Fermi National Accelerator Laboratory

September 1990

MAD/TH/90-19
FERMILAB-PUB-90/191-T

Feigin-Fuchs Derivation of $SU(1,1)$ Parafermion Characters

Paul A. Griffin†
Fermi National Accelerator Laboratory
P. O. Box 500, Batavia, IL 60510

Oscar F. Hernández††
Department of Physics
University of Wisconsin-Madison
Madison, WI 53706

Abstract

We use the Feigin-Fuchs Coulomb gas construction, with two free gaussian bosons, to study the representation theory of the $SU(1,1)$ parafermion models. We derive the chiral algebra and highest weights for unitary finitely reducible models, and calculate the irreducible parafermion characters which correspond to continuous and discrete unitary $SU(1,1)$ representations.

† bitnet pgriffin@fnal, decnet fnal::pgriffin
†† bitnet oscarh@wishep, decnet wishep::oscarh

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

1. Introduction

In previous work [1],[2], we showed how to use the free boson Feigen-Fuchs [3],[4] description of parafermion (PF) conformal field theories [5],[6] to describe the important properties of these models. Our work was based on the bosonization of the chiral algebra for the minimal Z_N PF models [7],[8]. We focused on the representation theory of the minimal models, although it was clear that many of our techniques could be applied to the non-minimal cases. It was shown in reference [9] that the non-minimal models describe the GKO [10] coset $SU(1,1)/U(1)$ of the $SU(1,1)$ affine (Kac-Moody) algebra, so we denote these as $SU(1,1)$ PF models. Reference [9] also discussed how the unitary $SU(1,1)$ PF models correspond to the unitary infinite dimensional representations of $SU(1,1)$. Relations between these models and other (super)conformal field theories are given in references [2],[6],[9].

In this paper we use the formalism developed in [2] to derive the representation theory of finitely reducible $SU(1,1)$ PF models. The finite reducibility requirement excludes models with irrational central charge. We describe the embedding of the different representations in the bosonic Fock space and use this to construct the irreducible PF characters which correspond to continuous and discrete unitary $SU(1,1)$ representations. Finally, we discuss the problem of constructing modular invariant $SU(1,1)$ theories.

2. Bosonic Representation of the Chiral Algebra

In this section we summarize results given in reference [2] which will be needed for the analysis in this paper. Consider two commuting free bosons ϕ_1 and ϕ_2 with propagators given by

$$\phi_1(z)\phi_1(w) = -\ln(z-w) , \qquad \phi_2(z)\phi_2(w) = +\ln(z-w) . \tag{2.1}$$

The holomorphic stress energy tensor is defined to be

$$T(z) = -\frac{1}{2}\partial\phi_1(z)\partial\phi_1(z) + \frac{1}{2}\partial\phi_2(z)\partial\phi_2(z) + i\frac{Q_o}{2}\partial^2\phi_1(z), \tag{2.2}$$

where the Q_o term corresponds to background charge $-Q_o$

$$Q_o = \sqrt{\frac{2b}{N+2b}} . \tag{2.3}$$

Finite reducibility of the algebra requires that all of the operator product exponents of the theory be given in units of the fraction $1/N$ where N is an integer [6]. In particular, the relative monodromy of ψ_1 with $\psi_1^\dagger$ is defined by the integer b. In this paper we focus on $SU(1,1)$ PF models, which correspond to $b<0$ [6].

The central charge is

$$c = 2 - 3Q_o^2 = 2(N-b)/(N+2b) \ . \tag{2.4}$$

We are interested in unitary theories and therefore require $c>0$. It is convenient to define $K \equiv N/|b|$. In terms of K, the $c>0$ condition becomes $K>2$.

It is convenient to parametrize vertex operators as

$$V_m^\ell(z) = \exp\left(\frac{-i\ell}{2b}Q_o\phi_1(z)\right)\exp\left(\frac{im}{2b}\sqrt{2b/N}\phi_2(z)\right) \ . \tag{2.5}$$

They have conformal dimension

$$\Delta_m^\ell = \frac{\ell(\ell+2b)}{4b(N+2b)} - \frac{m^2}{4bN} \ . \tag{2.6}$$

The Feigin-Fuchs conjugate $V_m^{-\ell-2b}(z)$, and the ϕ_2 conjugate $V_{-m}^\ell(z)$ have the same conformal dimension as $V_m^\ell(z)$.

