Fermi National Accelerator Laboratory

FERMILAB-PUB-90/202-T
November 1990

Quantization and BRST Cohomology of Pure Two-Dimensional Gravity

Paul A. Griffin†
Fermi National Accelerator Laboratory
P. O. Box 500, Batavia, IL 60510

Abstract

The path integral description of the continuum theory of pure 2-D gravity has a special local gauge symmetry, in addition to general coordinate invariance. Because of the additional symmetry, I argue that the gauge fixed path integral is not described by the non-local Polyakov action of the conformal anomaly. Instead, the quantum theory is described by a "topological gravity" model, which is treated in this paper in the framework of conventional conformal field theory. The construction of fields in the BRST cohomology is given in some detail, and the analysis leads to new candidate physical states.

† bitnet pgriffin@fnal

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

1. Introduction

There has been considerable recent progress in the matrix model approach to two dimensional quantum gravity[1]. The new work yields a complete perturbative (and possibly non-perturbative) description of pure gravity and gravity coupled to minimal matter[2], to all orders in the genus expansion of surfaces. Physical states of pure gravity are interpreted in the continuum scaling limit as finite loops (finite boundaries of the surface) or as infinitesimal loops (punctures of the surface). Correlation functions of these physical operators are exactly computable[3][4].

Continuum field theory descriptions of these systems have not to date been able to successfully reproduce all of the matrix model results. Quantization of gravity coupled to matter systems is based on the Polyakov model[5], which stresses the need to use a general coordinate invariant measure for the path integration over gravitational metrics. The critical exponents for minimal matter ($c < 1$) were first obtained by a clever choice of gauge[6], and later were derived in the more conventional conformal gauge[7]. In the conformal gauge, the description of pure gravity (the $c \rightarrow 0$ limit of gravity coupled to matter with central charge c), is given by the Liouville action which describes the conformal anomaly, and by the ghost action corresponding to the Fadeev-Popov determinant arising from the gauge fixing of diffeomorphisms.

In this paper, we will argue that the Liouville theory does not describe pure gravity, due to an additional local symmetry which is built into the pure gravity theory. The gauge fixed theory of pure gravity which takes this additional symmetry into account is essentially a gauge fixed version of Witten's topological gravity[8][9].

Witten has shown, by reproducing correlators at genus zero and one, that the matrix model results for pure gravity are described by the continuum theory of topological gravity, which is invariant with respect to arbitrary infinitesimal variations of the metric[1]. The field theoretic description of topological gravity is developed further in references [11][12][13][14]. In particular, references [8][14] construct field theories such that the observables correspond to the stable cohomology classes on the moduli space of Riemann surfaces.

In section 2 we consider the symmetries of pure 2-D gravity, and show that the Polyakov measure respects a subset of the additional symmetries found in the clas-

[1] In [10] it is also shown that topological matter systems coupled to topological gravity can reproduce the n-matrix model (gravity+minimal matter) correlators on the sphere.

sical action. The additional symmetry is seemingly lost when one transforms to the basis of metric fluctuations spanned by conformal, diffeomorphism, and Teichmüler variations in the standard way as first discussed by Polyakov. The non-trivial part of the Jacobian for this change of variables is the Liouville action of the Weyl mode given in the conformal gauge. In general covariant form it is the non-local conformal anomaly, and it is not invariant with respect to any symmetries other than diffeomorphisms. However, the quantum measure for the integration over the diffeomorphism fluctuations, which is usually divided out of the path integral to gauge fix, is also not invariant under the additional symmetry. The explicit gauge fixing described in section 3 will in fact show that the breaking of the additional symmetry by the Liouville action and by the measure over diffeomorphisms cancel each other. This implies that pure 2-D quantum gravity has no conformal anomaly.

We also discuss the number of representatives of the orbits of gauge transformations required to describe the 2-D gravity. All points in string moduli space are gauge equivalent. However, if we allow in the path integral surfaces on the boundary of the compactification of string moduli space, then additional gauge representatives are required.

In section 3 we quantize using the BRST formalism developed by Batalin and Vilkovisky[15], which can properly account for redundancies in gauge symmetries. The gauge fixed action for ghosts is a conformaly invariant model virtually equivalent to that previously discussed in the context of topological gravity in references [12][14]. However, the cosmological constant term in the action is a relevant perturbation to this model; this term differentiates 2-D gravity from topological gravity.

Physical states are found in the BRST cohomology of the gauge fixed quantum theory, so in section 4 we construct states which lie in the cohomology. The highest weight of the conformal field theory, has the natural interpretation as a puncture operator. As suggested by the matrix model approach, this is the basic primary state of the cohomology. A relatively complete search for possible physical states in the space of Virasoro highest weights is given. We find the states which were previously considered in the context of topological gravity models[13][14], and additional states which have not been previously considered.

Each state can be assigned an arbitrary amount of local "delta function" curvature. Curvature of this type must be inserted into the path integral to describe the contribution from surfaces at the boundary of the compactification of ordinary string moduli space.

Some implications of these results are given in section 5.

2. Is Pure Gravity the Liouville Model or "Topological Gravity"?

In this section we argue that pure gravity in two dimensions is essentially the topological gravity proposed by Witten. As shown by Witten, this is consistent with the matrix model results. A detailed description of the Liouville theory approach to pure gravity is given in reference [16].

