi(r^\mu)$ that arises in natural top condensation [3], of eq.(A8). We replace $\phi(r^\mu) \to \phi_\omega(r^\mu)$ in S_ϕ,

$$S_\phi \;\to\; M_0^4 \int d^4r \Big(Z\partial_\mu\phi_\omega^\dagger(r)\partial^\mu\phi_\omega(r) + 2g_0^2 N_c D_F(2r^\mu)\phi_\omega^\dagger(r)\phi_\omega(r) \Big) \tag{9}$$

where $\partial_\mu = \partial/\partial r^\mu$ and $D_F(2r)$ is defined in eq.(A10).

If we then vary S_ϕ with respect to $\phi_\omega + \delta\phi_\omega$ we obtain a formal, manifestly Lorentz invariant *integro-differential equation*:

$$M_0 \int \Big(-Z\frac{\partial^2\phi_\omega(r^\mu)}{\partial r^\mu \partial r_\mu} + 2g_0^2 N_c D_F(2r^\mu)\phi_\omega(r^\mu) \Big)\omega_\nu dr^\nu = ZM_0 \int \mu^2\phi_\omega(r^\mu)\omega_\nu dr^\nu = \mu^2\phi_\omega(r^\mu) \tag{10}$$

Note the presence of the overall line integral, $\int \omega_\mu dr^\mu$. This line integral remains in the equation of motion since $\phi_\omega(r^\mu)$ has no dependence upon $r_\mu \propto \omega_\mu$, hence the variation is constrained, $\delta\phi_\omega \sim \delta^3(r_\perp^\mu)$ where $\omega_\mu r_\perp^\mu = 0$ and does not produce a longitudinal variation, $\delta(\omega_\mu r^\mu)$. We define Z by the line integral normalization [3],

$$ZM_0 \int dr^\mu \omega_\mu = 1. \tag{11}$$

It is important to realize that eq.(10) is not a conventional Klein-Gordon equation due to the line integral constraint.

We can find solutions to eq.(10) as follows: Since S_ϕ is Lorentz invariant we can evaluate the action in the particular frame, $\omega^\mu = (1,0,0,0)$, where $\phi_\omega(r^\mu) \to \phi(0,\vec{r}) \equiv \phi(\vec{r})$ and hence $ZM_0 \int dr^\mu\omega_\mu \to ZM_0 \int dr^0 = 1$. In this frame the action becomes:

$$S_\phi = M_0^3 \int d^3r \left(-|\nabla_{\vec{r}}\phi(\vec{r})|^2 + \int dr^0 \; 2g_0^2 N_c M_0 D_F(2r^\mu)|\phi(\vec{r})\,|^2 \right) \tag{12}$$

Using eq.(11) converts the normalization of eq.(A3) to:

$$M_0^4 Z \int d^4r |\phi(r^\mu)|^2 \to \int d^3r \; M_0^3 \; |\phi(\vec{r})|^2 = 1, \tag{13}$$

and yields, in this frame, the Yukawa potential [3]:

$$V(2|\vec{r}|) = \int dr^0 2g_0^2 N_c D_F(2r^\mu) = -\frac{g_0^2 N_c e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|} \tag{14}$$

The SKG equation in the spherical ground state thus becomes:

$$-\nabla^2\phi(r) - g_0^2 N_c M \frac{e^{-2M_0 r}}{8\pi r}\phi(r) = \mu^2\phi(r) \qquad \nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) \qquad r \equiv \sqrt{\vec{r}^2}. \tag{15}$$

We study its properties and solutions in ref.[3].

In the solution to the SKG equations with the eigenvalue, μ^2, we see upon integrating by parts that $S_\phi \to \mu^2$, and the full bound state action then becomes:

$$S = \int d^4X \left(|D_H H(X)|^2 - \mu^2 |H(X)|^2 - \frac{\lambda}{2}(H^\dagger H)^2 - g_Y \left([\overline{\psi}_{iL}(X) t_R(X)]_f H^i(X) + h.c. \right) \right) + S'. \tag{16}$$

μ^2 is the physical mass of the bound state. The Yukawa coupling, g_Y and quartic coupling λ are defined in eqs.(A6-A11) in Appendix A. We emphasize that, while we have evaluated the action in the particular ω_μ frame in eq.(12), this is just a calculation in a simplifying frame in the overall Lorentz invariant action eq.(10). The result eq.(16) holds in any frame.

As shown in ref.[3], the SKG equation has a critical coupling, $g_0 = g_c$, for which $\mu^2 = 0$, very close to the quantum NJL critical coupling:

$$\frac{g_c^2 N_c}{8\pi^2} = 1.06940. \qquad \text{c.f, the NJL critical value} \qquad \left.\frac{g_0^2 N_c}{8\pi^2}\right|_{NJL} = 1.00 \tag{17}$$

The loop level (NJL-like) effects generate λ and also add to the formation of the bound state as discussed in [4]. This amplifies the coupling strength in the bound state channel and generates a renormalized coupling for the 4-fermion interaction, $\overline{g}_0^2$, where:

$$\overline{g}_0^2 = g_0^2 \left(1 - \frac{g_0^2 N_c}{8\pi^2} \right)^{-1} \tag{18}$$

When $\overline{g}_0 > g_c$ the eigenvalue is $\mu^2 < 0$. In such a solution the action eq.(16) for $H(X)$ then yields the "sombrero potential":

$$\mu^2 |H|^2 + \frac{\lambda}{2}(H^\dagger H)^2 \qquad \text{where,} \quad \mu^2 < 0. \tag{19}$$

The solution of the SKG equation for $\phi(r)$ can be obtained approximately analytically, or by numerical integration [3, 4]. At short distances $\phi(r) \sim \phi(0)$, and extends at large distances in the rest frame to, $\phi(r) \sim ce^{-|\mu| r}/r$, where $|\mu| < M_0$ and we are near critical coupling. The solution is normalized as in eq.(13), which dilutes the value of $\phi(0) \sim \sqrt{|\mu|/M_0}$ and suppresses the Yukawa coupling $g_Y \propto \phi(0)$ and $\lambda \propto g_Y^4 \propto |\phi(0)|^4$.