The parafermions are given by

$$\begin{aligned} \psi_p(z) =& n_p \oint \eta \ V_{-N+2bp}^{-(N+2b)}(z) \ , \quad n_p \equiv [b/N]^{p/2}p! \prod_{i=1}^{p-1}[c_{1,i}]^{-1} \ , \\ \psi_p^\dagger(z) =& n_p^\dagger \oint \tilde{\eta} \ V_{N-2bp}^{-(N+2b)}(z) \ . \end{aligned} \tag{2.7}$$

They satisfy the operator algebra

$$\begin{aligned} \psi_1(z)\psi_p(0) =& c_{1,p} z^{\Delta_{p+1}-\Delta_1-\Delta_p}\psi_{p+1}(0) + \cdots \ , \\ \psi_1^\dagger(z)\psi_p^\dagger(0) =& c_{1,p}^\dagger z^{\Delta_{p+1}-\Delta_1-\Delta_p}\psi_{p+1}^\dagger(0) + \cdots \ . \end{aligned} \tag{2.8}$$

Though $c_{1,p}$ can be determined by associativity of the PF algebra, our analysis does not require its evaluation. The fields η and $\tilde{\eta}$ are given by

$$\eta(z) = V_N^{(N+2b)}(z) \ , \qquad \tilde{\eta}(z) = V_{-N}^{(N+2b)}(z) \ . \tag{2.9}$$

They are dimension one fermions and two of the three screening operators in the bosonized PF theory. The third screening operator is given by

$$J = i\partial\phi_2 V_0^{-2b} \ . \tag{2.10}$$

The charge conjugates to η and $\tilde{\eta}$ are given by

$$\xi(z) = V_{-N}^{-(N+2b)}(z) \ , \qquad \tilde{\xi}(z) = V_{N}^{-(N+2b)}(z) \ . \tag{2.11}$$

They are conformal dimension zero fermions. Equations (2.9) and (2.11) are bosonic representations of two $c=-2$ fermion ghost systems [11], (η,ξ) and $(\tilde{\eta},\tilde{\xi})$. The U(1) currents of the two fermion systems are

$$j(z) = -\eta\xi \ , \qquad \tilde{j}(z) = -\tilde{\eta}\tilde{\xi} \ . \tag{2.12}$$

The charge operators j_0 and $\tilde{j}_0$ count the fermion charge of the vertex operators (2.5) as

$$j_0[V_m^\ell] = \frac{\ell - m}{2b} \ , \qquad \tilde{j}_0[V_m^\ell] = \frac{\ell + m}{2b} \ . \tag{2.13}$$

In particular, $j_0[\eta] = 1$, $j_0[\xi] = -1$. Vertex operators with integer j_0 ($\tilde{j}_0$) charge are local with respect to the (η,ξ) ($(\tilde{\eta},\tilde{\xi})$) system. Vertex operators with non-negative j_0 ($\tilde{j}_0$) charge are independent of the zero mode ξ_0 ($\tilde{\xi}_0$).

Finite reducibility requires the operator product of parafermion with the vertex operators V_m^ℓ, to have monodromy in units of $1/N$. This implies that m is an integer, and places no constraint on ℓ as can be easily checked by calculating the OPE of the parafermions and V_m^ℓ.

3. Parafermion Highest Weight Modules

The irreducible modules are obtained from the highest weight primary conformal fields ϕ_q which also satisfy the finite reducibility constraints. Define the monodromy parameters ω_q and $\omega_q^\dagger$ via the operator products between parafermions and PF/Virasoro highest weights

$$\psi_1(z)\phi_q(0) = z^{-\omega_q}\tilde{\phi}_{q+2b} \ , \qquad \psi_1^\dagger(z)\phi_q(0) = z^{-\omega_q^\dagger}\tilde{\phi}_{q-2b}^\dagger \ . \tag{3.1}$$

The independent parameters $\omega_q, \omega_q^\dagger \in R$ satisfy the finite reducibility constraints,

$$\omega_q = +q/N \mod 1 \ , \quad \omega_q^\dagger = -q/N \mod 1 \ , \quad q \in \text{ integers} \ , \tag{3.2}$$

and also satisfy the constraint that the ϕ_q are highest weights with respect to the PF algebra. This requires that the states $\tilde{\phi}_{q+2b}$ and $\tilde{\phi}_{q-2b}^\dagger$ of eqns. (3.1) have conformal dimension greater than or equal to the conformal dimension of the PF/Virasoro highest weight ϕ_q. This is equivalent to the constraints

$$\omega_q \le 1 - b/N \ , \qquad \omega_q^\dagger \le 1 - b/N \ . \tag{3.3}$$

We now search for PF/Virasoro primary fields which are of the form V^{ℓ}_{m}. The values of ℓ and m for these Virasoro primaries are constrained by requiring that they are parafermion highest weights. Although we have no proof that there are no other Virasoro PF/Virasoro highest weights in the boson Hilbert space for particular values of central charge c, we will find from the set we consider all representations of the unitary finitely reducible highest weights [9].

