A Tutorial on Nonlinear Modal Analysis in Structural Dynamics

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ABSTRACT

In structural dynamics, linear modal analysis has been the cornerstone of computational and experimental analysis for several decades. Thanks to the orthogonality properties of real, normal modes for linear systems, these solutions decompose complex vibratory responses into a small number of single degree-of-freedom oscillators that represent the dynamics with a low dimensional subspace. Structural nonlinearities such as frictional contact, nonlinear material behavior, and geometric nonlinearity may invalidate the assumptions necessary to model a structure's dynamics with linear system theory. Decades of research has focused attention on extending the concept of modal analysis to nonlinear mechanical systems using nonlinear normal mode theory. Nonlinear modes possess many unique properties that are not captured with linear theory, including amplitude or energy dependence, internal resonance, bifurcations, and localization. This tutorial provides a brief historical background of the topic of nonlinear modal analysis along with several important concepts and characteristics of nonlinear modes and how they differ from their linear counterparts. An overview of computational and experimental methods to obtain nonlinear modes for mechanical systems will be presented, along with examples of practical uses in engineering.

Keywords: structural dynamics, nonlinear modal analysis, computational methods, experimental identification, nonlinear vibrations

INTRODUCTION

Modal analysis serves as the cornerstone for structural dynamics to characterize, identify and quantify the key dynamic characteristics of a structure or dynamical system. Vibration modes are characterized by three quantities, namely the mode shape, natural frequency, and damping ratio and hence the primary objective of modal analysis is to identify of these modal properties. The mode shape is the specific deformation pattern that the structure vibrates at when oscillating at the natural frequency, the natural frequency is the rate of oscillation of the specific mode in the absence of any excitation, and the damping ratio is a dimensionless value that describe how the oscillations of the specific mode decay after removing the force. For linear systems, the real, normal vibration modes are orthogonal with respect to the structural mass and stiffness matrices and are referred to as linear normal modes. As a result of this orthogonality, linear normal modes are used to decompose complex vibration responses into a subset of modal responses whose linear combination describe the total vibration of the system. Knowledge of the vibration modes of a structure enables the scientist or engineer to better understand the dynamic behavior of the structure and the key characteristic properties that contribute to the response. For example, linear normal modes are used in practice to perform efficient numerical simulations via modal superposition, model updating between simulation and test, and design structures to survive known or expected shock and vibration environments.

Linear modal analysis requires that the system be linear and that there are no nonlinear physics present in the system or mathematical model. In mechanical systems, often these assumptions include small deformations, linear elastic materials, no friction, or intermittent contact, etc... These linearizing assumptions are not always appropriate since all real-world engineering structures possess some form of nonlinearity – it is only under certain conditions that these nonlinearities are negligible, and the linear assumptions are valid. Nonlinearity in mechanical systems commonly arise from frictional contact such as those arising in bolted joints, geometric nonlinearity from large deformations, material elasticity and plasticity, as well as multi-physics problems such as electro-mechanical devices. In the presence of nonlinearity, researchers over several decades have proposed and developed the concept of nonlinear modal analysis to enable the identification of the modal properties of a nonlinear mechanical system [1, 2]. In essence, nonlinear modes describe the amplitude or energy dependent mode shapes, natural frequencies and damping ratios and lead to unique properties not possessed by their linear counterparts, such as bifurcations, stability, internal resonance, and localization.

This tutorial is intended to introduce the concept of nonlinear modal analysis to scientists and engineers who are less familiar with the topic area and provide some key background and insight into the uses of nonlinear modes in practice. The tutorial will be organized as follows. The first part will give a brief historical overview of the foundation works of nonlinear modal analysis, describe the modal characteristics of a nonlinear system, and distinguish the unique properties of these modes from their linear counterparts. The second part of the tutorial focuses on the identification of nonlinear modes, with attention to numerical and experimental techniques applicable to practical engineering structures. Lastly the tutorial finishes by highlighting a few practical uses of nonlinear modal analysis in engineering and how they can be useful to the greater engineering and science community.