III. BROKEN PHASE, MANIFESTLY LORENTZ INVARIANT VACUUM, AND BEH EXCITATIONS

A. Formal Derivation

The action of eq.(9) is manifestly Lorentz invariant with dependence upon the 4-vector ω_μ. For the spherically symmetric ground state it must yield a Lorentz invariant expression in ω. Hence, under a Lorentz transformation, $\omega'_\mu = \Lambda_\mu^\nu \omega^\nu$ we can likewise perform $r'_\mu = \Lambda_\mu^\nu r^\nu$, however, we are then free to change the integration variable back to its original form, $r' \to r$ to obtain:

$$S_\phi = M_0^4 \int d^4r \left(Z \partial_\mu \phi^\dagger_{\omega'}(r) \partial^\mu \phi_{\omega'}(r) + 2 g_0^2 N_c D_F(2r^\mu) \phi^\dagger_{\omega'}(r) \phi_{\omega'}(r) \right) \tag{20}$$

We can see that there is orthogonality of the $\phi_\omega(r^\mu)$ solutions in the kinetic term (or in integrating over any extended smooth, approximately constant, Lorentz invariant function of r^μ). The orthogonality breaks down, however, in the pointlike limit of the interaction where the $\phi_\omega(r)$ will freely mix with any $\phi_{\omega'}(r)$ where $\omega' \neq \omega$.

Formally, for a smooth Lorentz invariant function, $F(r^\mu)$ (or differential operator, e.g, $F \sim \partial^2$):

$$Z M_0^4 \int d^4r \; \phi^\dagger_\omega(r^\mu) F(r^\mu) \phi_{\omega'}(r^\mu) = Z M_0^4 \int d^4r \; \delta^4(\omega - \omega') \phi^\dagger_\omega(r^\mu) F(r^\mu) \phi_{\omega'}(r^\mu) \tag{21}$$

If however, $F \sim \delta^4(r^\mu)$ we have mixing of ω and ω':

$$Z M_0^4 \int d^4r \; \phi^\dagger_\omega(r^\mu) \delta^4(r^\mu) \phi_{\omega'}(r^\mu) = Z M_0^4 \phi^\dagger_\omega(0) \phi_{\omega'}(0) = Z M_0^4 \int d^4r \; \delta^4(r^\mu) \phi^\dagger_\omega(r) \phi_{\omega'}(r) \tag{22}$$

This implies that we will have the BCS or BEC-like behavior for $\Phi(r)$. We derive the orthogonality in Appendix B.

B. Discrete Summation

We find it conceptually useful to begin by approximating an integral representation of the collective state by a discrete sum over $\omega_{i\mu}$, which approximates the continuous integrals in a large-N limit. Define a collective field by summing over a large set of N arbitrary $\omega_{i\mu}$ unit 4-vectors that span the future timelike hyperboloid:

$$\Phi(r^\mu) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\phi_{\omega_i}(r^\mu) \qquad \text{where,} \qquad \phi_{\omega_i}(r^\mu) = \phi(\omega_i^\mu(\omega_{i\nu}r^\nu) - r^\mu) \qquad \omega_i^2 = 1. \tag{23}$$

The $\phi_{\omega_i}(r)$ each have an internal $\omega_{i\mu}$ and each is a solution of the equation of motion, eq.(10). We will use ϕ_ω to represent any arbitrary ω field. In this framework the classical average of the ϕ_ω is:

$$\widetilde{\Phi}(r^\mu) = \frac{1}{\sqrt{N}}\Phi(r^\mu) = \frac{1}{N}\sum_{i=1}^{N}\phi_{\omega_i}(r^\mu); \qquad \widetilde{\Phi}(r^\mu) = \phi_{\omega_i}(0) \tag{24}$$

Consider the action for the single two-body bound state $H(X)\phi(r)$ of eqs.(A2 to A6), replacing $\phi(r)\to\Phi(r)$ and $H(x)\to H'(X)$. and denote renormalized parameters by primes $'$. For the sake of discussion we break the action into separate components:

$$S = S_1 + S_2 + S_3 + S_Y + S_\lambda \qquad \text{where,} \qquad H(X)\phi(r)\to H'(X)\Phi(r) \tag{25}$$

where:

$$S_1 = M_0^4\int d^4X\; d^4r\Big(Z'|DH'(X)|^2|\Phi(r)|^2\Big) \tag{26}$$

$$S_2 = M_0^4\int d^4X\; d^4r\Big(Z'|H'(X)|^2|\partial_r\Phi(r)|^2\Big) \tag{27}$$

$$S_3 = M_0^4\int d^4X\; d^4r\Big(2g'^2_0N_c\frac{1}{16M_0^2}\delta^4(r^\mu)|H'^\dagger H'||\Phi(r))|^2\Big) \tag{28}$$

and we have taken the pointlike limit of $D_F(2r^\mu)$ of eq.(A10),

$$D_F(2r^\mu)\to\frac{1}{M_0^2}\delta^4(2r^\mu) = \frac{1}{16M_0^2}\delta^4(r^\mu), \tag{29}$$

The Yukawa interaction is,

$$S_Y = \widehat{g}'_Y M_0^2\int d^4Xd^4r\;[\overline{\psi}_{iL}(X+r)\psi_R(X-r)]_f D_F(2r)\;H'^i(X)\Phi(\vec{r})+h.c.. \qquad \widehat{g}'_Y\approx g'^2_0\sqrt{2N_c/J} \tag{30}$$

and the quartic interaction is given in the point-like limit of $\Phi(r)$:

$$S_\lambda = -\frac{\widehat{\lambda}}{2}\int d^4X(H^\dagger H)^2|\Phi(0)|^4+h.c.. \tag{31}$$

Our problem is to verify that $S(H'\Phi)$, with renormalized parameters, is consistent with the underlying theory of the $\phi_\omega = \phi(r)$ as defined in eqs.(A2,A8). We therefore substitute the collective field definition of eq.23 and define the renormalized parameters.