Classical Symmetries

The continuum theory of gravity is governed by the action

$$S = \mu \int \sqrt{g} + \lambda \int \sqrt{g} R \ . \tag{2.1}$$

In two dimensions, the dynamics of the action simplifies because the Einstein-Hilbert term is a topological index – $\int \sqrt{g}R = 4\pi\chi$, where the Euler number $\chi = 2 - 2g$ and g is the number of handles (genus) of the closed surface. In addition to the local diffeomorphism invariance, under which the metric and coordinates transform as

$$\delta_d g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \ , \quad \delta_d x^\mu = \epsilon^\mu \ , \tag{2.2}$$

the action is also invariant with respect to *any* local variation of the metric which preserves the area. It will be convenient to parametrize these variations as

$$\begin{aligned} \delta_s g_{\mu\nu} =& g_{\mu\lambda} L^\lambda_\nu + g_{\nu\lambda} L^\lambda_\mu \ , \quad L^\lambda_\lambda = 0 \ , \\ \delta_w g_{\mu\nu} =& \alpha' g_{\mu\nu} \ , \quad \int \sqrt{g}\alpha' = 0 \ . \end{aligned} \tag{2.3}$$

The variation δ_s of the metric is an infinitesimal $SL(2,R)$ transformation. To see this, consider the zwei-bein parametrization of gravitational degrees of freedom, $g_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu$, where δ_{ab} is the flat Euclidean tangent space metric. A δ_s variation of the zwei-bein is given by

$$\delta_s e^a_\mu = L^\lambda_\mu e^a_\lambda = \Lambda^a_b e^b_\mu \ , \tag{2.4}$$

where we have defined the matrix $\Lambda^a_b = e^a_\mu (e^{-1})^\lambda_b L^\mu_\lambda$. In the Λ^a_b basis of the transformation, it is clear that δ_s is an infinitesimal left $SL(2,R)$ transformation on the zwei-bein, where the $U(1)$ subgroup is the local 2-D Euclidean Lorentz group. The metric is unchanged by local Lorentz transformations, and therefore the action of δ_s on the metric is the coset $SL(2,R)/U(1)$ of the gauge symmetry.

The variation δ_w of the metric is an area preserving conformal transformation. Any diffeomorphism acting on the metric can be expressed as a combination of these conformal and the $SL(2,R)/U(1)$ transformations.

Quantum Symmetries

The quantum path integral description of 2-D gravity requires a definition of the measure for the gravitational degrees of freedom. A multitude of results on this topic applied to string theory are found in [17] and citations therein.

The Polyakov measure [5] for integration over the metric is determined by the norm of infinitesimal fluctuations in the field space of metrics,

$$||\delta g_{\mu\nu}||^2 = \int \sqrt{g}\left[g^{\mu\nu}g^{\lambda\sigma}\delta g_{\mu\lambda}\delta g_{\nu\sigma} + Cg^{\mu\nu}g^{\lambda\sigma}\delta g_{\mu\nu}\delta g_{\lambda\sigma}\right] \ , \tag{2.5}$$

where C is a relative normalization constant. This is the unique norm which is quadratic in the fluctuations of the metric and invariant with respect to *background* diffeomorphisms. With respect to background diffeomorphisms, the metric transformation is given by (2.2) and the fluctuations transform as components of a second rank tensor,

$$\delta_d(\delta g_{\mu\nu}) = \nabla_\lambda(\delta g_{\mu\nu})\epsilon^\lambda + (\delta g_{\lambda\nu})\nabla_\mu\epsilon^\lambda + (\delta g_{\mu\lambda})\nabla_\nu\epsilon^\lambda \ . \tag{2.6}$$

The norm is not invariant with respect to background conformal transformations of the metric. It is however invariant with respect to the background $SL(2,R)/U(1)$ transformations, whose action on the metric is given by (2.3) and on the metric fluctuations by,

$$\delta_s(\delta g_{\mu\nu}) = (\delta g_{\mu\lambda})L^\lambda_\nu + (\delta g_{\lambda\nu})L^\lambda_\mu \ . \tag{2.7}$$

An arbitrary infinitesimal variation of the metric can be decomposed as a combination of conformal, diffeomorphism, and Teichmüller variations,

$$\delta g_{\mu\nu} = (\delta\phi + \nabla_\lambda\delta V^\lambda)g_{\mu\nu} + (P_1\delta V)_{\mu\nu} + \delta t_{(i)}\mu^{(i)\lambda}_\mu g_{\lambda\nu} \ , \tag{2.8}$$

where the operator P_1 is given by

$$(P_1\delta V)_{\mu\nu} = \nabla_\mu\delta V_\nu + \nabla_\nu\delta V_\mu - g_{\mu\nu}\nabla^\lambda\delta V_\lambda \ , \tag{2.9}$$

and the real parameters $\delta t_{(i)}$ and Beltrami differentials $\mu^{(i)\lambda}_\mu$ correspond to infinitesimal Teichmüller deformations of the metric. This decomposition can be used to rewrite the path integral measure over the metric $g_{\mu\nu} = e^\phi\hat{g}_{\mu\nu}$, where $\sqrt{\hat{g}} = 1$:

$$[dM] = [d\phi]_{\hat{g}}[dV]_{\hat{g}}[\text{Teich}]det(e^\phi)det'(P_1^\dagger P_1)^{\frac{1}{2}}\ e^{-\delta\mu\int\sqrt{g}} \ , \tag{2.10}$$

and where

$$||\delta\phi||_g^2 = \int \delta\phi\delta\phi \ ,$$
$$||\delta V_\mu||_g^2 = \int \sqrt{g} g^{\mu\nu} \delta V_\mu \delta V_\nu \ . \tag{2.11}$$

The cosmological constant term $\delta\mu$ is the ultralocal dependence on the Weyl mode ϕ that has been extracted from the measure $[dV]$. The Jacobian factors of the measure can be evaluated via the heat kernel regularization scheme[18]; in the conformal gauge, this procedure results in a Liouville action for the Weyl mode and the usual first order action for the diffeomorphism ghosts. The Liouville action is an expression of the conformal anomaly, which is non-local in covariant form. The volume of the diffeomorphism group connected to the identity is $\int[dV]$, and this term is divided out of the path integral to gauge fix the *quantum* diffeomorphism symmetry $\delta_d V^\mu = \epsilon^\mu$.