The OPE's of these operators with the parafermions ψ_1 and $\psi_1^\dagger$ are easily evaluated within the free field theory.

$$\begin{aligned}\psi_1(z)V^{\ell}_{m}(0) = \frac{n_1}{2b}\ z^{-m/N-1}\Big[&(\ell-m)\\ &+z\left(\sqrt{\frac{2b}{N}}\ (\ell-m+N)i\partial\phi_2(0)-\sqrt{2b(N+2b)}\ i\partial\phi_1(0)\right)\\ &+\mathcal{O}(z^2)\Big]V^{\ell}_{m+2b}(0)\ ,\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}\psi_1^\dagger(z)V^{\ell}_{m}(0) = \frac{n_1^\dagger}{2b}\ z^{+m/N-1}\Big[&(\ell+m)\\ &-z\left(\sqrt{\frac{2b}{N}}\ (\ell+m+N)i\partial\phi_2(0)+\sqrt{2b(N+2b)}\ i\partial\phi_1(0)\right)\\ &+\mathcal{O}(z^2)\Big]V^{\ell}_{m-2b}(0)\ .\end{aligned}\tag{3.5}$$

In contrast to the $b>0$ case [2] we find two type of solutions to (3.3). Either

$$m=\pm\ell \qquad \ell=b,b+1,\cdots,N-b\ ,\tag{3.6}$$

or

$$m=b,b+1,\cdots,-b \qquad \ell\in\text{complex numbers},\ \ell\neq\pm m\ .\tag{3.7}$$

Thus the holomorphic part of the PF/Virasoro highest weights are

$$\begin{aligned}\phi_0 &=V^0_0\ ,\\ \phi_q &=V^q_q\ , &&q=-|b|,-|b|+1,\cdots,N+|b|\ ,\\ {\phi'}_q &=V^{N-q}_{-(N-q)}\ , &&q=-|b|,-|b|+1,\cdots,N+|b|\ ,\end{aligned}\tag{3.8}$$
$$\gamma^{u,w}=V^u_w\ ,\ u\in\text{complex numbers},\ u\neq\pm w\ ,\quad w=-|b|,-|b|+1,\cdots,|b|\ .$$

We will see that ϕ_q and ${\phi'}_q$ correspond to the $D^{(\pm)}$ series of $SU(1,1)$ representations and that $\gamma^{u,w}$ correspond to the continuous series. The conformal dimension of these states is given by

$$\dim[\phi_q] = \dim[{\phi'}_q] = \frac{q(N-q)}{2N(N+2b)} \ , \tag{3.9}$$

$$\dim[\gamma^{u,w}] = \frac{u(u+2b)}{4b(N+2b)} - \frac{w^2}{4bN} \ . \tag{3.10}$$

A non-negative dimension for ϕ_q and ${\phi'}_q$ restricts us to

$$q = 0, 1, \cdots, N \ . \tag{3.11}$$

A non-negative dimension for $\gamma^{u,w}$ restricts us to

$$u = x \neq \pm w \ , \quad |b| - t \leq x \leq |b| + t \ , \quad \text{where } t \equiv \sqrt{b^2 + w^2(1 - 2/K)} \ , \tag{3.12}$$

or to

$$u = |b| + iy, \quad y \in \text{real numbers} \ . \tag{3.13}$$

Consider the action of the parafermions on the PF/Virasoro highest weights ϕ_q and ${\phi'}_q$. We use the same notation as Lykken [6] to label the PF descendants which are Virasoro highest weights. Using (3.4), (3.5) and the bosonized form of ψ_1 and $\psi_1^\dagger$ given in (2.7), we find the lowest dimension PF descendant, Virasoro highest weight state for each PF charge sector (up to normalization constants)

$$\begin{aligned}
\phi_p^q(z) &= A_{(-N+q-b+2bp)/N} \cdots A_{(-N+q+b)/N} \ \phi_q(z) &&= \oint \eta V^{q-(N+2b)}_{q-N+2bp}(z) \ , \\
\tilde{\phi}_p^q(z) &= A^\dagger_{(-N+q-b+2bp)/N} \cdots A^\dagger_{(-N+q+b)/N} \phi_{N-q}(z) &&= V^{N-q}_{N-q-2bp}(z) \ , \\
{\phi'}_p^q(z) &= A_{(-N+q-b+2bp)/N} \cdots A_{(-N+q+b)/N} \ {\phi'}_q(z) &&= V^{N-q}_{q-N+2bp}(z) \ , \\
\tilde{\phi}'^q_p(z) &= A^\dagger_{(-N+q-b+2bp)/N} \cdots A^\dagger_{(-N+q+b)/N} {\phi'}_{N-q}(z) &&= \oint \tilde{\eta} V^{q-(N+2b)}_{N-q-2bp}(z) \ .
\end{aligned} \tag{3.14}$$