The orthogonality of the ϕ_{ω_i} fields implies in a discrete sum:

$$\begin{aligned}&\int d^4r\;\phi^\dagger_{\omega_i}(r)\phi_{\omega_j}(r) = \delta_{ij}\int d^4r\;\phi^\dagger_\omega(r)\phi_\omega(r) = \delta_{ij}\\&\int d^4r\;\partial_\mu\phi^\dagger_{\omega_i}(r)\partial^\mu\phi_{\omega_j}(r) = \delta_{ij}\int d^4r\;\partial_\mu\phi^\dagger_\omega(r)\partial^\mu\phi_\omega(r)\end{aligned} \tag{32}$$

We use these relations and the following normalizations:

$g'^2_{c0} = g^2_{oc}$	$Z' = NZ$	$H' = \frac{1}{\sqrt{N}} H$	$\Phi(r^\mu) = \frac{1}{\sqrt{N}} \sum_{i=1}^N \phi_{\omega_i}(r^\mu)$	$\widetilde{\Phi}(r) = \frac{1}{\sqrt{N}}\Phi(r)$

S_1**:** The new composite bilocal field takes the form $H'(X)\Phi(r)$. The $H'(X)$ kinetic term should be canonical with the Lorentz invariant $\Phi(r)$, hence we renormalize the $H'(X)$ field as:

$$H' = H/\sqrt{N} \qquad \text{and then} \qquad Z' = NZ. \tag{33}$$

This choice preserves the product $Z'H'^\dagger H' = ZH^\dagger H$, and Z is still defined by the line integral relation of eq.(11) for the underlying ϕ_ω field. We have from eqs.(23,26):

$$\begin{aligned} S_1 &= M_0^4\!\int\! d^4X\; d^4r\Big(Z'|DH'(X)|^2|\Phi(r)|^2\Big) = M_0^4\!\int\! d^4X\; d^4r|DH'|^2\Big(\frac{Z'}{N}\sum_{i=1}^N\sum_{j=1}^N \phi^\dagger_{\omega_i}(r)\phi_{\omega_j}(r)\Big) \\ &= M_0^4\!\int\! d^4X\; d^4r|DH'|^2\Big(\frac{Z'}{N}\sum_{i=1}^N \phi^\dagger_{\omega_i}(r)\phi_{\omega_i}(r)\Big) = ZM_0^4\!\int\! d^4X\; d^4r|DH|^2\Big(\phi^\dagger_\omega(r)\phi_\omega(r)\Big) \text{ for any } \omega, \\ &= ZM_0^4\!\int\! d^4X\; d^4r|DH|^2|\widetilde{\Phi}|^2 = \!\int\! d^4X|DH|^2 \end{aligned} \tag{34}$$

where we have:

$$ZM_0^4\int d^4r|\widetilde{\Phi}(r^\mu)|^2 = ZM_0^4\int d^4r|\phi_\omega(r^\mu)|^2 \text{ (for any } \omega) = 1 \tag{35}$$

S_1 is therefore consistent with the underlying two-body theory. Note that $\widetilde{\Phi}_\omega$ and ϕ_ω integrates out with the normalization eq.(35).

S_2**:** Likewise the Φ kinetic term becomes:

$$\begin{aligned} S_2 &= M_0^4\!\int\! d^4X\; d^4r\Big(Z'|H'(X)|^2|\partial_r\Phi(r)|^2\Big) = ZM_0^4\!\int\! d^4X\; d^4r\Big(|H|^2\partial_\mu\phi^\dagger_\omega(r)\partial^\mu\phi_\omega(r)\Big) \text{ (for any } \omega) \\ &= ZM_0^4\!\int\! d^4X\; d^4r\Big(|H|^2\partial_\mu\widetilde{\Phi}^\dagger(r)\partial^\mu\widetilde{\Phi}(r)\Big) \end{aligned} \tag{36}$$

yielding consistency with the underlying kinetic term.

S_3**:** The interaction becomes:

$$\begin{aligned} S_3 &= M_0^4\!\int\! d^4X\; d^4r\Big(2g'^2_0N_cD_F(2r^\mu)|H'|^2|\Phi(r))|^2\Big) \approx M_0^4\!\int\! d^4X\; d^4r\;|H'|^2\Big(2g'^2_0N_c\frac{\delta^4(r^\mu)}{16M_0^2}\frac{1}{N}\sum_{i=1}^N\phi^\dagger_{\omega_i}(r)\sum_{j=1}^N\phi_{\omega_j}(r)\Big) \\ &= M_0^4\!\int\! d^4X\; d^4r\; N|H'|^2\Big(\frac{g'^2_0N_c}{8M_0^2}\delta^4(r)\phi^\dagger_\omega(0)\phi_\omega(0)\Big) = M_0^4\!\int\! d^4X\; d^4r\;|H|^2\Big(\frac{g'^2_0N_c}{8M_0^2}\delta^4(r)\phi^\dagger_\omega(r)\phi_\omega(r)\Big) \text{ (for any } \omega) \end{aligned} \tag{37}$$

The essential result is that the potential is approximately $\sim \delta^4(r^\mu)$, hence the line integral orientation ω_μ becomes irrelevant in the potential. In the second-to-last term we see the usual enhancement factor, N, that would normally lead to the BEC or BCS phenomena. However, in the last term we see that the normalization of $H' = H/\sqrt{N}$ undoes the N-fold enhancement and the coupling is not renormalized $g'^2_0 = g_0^2$. This is therefore different than the case of the BCS superconductor or the BEC where the coupling is enhanced by a factor of N. The nonrenormalization of g_0^2 owes to the bilocal field theory with the renormalization of $H' = H/\sqrt{N}$.

We can rewrite the interaction in the unprimed parameters as:

$$S_3 = M_0^4\!\int\! d^4X\; d^4r|H(X)|^2\Big(\frac{2g_0^2N_c}{M_0^2}D_F(2r)|\widetilde{\Phi}(r)|^2\Big). \tag{38}$$

To a good approximation, we can freely swap between $D_F(2r) \leftrightarrow \delta^4(r)/16M_0^2$.