There does not appear to be any residual local symmetry after the gauge fixing of diffeomorphisms which would correspond to the background $SL(2,R)/U(1)$ symmetry. One can verify that the Polyakov action (Liouville action in the conformal gauge) is not invariant with respect to the $SL(2,R)/U(1)$ transformations of the metric. This is actually not inconsistent with the claim that there is such additional symmetry in the full path integral however, because the measure $[dV]$ is also not invariant with respect to the background $SL(2,R)/U(1)$ transformations. By inspection of the diffeomorphism norm, this measure would be invariant only if $\delta_s \delta V_\mu = L_\mu^\nu \delta V_\nu$ were satisfied for arbitrary L_μ^ν. The explicit variation of the measure with respect to the $SL(2,R)/U(1)$ transformations can be calculated using equations (2.7) and (2.8),

$$\nabla_\mu(\delta_s \delta V_\nu - L_\nu^\lambda \delta V_\lambda) + (\mu \leftrightarrow \nu) = (\nabla_\nu L_\mu^\lambda - \nabla^\lambda L_{\mu\nu})\delta V_\lambda + L_\mu^\lambda \nabla_\lambda \delta V_\nu + (\mu \leftrightarrow \nu) \ . \tag{2.12}$$

The right hand side of this equation vanishes only if L_μ^λ corresponds to a transverse diffeomorphism. These are the diffeomorphisms that are redundant with $SL(2,R)/U(1)$ when acting on the metric. The norm for the measure $[dV]$ is therefore not invariant with respect to an arbitrary $SL(2,R)/U(1)$ variation. In fact, the $SL(2,R)/U(1)$ dependence of the diffeomorphism measure and the Liouville action should cancel one another. Although this analysis does not prove that there is explicit cancellation for the regularized theory, the BRST quantization in section 3 will indicate that this is indeed the case. Since conformal transformations of the metric can be expressed as a combination of diffeomorphism and

$SL(2,R)/U(1)$ transformations, it is possible to properly gauge fix both diffeomorphisms and $SL(2,R)/U(1)$ symmetries only if the pure gravity theory has no conformal anomaly.

To divide the path integral out by the volume of connected diffeomorphisms $\int[dV]$ breaks both the diffeomorphism and $SL(2,R)/U(1)$ symmetries. No accounting has been made however, for the redundancies between the two symmetries.

Representatives of Gauge Orbits

For a genus g surface, what are the representatives of the gauge orbits? Assume that the conformal anomaly of the gauge fixed theory vanishes without the addition of matter. Then all area-preserving variations of the metric are gauge variations of the quantum theory. In particular, if we take $L_\nu^\lambda = \delta t_{(i)} \mu_\nu^{(i)\lambda}$, we see that only one point in string moduli space is required to specify the orbits of the pure gravity theory for each genus. In addition, we must integrate over the area (i.e. the zero mode ϕ_0 of the Weyl mode). Since the decomposition given by equation (2.8) is complete, the moduli space of the pure gravity theory is one dimensional for each genus.

Any compactification of the moduli space generated by the infinitesimal Teichmüller deformations will include limit points of a certain codimension; the moduli space of stable curves corresponds to the compactification with a boundary of codimension one. One cannot perform a series of infinitesimal Teichmüller deformations on the boundary points to return to a generic representative in moduli space. This means that the boundary of moduli space requires separate representatives in the gauge fixing procedure of pure gravity.

For example, consider the genus one case. The conformal classes of tori are represented by the doubly periodic parallelogram with sides $(1,\tau)$ in the complex plane. The modulus is the complex parameter $\tau = \tau_1 + i\tau_2$, where $-\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}$, $\tau_2 > 0$, and $|\tau| > 1$. The limit point is $\tau_2 \to \infty$, for which the torus degenerates into an infinite cylinder. (This point is infinitely far from any generic point in moduli space, with respect to the Weil-Petersson metric $d^2s = 2|\delta\tau|^2/\tau_2^2$.) The cylinder can be conformally mapped to the Riemann sphere, with $z=0$ and $z=\infty$ "connected" by a tube of infinitely thin width. A combination of conformal transformations and diffeomorphisms can then map this to a finite sphere with two punctures. The two punctures carry curvature since the total integrated curvature of the sphere with punctures is zero for the "torus". There are no infinitesimal Teichmüller deformations (gauge transformations) that can deform this representation of the limit point back to a generic point in the moduli space of the torus. Therefore, there are two representatives of the gauge symmetry of pure gravity for genus one, if we consider the compactification of moduli space to be the relevant space of metric configurations.