The operators A and $A^\dagger$ are the modes of the parafermions ψ_1 and $\psi_1^\dagger$, in the appropriate charge sector, defined in the usual way via contour integration [5],[6]. The states ϕ_p^q and $\tilde{\phi}_p^{N-q}$ for $p = 1, 2, \ldots$ are parafermion descendants of $\phi_q \equiv \phi_0^q = \tilde{\phi}_0^{N-q}$, and states ${\phi'}_p^q$ and $\tilde{\phi}'^{N-q}_p$ are parafermion descendants of ${\phi'}_q \equiv {\phi'}_0^q = \tilde{\phi}'^{N-q}_0$. The parafermions $\phi_p^0 = \psi_p$ and $\tilde{\phi}'^0_p = \psi_p^\dagger$, have conjugates $\tilde{\phi}_p^0 = V^N_{N-2bp}$ and ${\phi'}_p^0 = V^N_{-N+2bp}$ as was discussed in [2]. The conformal dimensions h_q^p of the fields ϕ_p^q, $\tilde{\phi}_p^q$, ${\phi'}_p^q$ and $\tilde{\phi}'^q_p$ are $h_q^p = \Delta^{N-q}_{N-q-2bp}$, where Δ^ℓ_m is given by eqn. (2.6). Under the identification in (3.14), we have the trivial equivalence $\phi_0^q = \tilde{\phi}_0^{N-q}$ and ${\phi'}_0^q = \tilde{\phi}'^{N-q}_0$, as can be checked by explicit calculation. Note that there do not exist the Z_2 degeneracies or truncations encountered in the minimal $SU(2)$ PF modules [2].

The $\gamma^{u,w}$ states have no analogue with states in the minimal $SU(2)$ PF theories. As with the ϕ_q and ϕ'_q, we can write the bosonized form of the lowest dimension PF descendant (Virasoro highest weight state), in each PF charge sector. We find (up to normalization constants)

$$\begin{aligned} \gamma_p^{u,w}(z) &= A_{(w-b+2bp)/N} \cdots A_{(w+b)/N}\ \gamma^{u,w} = V^u_{w+2bp}\ , \\ \gamma_{-p}^{u,w}(z) &= A^\dagger_{(-w-b+2bp)/N} \cdots A^\dagger_{(-w+b)/N}\ \gamma^{u,w} = V^u_{w-2bp}\ . \end{aligned} \tag{3.15}$$

The conformal dimension of the $\gamma_p^{u,w}$ field is given by $h_p^{u,w} = \Delta^u_{w+2bp}$.

4. Relationship between Parafermion states and $SU(1,1)$ Kac-Moody Representations

The purpose of this section is to relate the PF states in the previous section to $SU(1,1)$ states. By adding a third boson ϕ_3 with the same propagator as ϕ_1 to the parafermions we can construct three dimension one currents

$$\begin{aligned} X^3 &= i\sqrt{N/2b}\ \partial_z\phi_3(z)\ , \\ X^+ &= \sqrt{N/b}\ \psi_1(z)\exp\left(i\sqrt{2b/N}\phi_3(z)\right)\ , \\ X^- &= \sqrt{N/b}\ \psi_1^\dagger(z)\exp\left(-i\sqrt{2b/N}\phi_3(z)\right)\ , \end{aligned} \tag{4.1}$$

that obey the SL(2,C) Kac-Moody algebra OPE. For $b>0$, identifying $J^3 \equiv X^3$ and $J^\pm \equiv X^\pm \equiv J^1 + iJ^2$ leads to an $SU(2)$ level N/b Kac-Moody Algebra with Cartan Killing form $g^{ij} = \text{diag}[1,1,1]$ and an adjoint Casimir of $Q_{ad} = 2$. In the $b<0$ case, $\psi_1^\dagger$ increases the J_3 charge, and we identify $J^3 \equiv -X^3$ and $J^\pm \equiv X^\mp \equiv iJ^1 \mp J^2$. This leads to an $SU(1,1)$ level $K \equiv N/|b|$ Kac-Moody Algebra with $g^{ij} = \text{diag}[1,1,-1]$ and $Q_{ad} = -2$. In terms of J^1, J^2, J^3, our $SU(1,1)$ notation now conforms to that used in [9],

$$\begin{aligned} J^i(z)J^j(w) &= \frac{\frac{1}{2}Kg^{ij}}{(z-w)^2} + \frac{i\epsilon^{ij}{}_k J^k(w)}{(z-w)} + :J^i(z)J^j(w):\ , \\ g^{ij} &= g_{ij} = \text{diag}[\ 1,\ 1,-1]\ , \qquad \epsilon^{ij}{}_k = g_{kr}\epsilon^{ijr}\ . \end{aligned} \tag{4.2}$$