S_Y**:** The Yukawa interaction is:

$$\begin{aligned} S_Y &= g_0^2\sqrt{2JN_c}M_0^2 \int d^4X d^4r\, [\overline{\psi}_{iL}(X+r)\psi_R(X-r)]_f D_F(2r)\; H'^i(X)\Phi(\vec{r}) + h.c.. \\ &\approx g_0^2\sqrt{2N_c/J}M_0^2 \int d^4X d^4r\, [\overline{\psi}_{iL}(X+r)\psi_R(X-r)]_f\; \delta^4(r)\; H^i(X)\widetilde{\Phi}(r^\mu) + h.c.. \\ &= g_Y M_0^2 \int d^4X\, [\overline{\psi}_{iL}(X)\psi_R(X)]_f\;\; H^i(X)\widetilde{\Phi}(0) + h.c.; \qquad\qquad g_Y = g_0^2\sqrt{2N_c/J} \end{aligned} \tag{39}$$

where we have taken the limit $D_F(2r) \to \delta^4(r)/16M_0^2$. (Here $J = 16$ is the Jacobian in passing from coordinates (x, y) to (X, r) which is derived in [3])

We see that g_Y is nonrenormalized. The nonrenormalization of g_Y implies that the result obtained for M_0 from the solution to the SKG equation remains intact, i.e., $M_0 \sim 6$ TeV with $|\mu| \sim 88$ GeV. If we don't take the strict δ-function limit we can do an expansion of the integrand in r^μ. This generates a series of higher dimension operators in inverse powers of M_0^2 providing potentially sensitive probes to M_0 and the shape of the wave-function $\Phi(r^\mu)$, briefly described in section IV.

S_λ**:** The quartic term likewise involves,

$$\frac{1}{2}\hat{\lambda}' \int d^4X d^4r\; \delta^4(r)|H'(X)\Phi(r)|^4 = \frac{1}{2}\hat{\lambda} \int d^4X|H(X)|^4 \left|\widetilde{\Phi}(0)\right|^4 \tag{40}$$

where $\lambda' = \lambda$ is therefere not renormalized. Essentally λ is determined by the Yukawa coupling, g_Y and the renormalization group running λ from M_0 to $|\mu|$ and the result we obtained from the underlying theory at one loop, e.g., $\lambda \approx 0.23$, is not modifed. This term would also permit expansion in r^μ and yield an operator expansion of new physics.

Finally, the result of inserting solutions to the SKG equation, integrating the kinetic term by parts, yields the full action:

$$S_\Phi = Z' M_0^4 \mu^2 \int d^4X d^4r |H'|^2\, |\widetilde{\Phi}|^2 = \mu^2 \int d^4X|H|^2 \tag{41}$$

We thus obtain the same eigenvalue for $\Phi(r)$ as for the underlying $\phi_\omega(r)$.

C. Continuous Integral Representation

It is straightforward to repeat the above with a continuous, manifestly Lorentz invariant, integral representation. To match definitions used above we have to reintroduce N, and require the matching condition to the classical sum:

$$\sum_i^N (c) = \mathcal{N}' \int d^4\omega\; \delta(\omega^2 - 1)\; \phi_\omega(r^\mu) \equiv \int_\omega (c) = N(c) \tag{42}$$

where c is an arbitrary constant. We then define the collective field, $\Phi(r^\mu)$, and match the discrete and continuous representations,

$$\Phi(r^\mu) = \frac{1}{\sqrt{N}} \sum_i^N \phi_{\omega_i} = \mathcal{N} \int d^4\omega\; \delta(\omega^2 - 1)\; \phi_\omega(r^\mu) \equiv \frac{1}{\sqrt{N}} \int_\omega \phi_\omega(r^\mu) \tag{43}$$

Hence $\mathcal{N}' = \sqrt{N}\mathcal{N}$.

Note that $\mathcal{N}$ is the normalization of a divergent integral over the hyperboloid, and requires, in principle, regularization. We will not enter into a detailed discussion of regularization here. We note, however, that we can usually perform a "Wick rotation" if the integral is an analytic function of the metric, $g_{\mu\nu} = (1, -1, -1, -1)$ it can be continued as $g_{\mu\nu} \to \eta_{\mu\nu} \sim (1, 1, 1, 1)$. Then the hyperboloid is replaced by a Euclidean 4-sphere and

$$N = \mathcal{N} \int d^4\hat{\omega}\; \delta(1 - \hat{\omega}^2) = \mathcal{N}\pi^2 \int \hat{\omega}^2 d\hat{\omega}^2 \delta(\hat{\omega}^2 - 1) = \pi^2\mathcal{N}, \tag{44}$$

We then replace η by g. Moreover, alternative averaging functions could be defined. For example, if we define $\phi_\omega(r^\mu) = \phi(\omega^\mu(\omega \cdot r) - r^\mu)$ then we could take $\int_\omega \to \int d^{4-\epsilon}\omega$ as in a momentum integral, and thus use dimensional regularization.

To apply this, consider the expression for the normalization of $H'\Phi$, similar to the S_1 calculation above:

$$Z'M_0^4\int d^4Xd^4r|H'|^2|\Phi(r)|^2 = Z'M_0^4\int d^4Xd^4r|H'|^2\frac{1}{\sqrt{N}}\int_\omega \phi^\dagger_\omega(r)\frac{1}{\sqrt{N}}\int_{\omega'}\phi^\dagger_{\omega'}(r)$$
$$= ZM_0^4\int d^4Xd^4r|H|^2\frac{1}{N}\int_\omega \phi^\dagger_\omega(r)\phi^\dagger_\omega(r)\times\left(\int_{\omega'}\right) = ZM_0^4\int d^4Xd^4r|H|^2 \tag{45}$$

where we use the orthogonality relation,

$$\int d^4r\;\phi^\dagger_{\omega_i}(r)\phi_{\omega_j}(r) = \delta_{ij}\int d^4r\;\phi^\dagger_\omega(r)\phi_\omega(r) = \delta_{ij}\int d^4r|\widetilde{\Phi}(r^\mu)|^2 \tag{46}$$

also eq.(42) for the dummy integral, $(\int_\omega) = 1$ and $Z'|H'|^2 = Z|H|^2$ and eq.(35).

The interaction becomes:

$$S_3 = M_0^4\int d^4X\;d^4r\;|H'|^2\left(2g'^2_0N_c\frac{\delta^4(r^\mu)}{16M_0^2}\frac{1}{\sqrt{N}}\int_\omega \phi^\dagger_\omega(r)\frac{1}{\sqrt{N}}\int_{\omega'}\phi_{\omega'}(r)\right)$$
$$= M_0^4\int d^4X\;d^4r\;|H'|^2\left(\frac{g'^2_0N_c}{8M_0^2}\delta^4(r)N\phi^\dagger_\omega(0)\phi_\omega(0)\right) = M_0^4\int d^4X\;d^4r\;|H|^2\left(\frac{g_0^2N_c}{8M_0^2}\delta^4(r)|\widetilde{\Phi}(r^\mu)|^2\right). \tag{47}$$

IV. FULL ACTION

The full action then becomes that of the standard model BEH boson coupled to top quarks with in the unprimed parameters:

$$S = \int d^4X\left(|DH(X)|^2 - \mu^2|H|^2 + g_Y H^{i\dagger}(X)[\overline{\psi}_R(X)\psi_{iL}(X) + h.c.]_f - \frac{1}{2}\lambda(H^\dagger H)^2\right) + S', \tag{48}$$

where S' contains the free (unbound) top quark action and interactions through coloron exchange.