Next, consider the higher genus cases. The boundary of moduli space, in the Fenchel-Nielson coordinates of Teichmüller space [19], are the surfaces obtained by pinching any combination of the $3g-3$ non-intersecting loops for genus g. Consider for instance the pinching of a homology cycle, which reduces the genus of the surface by one. The moduli of the genus $g-1$ pinched surface are the positions of the two points on the surface with the required curvature insertions, and the ordinary moduli of a generic genus $g-1$ surface. There are no moduli to unpinch the surface. The moduli space of pure 2-D gravity for the compactified string moduli space is therefore finite dimensional for each genus; it contains one unpinched surface, and pinched surfaces at the boundary of Teichmüller space which are inequivalent under global diffeomorphisms.

3. BRST Quantization and Conformal Field Theory

In this section the conformal, diffeomorphism, and $SL(2,R)/U(1)$ symmetries of the pure gravity model will be gauge fixed, leading to a conformal field theory description of the gauge fixed path integral. The gauge fixed quantum theory will have chiral BRST charges which square to zero, indicating that it is indeed consistent to assume the absence of the conformal anomaly. The analysis of this section overlaps substantially with [11][12], and the notation we use conforms to [14].

The initial expectation for the pure gravity theory is that surface area is the only observable for each topological class of metrics, so that gauge fixing is henceforth trivial. However, this expectation is modified by the matrix model analysis of the random lattice formulation of 2-D gravity. In this analysis, local physical degrees of freedom are associated with finite or infinitesimal punctures of the surface[4]. In the continuum field theory description of gravity, infinitesimal punctures are marked points which diffeomorphisms are required to leave fixed. Therefore to obtain a puncture operator in the continuum theory, one can gauge fix to an unpunctured surface and subsequently require the ghosts associated with diffeomorphism gauge fixing to vanish at certain marked points. It is not completely trivial to obtain diffeomorphism ghosts from the pure gravity theory we have described in the previous section because diffeomorphisms are completely redundant with the conformal and $SL(2,R)/U(1)$ symmetries when acting on the metric. One must use the BRST formalism of Batalin and Vilkovisky[15] (BV) to treat the redundant symmetries. In their language, 2-D gravity is a first-stage gauge theory.

We will choose a gauge which preserves background general coordinate invariance, and which leads to a gauge fixed action that is essentially an exactly soluble conformal field theory. We begin by fixing to a generic point in moduli space for a given genus. Generalizations of the procedure which gauge fixes to representatives on the boundary of moduli space will be discussed briefly at the end of the section.

Gauge Fermions

A gauge fermion Ψ with ghost number -1 contains information of the gauge constraints in the BV gauge fixing formalism. The BRST variation $\delta_b\Psi$ is the gauge fixing contribution which is to be added to the action, and it includes the Fadeev-Popov ghost terms automatically. We decompose the metric $g_{\mu\nu}$ as $g_{\mu\nu} = e^{\phi}\hat{g}_{\mu\nu}$, and denote the background representative as $\bar{g}_{\mu\nu}$. Consider the gauge fermion $\Psi = \Psi_1 + \Psi_2$, where

$$\begin{aligned}\Psi_1 &= \int \chi(\sqrt{g}R - \sqrt{\hat{g}}\hat{R}) \ , \\ \Psi_2 &= \int b_{\mu\nu}\sqrt{g}g^{\mu\nu}, \quad b_{\mu\nu}\bar{g}^{\mu\nu} = 0 \ .\end{aligned} \tag{3.1}$$

The gauge fermion Ψ_1 will fix the trace of the metric, up to the zero mode ϕ_0 which corresponds to the physically observable area. Note that the gauge constraint $\sqrt{g}R = \alpha\sqrt{\hat{g}}\hat{R}$ with $\alpha \neq 1$ violates Euler number conservation; we will apply such a choice to gauge fix to configurations on the boundary of the compactification of moduli space. The second gauge fermion will fix the traceless (with respect to $\bar{g}_{\mu\nu}$) components of the metric to be those of a fixed background representative metric $\bar{g}_{\mu\nu}$. The zero mode of the antighosts χ and $b_{\mu\nu}$ should be excluded from Ψ_0. All of the zero modes which arise from the gauge fixing will be treated when the path integral over the gauge fixed action is obtained below.

The BRST variations of the fields are

$$\begin{aligned}\delta_b\chi &= \pi + c^\mu\partial_\mu\chi \\ \delta_b b_{\mu\nu} &= \pi'_{\mu\nu}, \quad \pi'_{\mu\nu}\bar{g}^{\mu\nu} = 0 \ , \\ \delta_b g_{\mu\nu} &= 2\psi_{\mu\nu} + \nabla_\mu c_\nu + \nabla_\nu c_\mu \ , \\ \delta_b\psi_{\mu\nu} &= -(\nabla_\mu\gamma_\nu + \nabla_\nu\gamma_\nu) \ , \\ \delta_b c_\mu &= (\nabla_\nu c_\mu)c^\nu + (\nabla_\mu c_\nu)c^\mu + \gamma_\mu \ ,\end{aligned} \tag{3.2}$$

and the ghost numbers of $\chi, b_{\mu\nu}, \pi, \pi'_{\mu\nu}, \psi_{\mu\nu}, c_\mu, \gamma_\mu$ are $-1, -1, 0, 0, 1, 1, 2$ respectively. The BRST transformation laws for the fields $(\pi, \pi'_\mu, \gamma_\mu)$ are determined by nilpotency of the BRST charge (δ_b^2=0 on all fields). The BRST transformation law for χ has been unconventionally defined by adding the $c^\mu \partial_\mu \chi$ term to make χ transform as a scalar with respect to background coordinate transformations; this does not introduce extra degeneracies in the constraints imposed by (π, π'_μ). The advantage of this redefinition is that it decouples the (χ, π) ghost system from the (b, c) system. Physically, it means that the gauge fermion Ψ_1 is defined so as not to break general coordinate invariance. The fields $c^\mu, \psi_{\mu\nu}$ are diffeomorphism and topological ghosts respectively, and the field γ^μ expresses the redundancy between the two symmetries. This type of BRST transformation law for topological gravity was first given in reference [11].