The unitary representations [12] of $SU(1,1)$ are labeled by the value of their Casimir $\mathbf{J}^2 = g_{ij}J^iJ^j = \frac{1}{2}(-J^+J^- - J^-J^+) - (J^3)^2$ and their J^3 eigenvalue. First we have the two discrete series,

$$\begin{aligned}
D_\ell^{(+)} \qquad & \mathbf{J}^2 = -\ell(\ell-1)\ , \quad \ell = n + e_o\ , \quad n \in \{0,1,2\cdots\}\ , \quad 0 \le e_o < 1\ , \\
& J^3 \text{ eigenvalue } m \ge \ell\ , \quad m = \ell, \ell+1, \ell+2, \cdots\ , \\
D_\ell^{(-)} \qquad & \mathbf{J}^2 = -\ell(\ell-1)\ , \quad \ell = n - e_o\ , \quad n \in \{1,2,\cdots\}\ , \quad 0 \le e_o < 1\ , \\
& J^3 \text{ eigenvalue } m \le -\ell\ , \quad m = -\ell, -(\ell+1), -(\ell+2), \cdots\ .
\end{aligned} \tag{4.3}$$

There are also the continuous principle series and continuous supplementary series,

$$\begin{aligned}
C_\ell^{(p)e_o} \qquad & \mathbf{J}^2 = -\ell(\ell-1) > \tfrac{1}{4}\ , \quad \ell = \tfrac{1}{2} + iy\ , \quad y \in \text{real numbers}\ , \\
& J^3 \text{ eigenvalues } \ m = n + e_o\ , \quad n \in \{\cdots -1, 0, 1\cdots\}\ , \quad 0 \le e_o < 1\ , \\
C_\ell^{(s)e_o} \qquad & \mathbf{J}^2 = -\ell(\ell-1) > -e_o(e_o-1)\ , \quad 0 \le e_o < 1\ , \quad \ell \in \text{real numbers}\ , \\
& J^3 \text{ eigenvalues } \ m = n + e_o\ , \quad n \in \{\cdots -1, 0, 1\cdots\}\ .
\end{aligned} \tag{4.4}$$

Affine $SU(1,1)$ highest weight fields G^ℓ_m satisfy in general

$$\begin{aligned}
J^\pm(z) G^\ell_m(w) &\sim \frac{G^\ell_{m\pm1}(w)}{(z-w)} + \text{finite part}\ , \\
J^3(z) G^\ell_m(w) &= \frac{m\ G^\ell_m(w)}{(z-w)} + \text{finite part}\ .
\end{aligned} \tag{4.5}$$

The OPE's of the field G^ℓ_m are less singular if a field $G^\ell_{m\pm1}$ vanishes. This occurs for $D_\ell^{(+)}$ state $G_\ell^{(+)\ell}$ and the $D_\ell^{(-)}$ state $G_{-\ell}^{(-)\ell}$. Thus the relationship between the PF representation given in (3.14) and the $D^{(\pm)}$ highest weights is

$$\begin{aligned}
G_m^{(+)\ell}(z) &\sim \tilde{\phi}^{N-2|b|\ell}_{-(\ell-m)}(z) \exp\left(m\ \sqrt{\tfrac{2}{K}}\ \phi_3(z)\right)\ , \\
G_m^{(-)\ell}(z) &\sim \phi'^{N-2|b|\ell}_{-(\ell+m)}(z) \exp\left(m\ \sqrt{\tfrac{2}{K}}\ \phi_3(z)\right)\ .
\end{aligned} \tag{4.6}$$

The ϕ^q_p and $\tilde{\phi}'^q_p$ states correspond to descendants of the $D^{(\pm)}$ series which are at the boundary of the representations. For example, the ϕ^q_p PF states are the points on the (J^3_0, L_0) diagram, (figure 2. of ref. [9]), with $L_0 > 0$ and multiplicity one.

Let $G_m^{(e_o)\ell}$ correspond to the continuous series highest weight fields. Their relationship to the PF representations is given by

$$\begin{aligned}
& G_m^{(e_o)\ell}(z) \sim \gamma^{2|b|\ell, w}_{-m+w/(2|b|)}(z) \exp\left(m\ \sqrt{\tfrac{2}{K}}\ \phi_3(z)\right)\ , \\
& \text{for } w = \{-|b|, \cdots, -1\},\ e_o \equiv \tfrac{w}{2|b|} + 1; \qquad \text{for } w = \{0, \cdots, |b|\},\ e_o \equiv \tfrac{w}{2|b|}\ .
\end{aligned} \tag{4.7}$$

All of the PF/Virasoro highest weights we have found by the analysis of section 3 are in correspondence with the unitary $SU(1,1)$ parafermion theories [9], except for the

$\gamma_p^{x,w}$ with $x \leq \frac{1}{2} - \left|e_o - \frac{1}{2}\right|$ or $x \geq \frac{1}{2} + \left|e_o - \frac{1}{2}\right|$. For these values of x the constraint $-\ell(\ell-1) > -e_o(e_o-1)$ is not satisfied and the $\gamma_p^{x,w}$ correspond to non-unitary continuous representations.