Note the theory generates the usual "sombrero potential":

$$\mu^2|H|^2 + \frac{1}{2}\lambda(H^\dagger H)^2 \qquad \text{where,}\quad \mu^2 < 0 \tag{49}$$

We extremalize the sombrero potential to obtain the broken phase, i.e., for the vacuum of the SM, we therefore find:

$$H'(X)\Phi(r) = H(X)\widetilde{\Phi}(r) \to \exp(i\pi^a(X)\tau^a/2v_{weak})\begin{pmatrix} v_{weak}\widetilde{\Phi}(r^\mu) + \frac{h(X,r^\mu)}{\sqrt{2}} \\ 0 \end{pmatrix} \tag{50}$$

where $h(X, r^\mu) = h(X)\widetilde{\Phi}(r^\mu)$ is the physical BEH boson and $\pi^a(X)$ are Nambu-Goldstone bosons, which are phase factors of $H(X)$. The electroweak symmetry is broken spontaneously and the gauge fields absorb the Nambu-Goldstone phase factors and acquire mass in the usual way. The resulting broken phase of the BEH field, $h(X)$ is then the standard model and the canonical normalization of $h(X)$ follows from the Z normalization of the internal field $\widetilde{\Phi}(r)$ in eq.(26).

The Nambu-Goldstone modes have only X dependence, and the neutral component of H' is then

$$H'(X)\Phi = H(X)\widetilde{\Phi}(r) \to v_{weak}\widetilde{\Phi}(r) + h(X)\widetilde{\Phi}(r)/\sqrt{2} \tag{51}$$

where $h(X)$ is the SM BEH boson. One might be concerned that the "two-body" field, $h(X)$, is now associated with apparent relative time through $\widetilde{\Phi}(r)$, however, $\widetilde{\Phi}(r)$ is a vacuum feature and makes no reference to a particular ω_μ. Hence the SM BEH field, $h(X)$ is a collective object and its precise two-body nature is actually blended wth the collective vacuum Φ. Likewise, the Nambu-Goldstone bosons are "eaten" by the gauge fields and $\Phi(r)$ is a spectator to the usual combinatorics of that.

We obtain the BEH kinetic term eq.34) of the SM :

$$S_1 \to \frac{1}{2}ZM_0^4\int d^4X\;d^4r\;(\partial h(X))^2|\widetilde{\Phi}(r)|^2 = \frac{1}{2}\int d^4X(\partial h)^2 \tag{52}$$

where we integrate out $\widetilde{\Phi}$ in eqs.(34,52) using eq.(35). In fact, there is no way to discern the compositeness of the BEH field from the kinetic and mass terms (this requires the Yukawa interaction and quartic terms). Moreover, $\langle H\rangle \to v_{weak}$ with the covariant derivative, D_μ, of eq.(A1), leads to:

$$S_1 = M_0^4\int d^4X\; d^4r\left(Z|DH(X)|^2|\widetilde{\Phi}(r)|^2\right) \to$$
$$= ZM_0^4\int d^4X\; d^4r\left(M_W^2W^+W^- + \frac{1}{2}M_Z Z^2\right)|\widetilde{\Phi}(r)|^2 = \int d^4X\left(M_W^2W^+W^- + \frac{1}{2}M_Z Z^2\right) \tag{53}$$

Hence we generate the $W^\pm$ and Z^0 mass terms in the usual way, and the Nambu-Goldstone bosons have become their longitudinal gauge components.

The usual effective action for the BEH boson $h(X)$ emerges:

$$\int d^4X\left(\frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}m_h^2h^2 - \sqrt{\frac{\lambda}{2}}|\mu|h^3 - \frac{\lambda}{8}h^4\right) + \text{electroweak couplings} \tag{54}$$

where $m_h = \sqrt{2}|\mu| = 125$ GeV.

The structure of the collective field, $\Phi(r)$, can in principle be probed through the extended interactions, but the effects will be suppressed. The Yukawa coupling term in natural top condensation takes the form [3, 4] in the broken phase:

$$S'_Y = \sqrt{2N_cJ}g_0^2M_0^2\int d^4Xd^4r\;[\overline{\psi}_L(X+r)\psi_R(X-r)]_f D_F(2r^\mu)\left(v_{weak} + \frac{h(X)}{\sqrt{2}}\right)\widetilde{\Phi}(r^\mu) + h.c.. \tag{55}$$

where $\psi_L = (t,b)_L$ and $\psi_R = t_R$. $J = 16$ is a Jacobian. To a leading approximation we can take the pointlike limit of the potential, and we have $D_F(2r^\mu) \to (JM_0^2)^{-1}\delta^4(r)$. Hence

$$g_Y = \hat{g}_Y|\widetilde{\Phi}(0)| \qquad \text{where,} \qquad \hat{g}_Y \equiv g_0^2\sqrt{2N_c/J} = g_0^2\sqrt{3/8}. \tag{56}$$

This then leads to the conventional mass term for the top quark and BEH coupling:

$$S'_Y = g_Y\int d^4X\;[\overline{\psi}_L(X)\psi_R(X]_f\left(v_{weak} + \frac{h(X)}{\sqrt{2}}\right) + h.c. \tag{57}$$

V. SUMMARY

We have proposed a nontrivial vacuum for the natural top condensation theory. The vacuum is manifestly Lorentz invariant, composed of a collective Lorentz invariant sum over internal wave-functions, $\Phi = (1/\sqrt{N})\sum_\omega \phi_\omega(r^\mu)$. Though each internal wave-function, ϕ_ω, is independent of relative time, τ, they each have dependence upon the arrow of relative time, ω_μ, and are solutions to the Schrödinger-Klein-Gordon equation with eigenvalue $-|\mu^2|$. The sum over ϕ_ω makes the collective field, $\Phi(r^\mu)$, Lorenz invariant and completely independent of ω_μ.