To fix the redundant symmetry, consider the gauge fermion

$$\Psi_3 = \int \beta^{\mu\nu} \sqrt{g} \psi_{\mu\nu} \ , \quad \bar{g}_{\mu\nu} \beta^{\mu\nu} = 0 \ . \tag{3.3}$$

The BRST variations of the antighost multiplet $\beta^{\mu\nu}, \pi''^{\mu\nu}$ with ghost numbers $-2, -1$ are

$$\delta_b \beta^{\mu\nu} = \pi''^{\mu\nu} \ , \ \delta_b \pi''^{\mu\nu} = 0 \ . \tag{3.4}$$

The full gauge fixing term will be $S_{g.f.} = \delta_b(\Psi_1 + \Psi_2 + \Psi_3)$. There is no need to add the "extraghost" associated with the first-stage gauge fixing procedure of BV, since the constraints obtained by integrating over the Lagrange multipliers π and $\pi'_{\mu\nu}$ are non-degenerate.

Gauge Fixed Action

Integrating out the Lagrange multipliers $\pi'_{\mu\nu}$, $\pi''^{\mu\nu}$ imposes the constraints

$$g_{\mu\nu} = e^\phi \bar{g}_{\mu\nu} \ , \quad \psi_{\mu\nu} = \tfrac{1}{2} \psi g_{\mu\nu} \ . \tag{3.5}$$

The additional contribution to the action (2.1) from gauge fixing is then

$$\begin{aligned} S_{g.f.} = \int \sqrt{\bar{g}} [& \pi \bar{\Delta} \phi - \chi \bar{\Delta}(\psi + \bar{\nabla}_\lambda c^\lambda) \\ & + b_{\mu\nu}(\bar{\nabla}^\mu c^\nu + \bar{\nabla}^\nu c^\mu) + \beta^{\mu\nu}(\bar{\nabla}_\mu \gamma_\nu + \bar{\nabla}_\nu \gamma_\mu)] \ , \end{aligned} \tag{3.6}$$

where $\bar{\Delta} = -\bar{\nabla}_\mu \bar{\nabla}_\nu \bar{g}^{\mu\nu}$. To simplify the second term of (3.6), we redefine the field ψ by making the linear shift

$$\psi \to \psi - \bar{\nabla}_\lambda c^\lambda \ . \tag{3.7}$$

Note that the redefinition will effect the BRST transformation law of ϕ in the quantized theory.

The bc ghost system exponentiates the standard Fadeev-Popov determinant of the diffeomorphisms (see equation (2.10)), and the $\beta\gamma$ system exponentiates the inverse determinant. This occurs because gauge fixing the $SL(2,R)/U(1)$ symmetry requires no determinant factor; there are no derivatives in the metric variation given by equation (2.3). For an unpunctured surface, the two ghost systems are expected to cancel. One can in principle integrate out the π, ϕ, χ, ψ fields, which we denote as the *Weyl sector* of the theory, to impose the constraint $\phi = \phi_0$, where ϕ_0 is the zero mode of the field ϕ. However, it will be useful to have the Weyl sector in the path integral when we attempt to construct physical states using standard conformal field theory (CFT) technology.

The remaining action describes CFT of free fields with central charges, $-26, +26, -2$ and $+2$ from the $bc, \beta\gamma, \chi\psi$ and $\pi\phi$ systems respectively, and therefore one expects that the heat kernel regularized partition function has no conformal anomaly. This is indeed the case, although the Weyl mode contribution requires special attention. A careful evaluation of the Jacobian $det(e^{\phi})$ which arises from the measure[18], is required. In the calculation, the $\pi\phi$ and $\chi\psi$ systems are properly regarded as a representation of the covariant delta function $\delta(\phi')$.

From the action (3.6), one can read off the ghost zero modes, which require special treatment in the definition of the path integral. First, recall that the surface area is a physical observable, and the zero mode ϕ_0 was not gauge fixed. Consequently, the zero modes of π,χ and ψ are not in the irreducible space of states; we must insert delta functions in the covariant path integral to soak up these modes. Secondly, the modes of $b_{\mu\nu}$ in $Ker P_1^{\dagger}$ are also not in the irreducible space of states. This is because the constraints on the topological ghost $\psi_{\mu\nu}$ obtained by the BRST variation of Ψ_3 do not fix the modes of $\psi_{\mu\nu}$ in $Ker P_1^{\dagger}$. The path integral over these modes eliminates the modes of $b_{\mu\nu}$ in $Ker P_1^{\dagger}$. We also need to fix the modes of $\beta^{\mu\nu}$ in $Ker P_1^{\dagger}$ because the Teichmüller variations of the $SL(2,R)/U(1)$ symmetry are not redundant with diffeomorphisms. And, as is standard in string theory, the remaining conformal Killing symmetries of the sphere and torus are fixed by inserting modes of c^{μ} in $Ker P_1$ into the path integral.