DEFINITIONS OF NONLINEAR MODES

Some of the earliest theoretical works related to the development of nonlinear modal analysis dates back to the work by Rosenberg in the 1960s [3, 4], where he defined a nonlinear mode for a conservative system as a periodic oscillation that oscillates in unison. In this way, all the material points of the system would reach their extrema (maximum or minimum positions) and equilibrium states simultaneously. Decades later, the definition of a nonlinear normal mode (NNM) was further extended for conservative systems to be a “not necessarily synchronous periodic response of the undamped nonlinear system” [1, 2]. This serves as one of the most adopted definitions of NNMs to date for conservative, undamped systems, which has likeness to real, normal modes for undamped linear systems.

A challenge arises when considering nonlinear modes for non-conservative, damped systems, in which case the periodic motion concept is generally no longer applicable in the presence of dissipation mechanisms. With nonlinear systems, it is not often straightforward to separate the conservative and non-conservative terms in the equations-of-motion (e.g. frictional contact) and thus one cannot simply perform nonlinear modal analysis of a non-conservative system by omitting the damping terms. Several researchers over many decades have developed definitions of nonlinear modes for damped systems. The earliest work was done by Shaw and Pierre in the 1990s [5, 6] where they defined a damped nonlinear mode as a two-dimensional invariant manifold in phase space. More recently, Haller developed a unified approach for nonlinear modal analysis of damped systems using spectral submanifolds [7], which are considered the smoothest invariant manifold that are asymptotic to the generalized concepts of NNMs. Other definitions and concepts of nonlinear modes for damped systems include complex nonlinear modes [8], extended periodic motion concept [9], and phase resonance nonlinear modes [10].

A brief demonstration of the concept of nonlinear modal analysis for a dissipative system is shown in Fig. 1. The amplitude dependent natural frequency and damping ratio, based on the phase resonance nonlinear mode (PRNM) concept [10], are plotted for a cantilever beam with a cubic spring at the free end and proportional viscous damping. It is evident from these plots that the modal characteristics are no longer invariant properties with amplitude, as they are for linear modal properties, and thus depend on the energy of the oscillatory motion in the system. The plots in Fig. 2 demonstrate the relationship between the NNMs and the harmonically forced response near resonance. The NNMs correspond to the motion that occurs at resonance, and thus describes the deformations, dissipation, and oscillation frequency at the most damaging forcing conditions. It is evident that the backbone curve follows the locus of the forced response curves at resonance, thus providing a family of solutions that relate to the resonant behavior of the system. There also exists a 90-degree phase relationship between the forced and unforced response at resonance as shown in Fig. 2b. Other important and unique properties of nonlinear modes will be presented in the tutorial.