The bilocal BEH field in the vacuum is then a minimum of the sombrero potential and takes the form:

$$H'(X)\Phi(r) = H(X)\widetilde{\Phi}(r) \to \exp(i\pi^a(X)\tau^a/2v_{weak})\begin{pmatrix} v_{weak}\widetilde{\Phi}(r^\mu) + h(X,r^\mu) \\ 0 \end{pmatrix} \tag{58}$$

where $v_{weak} = |\mu|/\sqrt{2}$ and the observed BEH boson is:

$$h(X,r^\mu) = h(X)\widetilde{\Phi}(r^\mu) \tag{59}$$

Much can be done to explore this theory further, including the bilocal formalism itself. For a more complete theory, we note that the b_R quark also participates in topcolor in an anomaly-free scheme, *e.g.*, see [7], and this should be adapted to the present dynamics. An additional $Z^{0\prime}$ interaction can be introduced that makes the $\overline{b}b$ channel sub-critical, hence non-binding. Light fermion masses are presumably then generated in analogy to extended technicolor models [13]. So far, we have relied upon the intuition of the 1990's topcolor scheme, but a more complete natural top condensation theory, including the light particle masses and mixing angles, could be readily formulated.

We comment briefly on some phenomenological implications of this theory. If we go beyond the pointlike limit in, e.g., the BEH-Yukawa interaction, we will have Lorentz invariant $O(1/M_0^2)$ corrections. Before taking the pointlike limit of the Yukawa interaction, as in eq.A4), we can rewrite this in the broken phase as:

$$S'_Y = m_{top} \int d^4X d^4r\, [\overline{\psi}_L(X+r)\psi_R(X-r)]_f (M_0^2 D_F(2r^\mu))\, F(r^\mu) + h.c. \qquad F = \frac{\widetilde{\Phi}(r)}{\widetilde{\Phi}(0)} \sim 1 + a_1 r^2 + ... \tag{60}$$

An expansion in r^μ in all terms in the integrand generates a tower of assorted higher dimension operators such as:

$$\frac{m_t}{M_0^2}\left(1 + \frac{h(X)}{\sqrt{2}v}\right)\left([\bar{b}_L(X)D^2 t_R(X)] + [\bar{t}_L(X)D^2 t_R(X)] + ...\right) + h.c. \tag{61}$$

where D^2 is the covariant derivative, including the gluon, γ, $W^\pm$, Z couplings. These operators represent new contact terms and processes such as:

$$\bar{t} \to b + W + (g, \gamma, Z)) \qquad \text{and,} \qquad \bar{t} \to t + (g, \gamma, Z)) \tag{62}$$

These can be probed in decays or in production in, e.g., a lepton collider via:

$$(\ell^+\ell^-) \to (\gamma^*, Z^*) \to t + b + (W, g, \gamma, Z)) \tag{63}$$

Determining the full set of effective operators is straightforward, but beyond the scope of the present paper.

The theory is therefore testable, mainly by direct discovery of the octet of colorons at $M_0 \approx 6$ TeV. The theory points toward an $SU(3)\times SU(3)\times SU(2)\times U(1)_Y$ gauge structure emerging at the ~ 10 TeV scale with likely additional $U(1)'s$. The theory also offers sensitive probes of new contact interactions involving $t-$ and b-quarks, and though flavor mixing there may be induced rare processes involving the other quarks and leptons [13].

Note that bilocality of the wave-function is an important naturalness constraint. One might be tempted to, e.g., loop the Yukawa interaction and argue for a problematic large correction to the BEH mass $\propto -\overline{g}_Y^2 N_c M_0^2$. This would lead to the *false conclusion* that the effective theory is "unnatural." The loop actually generates an enhancing correction to the bilocal potential, i.e., $\sim g_Y^2 \delta^4(2r^\mu)/M_0^2 \sim g_Y^2 D_F(2r^\mu)$ for large M_0 [4], and we show that this leads to a "critical amplification" of the effective 4-fermion coupling, $\overline{g}_0^2$, where: $\overline{g}_0^2 = g_0^2\left(1 - g_0^2 N_c/8\pi^2\right)^{-1}$. Hence, while $\overline{g}_0^2$ is supercritical, the underlying topcolor coupling, g_0^2, is smaller and subcritical.

In conclusion, we emphasize this theory is natural and manifestly Lorentz invariant. The approximate scale symmetry near critical coupling provides the custodial symmetry of the small $|\mu|^2$. The low energy physics is controlled by the $\Phi(r^\mu) \sim \phi(r^\mu)$ wave-function spreading, rather than the renormalization group of [6]. The hierarchy is protected against additive radiative corrections by the bilocality, i.e., there is no "additive quadratic divergence," but only additive and enhancing renormalizations of the bilocal binding interaction [4]. We used the source/Legendre-transform methods of Jackiw et.al., [16], to derive the effective semiclassical theory used here, which leads to critical amplification of the potential coupling $\overline{g}_0^2$ [4]. The fine-tuning is at the few % level, and, indeed, this may be the first and only mnimally-fine-tuned theory of the BEH boson that is consistent with experiment and testable in the not-too distant future.