5. Character Analysis

The parafermion character for each PF/Virasoro highest weights listed in sec. 3 is evaluated by truncating free boson characters as discussed in [2]. This is a two step process. Since the chiral algebra is independent of both of these zero modes, we first extract from the boson characters states which do not have the proper dependence on the fermion zero modes ξ_0 and $\tilde{\xi}_0$. Secondly, we need to subtract any additional null vector modules in the highest weight module which can be constructed with screening operators. We will determine the number of these null vectors by explicit construction. Because there are fewer null vectors than the minimal $SU(2)$ case, the calculation of the characters is more straightforward than that presented in [2].

The PF/Virasoro highest weight states of sec. 3 cannot in general be diagonalized with respect to one or both (η, ξ) systems. A state is local with respect to a fermion system if it can be diagonalized in terms of the fermions and an auxiliary boson [2]. Since the parafermions are local with respect to either system, we can address the locality of parafermion modules by considering the PF highest weights of eqn. (3.8).

First consider the $D^{(\pm)}$ parafermion highest weights. The state ϕ_q is local with respect to the (η, ξ) system and independent of the ξ_0 zero mode. The state ϕ'_q is local with respect to the $(\tilde{\eta}, \tilde{\xi})$ system and independent of the $\tilde{\xi}_0$ zero mode. When $\frac{q}{b}$ is an integer ϕ_q and ϕ'_q are also local with respect to the other (η, ξ) system and depend on $\tilde{\xi}_0$ and ξ_0 respectively. Denote the holomorphic subspace of states in the boson Fock space as $\mathcal{H}$. Consider the subspace $\mathcal{H}_{\rm local} \subset \mathcal{H}$ which contains states which are relatively local with respect to the (η, ξ). Likewise we define $\tilde{\mathcal{H}}_{\rm local} \subset \mathcal{H}$ which contains states which are relatively local with respect to the $(\tilde{\eta}, \tilde{\xi})$ system. We define the small Hilbert space $\mathcal{H}_{\rm small} \subset \mathcal{H}_{\rm local}$ ($\tilde{\mathcal{H}}_{\rm small} \subset \tilde{\mathcal{H}}_{\rm local}$) to be the restriction of states in $\mathcal{H}_{\rm local}$ ($\tilde{\mathcal{H}}_{\rm local}$) to only those states which are independent of the fermion zero mode ξ_0 ($\tilde{\xi}_0$). We can decompose the relatively local Hilbert space $\mathcal{H}_{\rm local}$ ($\tilde{\mathcal{H}}_{\rm local}$) as

$$\mathcal{H}_{\rm local} = \mathcal{H}_{\rm small} \oplus \xi_0 \mathcal{H}_1 \ , \qquad \tilde{\mathcal{H}}_{\rm local} = \tilde{\mathcal{H}}_{\rm small} \oplus \tilde{\xi}_0 \tilde{\mathcal{H}}_1 \tag{5.1}$$

The $D^{(+)}$ ($D^{(-)}$) PF modules are in $\mathcal{H}_{\rm small}$ ($\tilde{\mathcal{H}}_{\rm small}$) . Non-vanishing two point functions for the Virasoro highest weights are given by

$$\langle \xi(w)\phi^q_p(z_1)\tilde{\phi}^q_p(0)\rangle_{2b} = z^{-2h^q_p} \quad \text{and} \quad \langle \tilde{\xi}(w)\phi'^q_p(z_1)\tilde{\phi}'^q_p(0)\rangle_{2b} = z^{-2h^q_p} \ . \tag{5.2}$$

The operators $\xi(w)$ ($\tilde{\xi}(w)$) inserted into the correlator soak up the path integral over the fermion zero mode [11]. Therefore the conjugate field to ϕ^q_p is $\tilde{\phi}^q_p$, and the conjugate to ϕ'^q_p is $\tilde{\phi}'^q_p$.