Conformal Field Theory of 2-D Gravity

For each topology of surfaces we can use standard CFT techniques to obtain a current algebra solution for the gauge fixed theory (see ref. [20] and citations therein). In each patch of the surface, a complex coordinate system $(z, \bar{z})$ can be defined. The dynamics of each conformal field $W(z, \bar{z})$ in the gauge fixed path integral are governed by its holomorphic and antiholomorphic components $w(z)$ and $\bar{w}(\bar{z})$, and the current algebra of the theory is locally reducible into two chiral algebras. The local operator algebra for the holomorphic sector is described in this section; completely analogous results are obtained for the antiholomorphic sector.

The local operator product expansions of the free field theories described by the action (3.6) are given by

$$b(z)c(w) \sim \beta(z)\gamma(w) \sim \partial_z\phi(z)\pi(w) \sim \partial_z\chi(z)\psi(w) \sim \frac{1}{z-w} , \tag{3.8}$$

and the holomorphic component of the traceless stress tensor is

$$T = \partial\pi\partial\phi + \partial\chi\partial\psi + c\partial b + 2\partial c b + \gamma\partial\beta + 2\partial\gamma\beta . \tag{3.9}$$

The cosmological constant term $\mu\int\sqrt{g}e^{\phi}$ of the action is treated in this approach as a relevant perturbation. With respect to the conformal stress tensor, the dimensions of the primary fields are

$$\Delta\left[c, \gamma, \pi, \phi, \chi, \psi, b, \beta\right] = \left[-1, -1, 0, 0, 0, 0, 2, 2\right] . \tag{3.10}$$

The chiral splitting of the current algebra implies that there are BRST charges Q and $\bar{Q}$ for the holomorphic and antiholomorphic sectors. BRST transformations for the holomorphic fields are

$$\begin{aligned}
\delta_b c =& c\partial c + \gamma ,\\
\delta_b b =& T ,\\
\delta_b \gamma =& c\partial\gamma - \gamma\partial c ,\\
\delta_b \beta =& b + \partial\chi\partial\phi + c\partial\beta + 2\partial c\beta ,\\
\delta_b \phi =& c\partial\phi + \psi ,\\
\delta_b \psi =& c\partial\psi - \gamma\partial\phi ,\\
\delta_b \chi =& \pi + c\partial\chi ,\\
\delta_b \pi =& c\partial\pi - \gamma\partial\chi ,
\end{aligned} \tag{3.11}$$

and the nilpotent BRST charge which generates these transformations is given by[14],

$$Q = \oint\{c[T - \tfrac{1}{2}(c\partial b + 2\partial c b)] + \gamma\partial\chi\partial\phi + \partial\pi\psi + b\gamma\} . \tag{3.12}$$

We have taken into account the redefinition (3.7) of the ψ field in the BRST transformation laws for ϕ and ψ.

Physical states are those annihilated by Q and $\bar{Q}$, but not proportional to either, i.e. in the cohomology. The BRST charge enforces the constraints of the gauge theory. In dimension $d \geq 3$, the gravitational constraints are known as the lapse and shift constraints, which require that the theory be invariant with respect to reparametrizations of time and the patching together from one time to the next of spatial coordinate systems. For 2-D gravity with matter, these constraints extend to the invariance with respect to arbitrary conformal transformations of coordinates. For the pure gravity case, the constraints are even stronger, as can be seen by the inspection of the BRST charge. The theory has a global supersymmetry[14]. The supercharge operator is given by

$$Q_s = \oint (\partial \pi \psi + b\gamma) \ , \tag{3.13}$$

and the generator of local superconformal transformations is

$$G = \partial\chi\partial\phi + c\partial\beta + 2\partial c\beta \ . \tag{3.14}$$

It is also useful to define the supercharge

$$\tilde{Q}_s = Q_s + \oint \gamma\partial\chi\partial\phi \ , \tag{3.15}$$

which when acting on the fields squares to zero, up to a diffeomorphism. By inspection of the BRST charge (3.12), we see that conformally invariant physical states must also be annihilated by the supercharge $\tilde{Q}_s$.

Gauge fixing on the Boundary of Moduli Space

Recall from section 2 that a genus one surface at the boundary of moduli space could be represented as the doubly punctured sphere with curvature insertions. To gauge fix to this surface, we must modify the gauge fixing constraints. Consider the gauge fermion

$$\Psi_1^{(\alpha)} = \int \chi(\sqrt{g}R - (1-\alpha)\sqrt{\hat{g}}\hat{R}) \ . \tag{3.16}$$

Since we want to fix to the sphere and still satisfy the Euler number theorem for the torus, we need to use $\Psi_1^{(1)}$ as the gauge fermion which fixes the Weyl mode. This modifies the gauge fixed Lagrangian by adding a background charge term for the π field; subsequently the holomorphic stress tensor is modified by the addition of

$$\delta T^{(\alpha)}_{zz} = -\alpha\partial^2\pi \ . \tag{3.17}$$

Non-vanishing correlation functions on the "torus" require the addition of BRST invariant operators $V^{q\bar{q}}$ proportional to $e^{q\pi+\bar{q}\bar{\pi}}$ since the $U(1)$ currents of the $\pi\phi$ system enforce charge neutrality of the path integral. These operators create curvature at their insertions. To see this, recall that the total curvature is

$$\sqrt{g}R = \sqrt{\bar{g}}(\bar{\Delta}\phi(z,\bar{z}) + \bar{R}) \ . \tag{3.18}$$

Taking the operator product of $\sqrt{\bar{g}}\bar{\Delta}\phi$ with $e^{q\pi\bar{q}\bar{\pi}}$ in local conformal coordinates shows that the operator inserts curvature $2\pi(q+\bar{q})\delta^2(Y-X)$ at the point X. The total integrated curvature sphere with two punctures must vanish, since it represents a torus with zero Euler number. As is appropriate for a torus, the two operator insertions break the $SL(2,C)$ invariance of the sphere down to the translation subgroup.