Appendix A: Brief Summary of Natural Top Condensation

Our new formalism, "Natural Top Condensation," [3, 4], is Lorentz invariant and postulates an attractive "topcolor" interaction [7] of strength g_0^2 at a high scale M_0. This yields a bound state given by a Schrödinger-Klein-Gordon (SKG) equation satisfied by an internal wave-function, $\phi(r)$. This has eigenvalue μ^2, which is the Lagrangian mass of the BEH boson and we have a Yukawa interaction between the bound state and unbound fermions. For supercritical coupling, $g_0 > g_c$ we find $\mu^2 < 0$, and which implies spontaneous symmetry breaking. For small $|\mu| < M_0$, near critical coupling, we have significant wave-function spreading and "dilution" of $\phi(0) \sim \sqrt{|\mu|/M_0}$. The resulting top quark Yukawa, $g_Y \propto \phi(0)$, and quartic couplings, $\lambda \propto |\phi(0)|^4$, are subject to power law suppression, rather than the relatively slow renormalization group (RG) evolution in the old Nambu–Jona-Lasinio (NJL [9]) based top condensation model. The dilution effect significantly reduces the hierarchy and, remarkably, the standard model (SM) quartic coupling, $\lambda \approx 0.25$, becomes concordant with experiment. The fine tuning of the model is also vastly reduced by dilution to $\sim \phi(0)^2 \sim |\mu|/M_0 \sim$ few %. Our central prediction is the existence of a binding interaction due to a color octet of massive gluon-like objects, called "colorons," [7][8][17], with mass $M_0 \sim 6$ TeV. The colorons may be accessible to the LHC. Moreover, loop effects enhance the binding in the 0^+ channel, and the requisite $\mu^2 < 0$ can occur for significantly weaker coloron coupling [4]. The top condensation model of a composite BEH boson therefore becomes a compelling theory. Inputting the induced Yukawa coupling $g_Y \approx 1$ we obtain the resulting prediction $M_0 \sim 6$ TeV. This construction was confirmed by applying the formal source/Legendre-transformation methods of Jackiw, *et. al,* [4][16].

Starting with third generation fermions, $\psi_{L,R}$, coupled to a coloron exchange potential, we obtain an effective, Lorentz invariant interaction structure for the bilocal BEH boson $H(X^\mu)\phi(r^\mu)$ in the symmetric phase of the standard model (SM) [3]. This was independently derived using techniques of Jackiw, and Cornwall, Jackiw and Tomboulis, [4][16]. The kinetic terms follow by electroweak gauge invariance, and in ref.[3] we introduced Wilson lines to "pull-back" the electroweak gauging to X. Hence the covariant derivative of H becomes the standard BEH form:

$$D_{H\mu} = \frac{\partial}{\partial X^\mu} - ig_2 W^A(X)_\mu \frac{\tau^A}{2} - ig_1 B(X)_\mu \frac{Y_H}{2}. \tag{A1}$$

With the pullback, $\phi(r)$ is a dimensionless complex scalar and has no gauge charges. The Wilson line pullback is essentially a low energy approximation for the electroweak interactions, but makes the effects of symmetry breaking transparent.

In the barycentric coordinates we have:

$$S = M_0^4 \int d^4X\; d^4r \left(Z|D_H H(X)|^2|\phi(r)|^2 + Z|H(X)|^2|\partial_r \phi(r)|^2 + 2g_0^2 N_c D_F(2r)|H^\dagger H||\phi(r))|^2 \right) + S_Y + S_\lambda + S' \tag{A2}$$

where the $H(X)$ kinetic term is canonical with the Lorentz invariant normalization of $\phi(r)$ is:

$$1 = M_0^4 Z \int d^4r |\phi(r^\mu)|^2 \tag{A3}$$

and Z as defined independently below. The Yukawa interaction is also generated at tree level,

$$S_Y = \widehat{g}_Y M_0^2 \int d^4X d^4r\; [\overline{\psi}_{iL}(X+r)\psi_R(X-r)]_f D_F(2r)\; H^i(X)\phi(\vec{r}) + h.c.. \tag{A4}$$

and a quartic interaction is generated at loop level, given in the point-like $\phi(r)$ approximation by:

$$S_\lambda = -\frac{\widehat{\lambda}}{2} \int d^4X (H^\dagger H)^2 |\phi(0)|^4 + h.c.. = -\frac{\lambda}{2} \int d^4X (H^\dagger H)^2 + h.c.. \tag{A5}$$

In the above, the Yukawa and quartic couplings g_Y and λ are derived quantities from the underlying theory. In a pointlike approximation for the interactions, $D(2r) \sim \delta^4(r)/M_0^2$, we have.

$$g_Y \approx g_0^2 \sqrt{2N_c/J}\; \phi(0) \qquad\qquad \lambda \approx (g_Y^4 - g_Y^2\lambda)\frac{N_c}{4\pi^2} \ln\left(\frac{M_0}{\mu}\right), \tag{A6}$$

Note that g_Y is classical and λ arises at loop level ($\mathcal{O}(\hbar)$). We thus see that these are subject to dilution in an extended solution with the internal wave-function $\phi(r)$ and $g_Y \propto \phi(0)$ and $\lambda \propto |\phi(0)|^4$. Experimentally we have $g_Y \approx 1$ and $\lambda \approx 0.25$.

The full action then takes the form:

$$S = \int d^4X \left(|D_H H(X)|^2 + |H(X)|^2 S_\phi + \; g_Y H^{i\dagger}(X)[\overline{\psi}_R(X)\psi_{iL}(X) + h.c.]_f - \frac{1}{2}\lambda(H^\dagger H)^2 \right) + S', \tag{A7}$$

where [...] denotes color indices are contracted, and i is an $SU(2)_{weak}$ index. Here S_ϕ describes the internal wave-function field $\phi(r^\mu)$, and S' describes the coupling of the bound state to external free fermions:

$$S_\phi \;=\; M_0^4 \int d^4r \left(Z\partial_\mu\phi^\dagger(r)\partial^\mu\phi(r) + 2g_0^2 N_c D_F(2r^\mu)\phi^\dagger(r)\phi(r) \right) \tag{A8}$$

$$S' \;=\; \int d^4x \left([\overline{\psi}_L(x) i\not{D}\psi_L(x)]_f + [\overline{\psi}_R(x)\; i\not{D}\psi_R(x)]_f \right) + g_0^2 \int d^4x d^4y\; [\overline{\psi}^i_L(x)\psi_R(y)]_f D_F(x-y)[\overline{\psi}_R(y)\psi_{iL}(x)]_f \tag{A9}$$

In S' we have free unbound fermions, $\psi_{Lif} \sim (t,b)_L$ and $\psi_{Rf} \sim t_R$ (the minimal model omits b_r but this can be readily incorporated as in [7]). Note that the internal field action S_ϕ is nested within the action for a conventional pointlike BEH boson, $H(X)$. $D_F(2r^\mu)$ is the Feynman propagator function for the massive colorons:

$$D_F(x-y) = -\int \frac{1}{q^2 - M_0^2} e^{2q_\mu r^\mu} \frac{d^4q}{(2\pi)^4}, \tag{A10}$$

where r^μ is a radius hence the factor of $2r^\mu$. Note the BCS-like enhancement factor of N_c in eq.(A8)) in the $\phi^\dagger D_F(2r^\mu)\phi$ interaction term.