Consider the $D^{(+)}$ Virasoro primary field $\tilde{\phi}^q_p = V^L_M$, where $L = N - q$ and $M = N - q - 2bp$, with conformal dimension $h^q_p = \Delta^L_M$. The character of the boson module $[V^L_M]$ is

$$\chi^L_M = \frac{q^{\Delta^L_M - c/24}}{[\varphi(q)]^2} \ , \tag{5.3}$$

where $\varphi(q) = \prod_{n=1}^{\infty}(1-q^n)$. Let $\{V^L_M\}$ be the submodule of $[V^L_M]$ which contains only descendants of V^L_M restricted to the small Hilbert space $\mathcal{H}_{\mathrm{small}}$, and denote $\widehat{\chi}^L_M$ as its character. By construction, the sum of this character and similar characters shifted by the charge of the zero mode must be the boson character χ^L_M

$$\chi^L_M = \widehat{\chi}^L_M + (\xi_0)\widehat{\chi}^{L+(N+2b)}_{M+N} \ , \tag{5.4}$$

where (ξ_0) denotes the charge of the zero mode. The conformal dimension Δ^{L+N+2b}_{M+N} of the "second level" highest weight is greater than Δ^L_M; it would not be included in eqn. (5.4) otherwise. (Eqn. (5.4) is unaffected if the Virasoro primary is local with respect to the $(\tilde{\eta}, \tilde{\xi})$ system. In these cases, the primary is $\tilde{\xi}_0$ dependent. However, there is no selection rule that requires subtracting from the boson character states which are $\tilde{\xi}_0$ independent. This is because the parafermions contain the mode $\tilde{\eta}_0$ and can therefore annihilate the $\tilde{\xi}_0$ mode.) The character $\widehat{\chi}^L_M$ is found recursively by continuing this process,

$$\widehat{\chi}^L_M = \sum_{r=0}^{\infty}(-1)^r \chi^{L+r(N+2b)}_{M+rN} \ . \tag{5.5}$$

In principle this is not the end of the analysis for the irreducible character since we must also subtract out PF null vectors in the module $\{\tilde{\phi}^q_p\}$. Following Feigin and Fuchs [3], we assume that for the values of c given by (2.4) the only null vectors in the bosonic theory are parafermion null vectors. We will now see that there are no such null vectors. It was shown in ref. [13] that these type of null vectors could be explicitly constructed with screening operators. This analysis was carried out in more detail than we present here for the minimal PF models in ref. [2].

Again consider the $D^{(+)}$ Virasoro primary $\tilde{\phi}^q_p = V^L_M$, where $L = N - q$, $M = N - q - 2bp$. Null vectors are constructed by multiplication by a set of screening operators and subsequently shifting the charges of the vertex operator. For instance, we can formally construct $\tilde{\Omega}^{(\eta)q}_p = \oint \eta V^{L-(N+2b)}_{M-N}$. For this to be a well defined non-vanishing null state, the contour integral must close and be singular. The contour integral over η closes, however it is non-singular unless $p = 0$, in which case $\tilde{\Omega}^{(\eta)q}_0 = \tilde{\phi}^q_0$. Similarly, we can construct $\tilde{\Omega}^{(\tilde{\eta})q}_p$ using $\tilde{\eta}$ as a screening operator. However, we find that non-vanishing states of this type are Virasoro primaries of the type $\tilde{\phi}'$, proportional to ξ_0 when diagonalized in the (η, ξ) basis and therefore not in $\mathcal{H}_{\mathrm{small}}$. We must also consider operators of the type $\tilde{\Omega}^{(r)q}_p = \prod^r_{i=1} [\oint J(z_i)]\ V^{L+2br}_M$, where J is the third screening operator given by eqn. (2.10). All of these null vectors vanish even if contours close since $\Delta^{L+2br}_M < \Delta^L_M$. One can also show that further combinations of the three screening operators do not yield non-trivial null vectors in $\mathcal{H}_{\mathrm{small}}$.

Thus the irreducible character $\tilde{c}^q_p$ of $\{\tilde{\phi}^q_p\}$ is given by eqn. (5.5) with $L = N - q$, $M = N - q - 2bp$. Since the conjugate of ϕ^q_p is $\tilde{\phi}^q_p$ the character of $\{\phi^q_p\}$ is also given by eqn. (5.5), and $c^q_p = \tilde{c}^q_p$. This is the parafermionic contribution to the irreducible $D^{(+)}$ characters. The ϕ_2 conjugation symmetry yields the same result for the parafermionic contribution to the irreducible $D^{(-)}$ characters; $c'^q_p = \tilde{c}^q_p$ and $\tilde{c}'^q_p = c^q_p$.

Now consider the continuous series. In general PF states in the continuous series are not local with respect to either of the (η, ξ) systems. Using eqn. (2.13) and incorporating the restriction in (3.12) we arrive at three exceptional cases:
Case I: $w = 0$, $u = x$, and $\frac{x}{2b} = -1$. We have both ξ_0 and $\tilde{\xi}_0$ dependence.
Case II: $w < 0$, $u = x$, and $\frac{x-w}{2b} = -1$. We have ξ_0 dependence and the state is not local with respect to the $(\tilde{\eta}, \tilde{\xi})$ system.
Case III: $w > 0$, $u = x$, and $\frac{x+w}{2b} = -1$. We have $\tilde{\xi}_0$ dependence and the state is not local with respect to the (η, ξ) system.