4. States in the BRST Cohomology

The goal of this section is to make a relatively complete list of states in the cohomology of the BRST operators Q and $\bar{Q}$. We will find the states discussed previously in references [13][14], which correspond to the stable cohomology classes on the moduli space of Riemann surfaces found by Mumford[21]. We at present have no similar interpretation for the additional states that exist in the cohomology; a careful study of their correlators may lead to insight on this question.

States are local operators of the form

$$\mathcal{O}^{(0)}_{n,\bar{n}}(X) = O_n\bar{O}_{\bar{n}}c\bar{c}(X), \tag{4.1}$$

where the fields O_n and $\bar{O}_{\bar{n}}$ are primary conformal fields of dimension $(1,0)$ and $(0,1)$ respectively. Note that the unfilled vacuum state 1 is not a highest weight with respect to the bc ghost system because c has negative conformal dimension. The c contribution to (4.1) fills the mode of the negative dimension fermionic ghost raising operator. The bosonic ghost γ is another negative conformal dimension primary field in the theory. The states (4.1) must also be highest weights with respect to this field,

$$\lim_{z\to 0}\gamma(z)O_n(0) = 0 \ . \tag{4.2}$$

The simplest solution to these constraints is the puncture operator,

$$\mathcal{P}^{(0)} = \delta(\gamma)\delta(\bar{\gamma})c\bar{c}(X) \ , \tag{4.3}$$

where

$$\int_X [d\gamma]\delta(\gamma)(X) = 1 \ . \tag{4.4}$$

The BRST transformation property of $\delta(\gamma)$ is fixed by (4.4) to be

$$\delta_b\delta(\gamma) = c\partial\delta(\gamma) + \delta(\gamma)\partial c \ . \tag{4.5}$$

The puncture operator is BRST invariant and not exact (more precisely, it will be shown below that it is the BRST variation of an unphysical state), and hence it is in the BRST cohomology of physical states. It is the simplest possible Virasoro highest weight state of the theory, and it has a simple physical interpretation. An infinitesimal boundary, or puncture at a point X of the surface constrains diffeomorphisms $\delta_d x^\mu = \epsilon^\mu$ to satisfy $\epsilon^\mu(X) = 0$. This means that there is less symmetry to gauge fix, and the field configurations of ghosts c, γ which do not vanish at X must be eliminated from the path integral. These configurations clearly are projected out by inserting the puncture operator (4.3).

The basic building blocks of the physical states are the holomorphic 1-forms $O_n dz$. From these, the 2-form representation of a state can also be constructed,

$$\mathcal{O}^{(2)}_{n,\bar{n}} = \int \sqrt{g}\, O_n \bar{O}_{\bar{n}} \ . \tag{4.6}$$

There will be cases where there exist 2-form operators in the BRST cohomology which do not correspond to a 0-form physical state. These type of operators will be denoted as *screening operators.*

"Bosonization" of the $\beta\gamma$ system

It will be useful to represent the $\beta\gamma$ Virasoro highest weights in terms of bosonic vertex operators in order to easily evaluate operator products. Following reference [13], we define new fields via the relations

$$\beta = e^{-\varphi}\partial\xi \quad , \quad \gamma = e^{\varphi}\eta \ . \tag{4.7}$$

The new fields have stress tensors given by

$$\begin{aligned} T_\varphi &= -\tfrac{1}{2}\left(\partial\varphi\partial\varphi + 3\partial^2\varphi\right) \ , \\ T_{\eta\xi} &= -\eta\partial\xi \ , \end{aligned} \tag{4.8}$$

and the conformal dimensions of η, ξ and φ primary fields are

$$\Delta\left[\eta, \xi, e^{q\varphi}\right] = \left[1, 0, -\tfrac{1}{2}q(q+3)\right] \ . \tag{4.9}$$

All states in the η, ξ, φ systems can be re-expressed in terms of the original $\beta\gamma$ system by use of the identities

$$\begin{aligned} e^{-\varphi} &= \delta(\gamma) \ , \ \eta = \partial H(\gamma) \ , \\ e^{\varphi} &= \delta(\beta) \ , \ \xi = H(\beta) \ , \end{aligned} \tag{4.10}$$

where $H(\gamma)$ is the unit stepfunction. The φ system is a Gaussian model coupled to background charge $3(1-g)$, and the simplest non-vanishing correlator on the sphere is the three-puncture configuration

$$\langle \xi(w) P(z_1) P(z_2) P(z_3) \rangle = 1 \ , \tag{4.11}$$

where $\xi(w)$ soaks up the zero mode ξ_0, which is not part of the irreducible space of states. It will also be useful to have a bosonic representation of the $\eta\xi$ system via the relations $\eta = e^{i\theta}$ and $\xi = e^{-i\theta}$. The θ system stress tensor is

$$T_\theta = -\tfrac{1}{2}(\partial\theta\partial\theta + i\partial^2\theta) \ , \tag{4.12}$$

and primary fields $e^{in\theta}$ have conformal dimension $\frac{1}{2}n(n+1)$. Although the θ system has background charge $(1-g)$, it is not correct to to use this charge cancellation as a selection rule for higher genus correlators in this case. The bosonization of the $\beta\gamma$ system of the superstring (with different conformal dimensions) in reference [22] shows that the correct prescription is given by writing $\eta = \partial\sigma$ and considering the anomaly of the $\xi\sigma$ scalar ghost system.