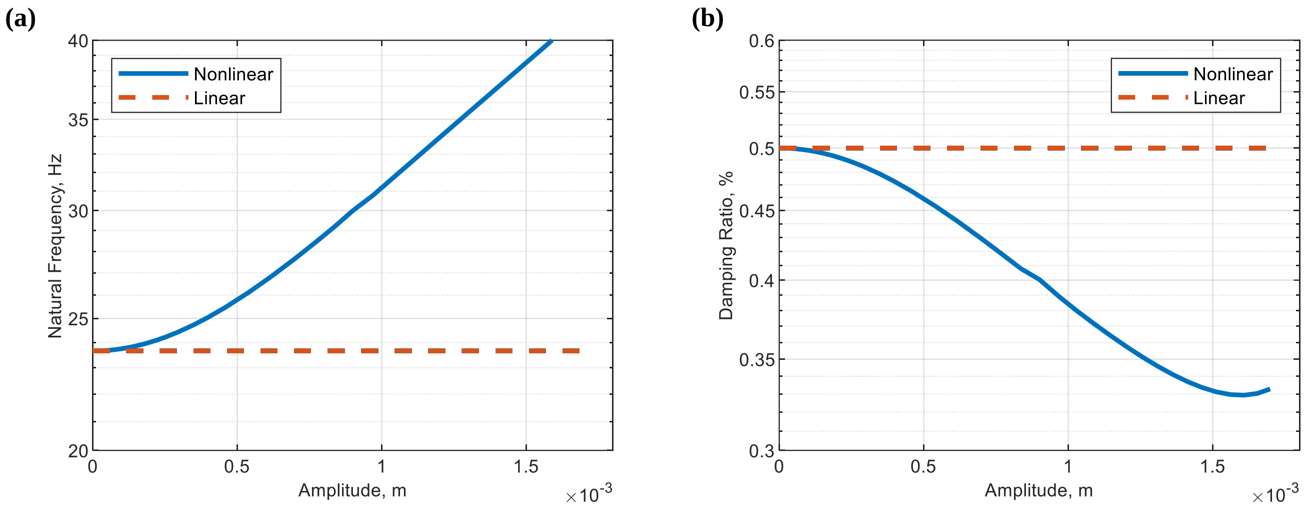

Fig. 1 Example of amplitude dependent (a) natural frequency and (b) damping ratio for a cantilever beam with a cubic spring nonlinearity.

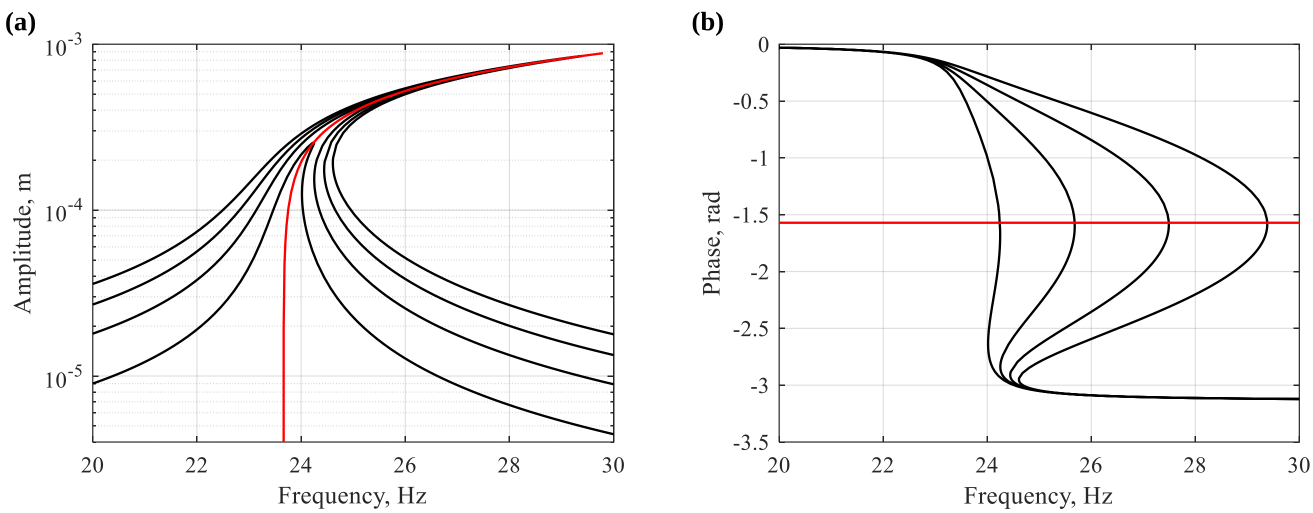

Fig. 2 Example of the relationship between the (black) harmonically forced periodic response and the (red) nonlinear mode for a cantilever beam with a cubic spring nonlinearity. (a) Relationship between the backbone curve and forced response curves, and (b) phase relationship between the nonlinear mode and forced response at resonance.