Case I is on the boundary of the unitary continuous supplementary series. and corresponds to the descendant $\phi^{2|b|}_1 = \tilde{\phi}'^{2|b|}_1$. Therefore it is a null state in the $D^{(\pm)}$ representations. Such a situation is mentioned in [9].

For the remaining $\gamma^{u,w}$ representations, even if they are local with respect to an (η, ξ) system (cases II and III) they are not independent of that systems ξ zero modes. Since the parafermions can annihilate these zero modes, the embedding of the continuous series representations fills up the entire two boson Fock space. The conjugate field of $\gamma^{u,w}$ is given by Feigin-Fuchs conjugation of the ϕ_1

boson and the usual conjugation of the ϕ_2 boson. The conjugate field to $\gamma_p^{u,w}$ is $(\gamma_p^{u,w})^\dagger = \gamma_{-p}^{-u-2b,-w}$. Note that if $\gamma_p^{u,w}$ satisfies (3.12) or (3.13), so does its conjugate field. Since there is no zero mode independence in the continuous representations, correlation functions are computed straightforwardly by using the bosonized form of the fields. The non-vanishing two-point function is

$$\langle \gamma_{-p}^{-u-2b,-w}(z)\gamma_p^{u,w}(0)\rangle = z^{-2h_p^{u,w}} \tag{5.6}$$

As in the discrete cases, one can check that there are no null vectors which can be constructed with screening operators for the continuous representations. Hence their character will be the same as that for the two boson Fock space.

In summary the irreducible discrete and continuous characters are given by

$$\begin{aligned} c_p^q &= \frac{q^{-c/24+h_p^q}}{[\varphi(q)]^2}\sum_{r=0}^{\infty}(-1)^r q^{\frac{1}{2}r(r+1)+rp} , \\ c_p^{u,w} &= \frac{q^{-c/24+h_p^{u,w}}}{[\varphi(q)]^2} . \end{aligned} \tag{5.7}$$

where the other discrete characters are given by the relation $\tilde{c}_p^q = c'^q_p = \tilde{c}'^q_p = c_p^q$. These results agree [14] with the determinant formula of ref. [9].

6. Discussion

One would like to combine these characters to find new modular invariant parafermion models. It is clearly more difficult to construct non-minimal unitary modular invariants than minimal ones. All PF highest weight representations in the minimal case are unitary and embedded into the bosonic Fock space in the same way, i.e. they are independent of both fermion zero modes. This embedding structure plays a crucial role in closing the operator algebra onto unitary states. However, in the non-minimal case, there is no unique embedding (specified by fermion zero mode dependence) of the highest weight representations into the bosonic Fock space. Furthermore, the highest weight analysis of sec. 3 admits non-unitary representations. One obvious approach to constructing modular invariant theories is to use the continuous series only, since their characters are just the two boson characters. Preliminary calculations using the methods developed in ref. [2], show that generic continuous unitary states fuse onto non-unitary states [15]. Similar calculations show that discrete highest weights can fuse onto continuous highest weights. A careful study of the fusion rules may determine the combinations of characters that form unitary modular invariants.

We would like to thank J. Lykken, D. Nemeschansky, and T. Morris for useful discussions. O.F.H thanks Leonard Peltier for encouragement.

References

[1] O.F. Hernández, Physics Letters B233 (1989) 355.

[2] P. Griffin and O.F. Hernández, MAD/TH/90-11 FERMILAB-PUB-90/117-T, June 1990.

[3] B.L. Feigin and D.B. Fuchs, " Representations of the Virasoro Algebra", Moscow preprint 1984.

[4] Dotsenko and Fateev Nucl. Phys. B 240 (1984) 312; Nucl. Phys. B999 (1984) 999 .

[5] A.B. Zamolodchikov and V.A. Fateev, Sov. Phys. JETP 62 (1985) 215.

[6] J.D. Lykken, Nucl. Phys. B313 (1989) 473.

[7] D. Nemeschansky, Phys. Lett. B224 (1989) 121, and USC-89/012 5/89.

[8] J. Distler and Z. Qiu, Nucl. Phys. B 336 (1990)553.

[9] L.J. Dixon, J.Lykken, M.E. Peskin, Nucl. Phys. B325 (1989) 329.

[10] P. Goddard, A. Kent, and D. Olive, Comm. Math. Phys. 103 (1986) 105.

[11] D. Friedan, E. Martinec, S. Shenker, Nucl. Phys. B271 (1986) 93.

[12] B.G. Wybourne, *Classical Groups for Physicists*, (John Wiley & Sons, 1974).

[13] M. Kato and S. Matsuda, Phys. Lett. B172 (1986) 216, Phys. Lett. B184 (1987) 184, KEK 86-112 3/87.

[14] J.D. Lykken, private communication.

[15] P. Griffin and O.F. Hernández, work in progress.