The value of M_0 is then determined by inputting $g_Y = 1$, and the known value of the symmetric phase (Lagrangian mass) of the BEH boson, which is $-|\mu|^2 = -(88)^2$ GeV^2. We find that *the scale M_0 is predicted,* $M_0 \approx 6$ *TeV* and is no longer the nonsensical 10^{15} GeV in the old top condensation based upon the NJL-model [6]. This is due to the faster power-law running of $g_Y \propto \phi(0) \sim \sqrt{|\mu|/M_0}$, rather than the slow, logarithmic RG running of g_Y in the NJL model.

Moreover, a stunning result of this model is the quartic coupling λ. Experimentally, in the SM using the value of $m_{BEH} \approx 125$ GeV and $v_{weak} \approx 175$ GeV we find $\lambda \approx 0.25$. In the present bilocal scheme, owing to dilution of $\phi(0)$, the quartic coupling is also suppressed and is now generated in RG running from a value of $\lambda(M_0) = 0$ at $M_0 \approx 6$ TeV, down to $\lambda(|\mu|)$ with $|\mu| \sim 88$ GeV, using $g_Y \approx 1$. The prefactor at one loop level reflects the full RG running of λ, and at leading log the RG equation for λ yields [18]:

$$\lambda \approx (g_Y^4 - g_Y^2\lambda - [\lambda^2])\frac{N_c}{4\pi^2}\ln\left(\frac{M_0}{\mu}\right) \approx 0.23 \quad \text{(cf., 0.25 experiment.)}; \tag{A11}$$

where we solve for λ self-consistently with $g_Y = 1$ and $M_0 \sim 6$ TeV. Note that the $[\lambda^2]$ term should be omitted since it involves internal propagation in loops of the composite BEH boson (and only slightly affects the result). The $g_Y^2\lambda$ terms are fermion loop leg renormalizations. This is in excellent agreement with experiment at one loop precision and significantly contrasts the prediction of the old NJL-based top condensation model where the quartic coupling was determined by the RG and found to be $\lambda \sim 1$ [6], much too large.

The degree of fine-tuning of the theory is remarkably suppressed by $|\phi(0)|^2$ in a subtle way. Rather than the naive result one would expect from the NJL model, $\delta g_0^2/g_c^2 \sim |\mu|^2/M_0^2 \sim 10^{-4}$, we now obtain a linear relation: $\delta g_0^2/g_c^2 \sim |\phi(0)|^2 \sim |\mu|/M_0 \sim 1\%$. This is derived in [3], but was accidentally noticed when the bound state was treated by a variational "spline approximation" in earlier papers [4].

Appendix B: Orthogonality

Recall that the $\phi_{\omega_i}(r^\mu)$ solutions of the Lorentz invariant integro-differential equation are normalized as in eq.(13):

$$1 = ZM_0^4\int d^4r\ \phi^\dagger_{\omega_i}(r)\phi_{\omega_i}(r) = 1 = M_0^3\int (ZM_0)\omega_{\mu i}dr^\mu\ d^3r_\perp\ \phi^\dagger_{\omega_i}(r)\phi_{\omega_i}(r) \tag{B1}$$

The latter Lorentz invariant expression can be evaluated in the rest frame, where $\omega_\mu = (1, 0, 0, 0)$:

$$1 = M_0^3\int d^3r|\phi_\omega(\vec{r})|^2 \qquad \text{where,} \qquad 1 = ZM_0\int dr^0 \equiv ZM_0T \tag{B2}$$

In the small $|\mu| < M_0$ limit the normalization integral eq.(13) is dominated by large r, and we have the large distance solution in the rest frame, $\phi(r) \sim Ne^{-|\mu|r}/r$, hence:

$$1 = M_0^3\int 4\pi r^2 dr\frac{N^2e^{-2|\mu|r}}{r^2} \sim 2\pi\frac{N^2M_0^3}{\mu} \qquad \text{hence,} \quad N^2 = \mu/2\pi M_0^3 \tag{B3}$$

If we consider $Z \sim 1/M_0T \ll 1$, then $M_0\int \omega_\mu dr^\mu \sim M_0T \gg 1$, then the $\phi^\dagger_\omega(r^\mu)$ become orthogonal in ω,

$$\int d^4r\ \phi^\dagger_\omega(r)\phi_{\omega'}(r) = 0 \qquad \omega_\mu \neq \omega'_\mu \tag{B4}$$

To see orthogonality, consider the timelike hyperboloid defined by $\omega^2 = 1$ and choose $\omega_0 = (1, 0, 0, 0)$, and $\omega' = (\cosh\theta, \sinh\theta, 0, 0)$ where θ defines a boost in the x direction. Then $\phi_{\omega_0}(r^\mu) \to \phi(0, r_x, r_y, r_z)$, with r^0 as the flat direction for ω_0, and $\vec{r} = (r_x, r_y, r_z)$. Then,

$$\phi_{\omega'}(r^\mu) = \phi(\omega'^\mu(\omega'_\nu r^\nu/\omega'^2) - r^\mu) = \phi(r^0\sinh^2\theta, r_x\cosh^2\theta, r_y, r_z) \tag{B5}$$

We consider small θ, hence $\phi_{\omega'}(r^\mu) \approx \phi(r^0\theta^2, \vec{r})$. The overlap integral is dominated by the large $r = |\vec{r}|$ component and in this limit with eq.(B2), and flat direction r^0,

$$1 = ZM_0^4\int d^4r\ \phi^\dagger_{\omega_0}(r)\phi_{\omega'}(r) = ZM_0^4\int 2\pi r^2 dr dr^0\frac{N^2e^{-2|\mu|r}}{\sqrt{r^2((r^0)^2\sinh^2\theta + r^2)}}$$

$$\approx \frac{\pi ZN^2M_0^4}{\theta\mu^2}\ln(M_0T) = \frac{1}{2\mu\theta T}\ln(M_0T) \tag{B6}$$

In the $T \to \infty$ limit this approaches zero.

The result is not identically zero. In the vacuum, however, where these fields will be clustered into a stable collective state and the system cannot decay, then T can go to infinity with impunity.

Acknowledgments

I thank Bill Bardeen, Bogdan Dobrescu, Tao Han and other participants of Photon-Lepton 2025, for discussions and comments.

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