IDENTIFICATION OF NONLINEAR MODES

Several methods have been developed over the last two to three decades to experimentally identify nonlinear modes from hardware. The methods are classified as either forced or free response techniques. An overview of these techniques will be summarized in the tutorial, such as phase resonant methods using force appropriation and ring-down experiments combined with time-frequency analysis. In addition to the experimental identification of nonlinear modes, the tutorial will highlight some of the analytical and numerical methods available to identify the nonlinear modes of a mathematical model [11]. The tutorial will focus on computational techniques that have been developed to solve larger-scale problems that are of practical interest for engineering structures.

PRACTICAL USES IN SCIENCE AND ENGINEERING

While nonlinear modes do not possess the orthogonality properties of their linear counterparts that lead to the concept of modal superposition, there have been several studies and applications of nonlinear modal analysis in science and engineering. The tutorial will highlight some of the practical uses of nonlinear modes, such as their importance to model updating for nonlinear mechanical systems and the identification of complex behavior observed in shock and vibration loading. The understanding of nonlinear normal modes has led to the design of energy harvesting devices as well as vibration absorbers to mitigate damaging vibrations over a broad spectrum of environments.

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REFERENCES

[1] G. Kerschen, M. Peeters, J. C. Golinval, and A. F. Vakakis, "Nonlinear normal modes. Part I. A useful framework for the structural dynamicist," *Mechanical Systems and Signal Processing,* vol. 23, no. 1, pp. 170-94, 2009, doi: 10.1016/j.ymssp.2008.04.002.
[2] A. F. Vakakis, "Non-linear normal modes (NNMs) and their applications in vibration theory: an overview," *Mechanical Systems and Signal Processing,* vol. 11, no. 1, pp. 3-22, 1997, doi: 10.1006/mssp.1996.9999.
[3] R. M. Rosenberg, "Normal modes of nonlinear dual-mode systems," *Journal of Applied Mechanics,* vol. 27, pp. 263–268, 1960.
[4] R. M. Rosenberg, "The Normal Modes of Nonlinear n-Degree-of-Freedom Systems," *Journal of Applied Mechanics,* vol. 29, no. 1, pp. 7-14, 1962, doi: 10.1115/1.3636501.
[5] S. Shaw and C. Pierre, "Non-linear normal modes and invariant manifolds," *Journal of sound and Vibration,* vol. 150, no. 1, pp. 170-173, 1991.
[6] S. W. Shaw and C. Pierre, "Normal Modes for Non-Linear Vibratory Systems," *Journal of Sound and Vibration,* vol. 164, no. 1, pp. 85-124, 1993, doi: 10.1006/jsvi.1993.1198.
[7] G. Haller and S. Ponsioen, "Nonlinear normal modes and spectral submanifolds: existence, uniqueness and use in model reduction," *Nonlinear Dynamics,* vol. 86, no. 3, pp. 1493-1534, 2016, doi: 10.1007/s11071-016-2974-z.
[8] D. Laxalde and F. Thouverez, "Complex non-linear modal analysis for mechanical systems: Application to turbomachinery bladings with friction interfaces," *Journal of Sound and Vibration,* vol. 322, no. 4, pp. 1009-1025, 2009, doi: https://doi.org/10.1016/j.jsv.2008.11.044.
[9] M. Krack, "Nonlinear modal analysis of nonconservative systems: Extension of the periodic motion concept," *Computers & Structures,* vol. 154, pp. 59-71, 2015, doi: https://doi.org/10.1016/j.compstruc.2015.03.008.
[10] M. Volvert and G. Kerschen, "Phase resonance nonlinear modes of mechanical systems," *Journal of Sound and Vibration,* vol. 511, p. 116355, 2021.
[11] L. Renson, G. Kerschen, and B. Cochelin, "Numerical computation of nonlinear normal modes in mechanical engineering," *Journal of Sound and Vibration,* vol. 364, pp. 177-206, 2016, doi: https://doi.org/10.1016/j.jsv.2015.09.033.