Puncture Operators

We can obtain an infinite number of representations of the puncture operator, each with different ghost number, by picture changing[20] the basic puncture operator P. Recall that the zero mode of χ is not in the irreducible space of states, so the picture changed state $[Q, \chi P] = \pi P$ is not BRST exact in the irreducible space of states. By taking infinite linear combinations of the puncture operator in the different pictures, one obtains the coherent states specified by the 1-forms

$$P^q = e^{q\pi}\delta(\gamma) \ . \tag{4.13}$$

As discussed in the previous section, these operators correspond to insertions of finite curvature at a point.

We can picture change these operators with respect to the zero mode of ψ,

$$
\begin{aligned}
[Q, \psi_w(P^q c)_w] &= \lim_{z\to w}(c\partial\psi - \gamma\partial\phi)_z(e^{q\pi}\delta(\gamma)c)_w \ , \\
&= -\lim_{z\to w} q : (e^{\phi}\eta)_z(e^{q\pi}e^{-\phi}c)_w : \ , \\
&= -q(\eta e^{q\pi}c)_w \ .
\end{aligned} \tag{4.14}
$$

Repeating this process for both Q and $\bar{Q}$ yields the generally left-right asymmetric picture-changed puncture operators $\mathcal{P}^{(0)q,\bar{q}}_{n,\bar{n}} = P^q_n \bar{P}^{\bar{q}}_{\bar{n}} c\bar{c}$, where

$$
\begin{aligned}
P^q_n =& (-q)^{-n}(\gamma\partial\phi)^n\delta(\gamma)e^{q\pi} \\
=& P_n e^{q\pi} \ ,
\end{aligned} \tag{4.15}
$$

and the zero curvature 1-forms P_n, with ghost charge $(2n, 0)$, are given by

$$
\begin{aligned}
P_n =& (\partial\gamma)^n\delta(\gamma) \ , \\
=& \frac{1}{\Gamma(n)}\eta\partial\eta\cdots\partial^{n-1}\eta e^{(n-1)\varphi} \ .
\end{aligned} \tag{4.16}
$$

The 1-forms with no curvature ($q = 0$) are not related to the basic puncture operator via picture changing. The Symmetric ($n = \bar{n}$) zero curvature operators are in fact the physical states of Distler[13] and of Verlinde and Verlinde[14] in the -1 picture. Selection rules for correlators of these operators reduce all correlation functions of $\mathcal{P}^{(2)}_{n,n}$ for genus $g = 0$ to specification of the the simplest non-vanishing correlator given by equation (4.11); in this sense, the P_n are redundant operators [23][24].

Physical States of the $\beta\gamma$ Sector

We now make a more general search for physical states constructed from 1-forms in the $\beta\gamma$ system. Consider the Virasoro highest weights

$$
\begin{aligned}
V_{a,b} &= e^{ia\theta}e^{i(b-1)\varphi} \\
\Delta[V_{a,b}] &= \tfrac{1}{2}(a(a+1) - b(b+1)) + 1 \ .
\end{aligned} \tag{4.17}
$$

These are probably the most general forms of Virasoro Highest weights in the $\beta\gamma$ system, up to the application of screening operators. Although we have no completeness proof, this type of ansatz has proved successful in describing the Virasoro highest weight spectrum in many CFT problems. To use these as building blocks for the physical states $|V_{a,b}c|^2$, the vertex operators must be dimension one, and they must be highest weights with respect to γ. The second constraint assures that the state is annihilated by the $\tilde{Q}_s$ part of the BRST operator. These constraints in terms of a and b are that $a = b$ or $b = -(a+1)$, and for both cases $a - b \geq 0$. The solutions are $V_{n,n}$ for $n \in Z$ and $V_{m,-(m+2)}$ for $m = 0, 1, \cdots$. The vertex operators $V_{n,n}$ for n negative are actually not quite physical states, because they explicitly depend upon the zero mode ξ_0, which is not part of the irreducible space of states. This problem is overcome by multiplying these states by the operator $\oint \eta$, which commutes with β, γ, and the stress tensor. The vertex operators $P_n = V_{n,n}$ for $n = 0, 1, \cdots$ were already discussed, and are given by equation (4.16) in the η, ξ, φ basis. The new operators obtained are denoted as

$$R_n = [\oint_z \eta] P_{-(n+1)}(z) = \partial\xi \cdots \partial^n \xi e^{-(n+2)\varphi} \ , \qquad S_{n+1} = V_{n+1,-(n+2)} = \eta \cdots \partial^n \eta e^{-(n+3)\varphi} \ , \tag{4.18}$$

for $n = 0, 1, \cdots$. These operators are various Feigen-Fuchs conjugates of the P_n operators[25]. Note that although the 1-forms S_n are not β highest weights, they are acceptable building blocks for physical states. We can multiply the 1-forms of (4.18) by $e^{q\pi}$ to generate the 1-forms with curvature R_n^q and S_n^q, which are also suitable for creating physical states.

Are these new states independent, or are they picture changed versions of one another? First consider the R_n^q states; the picture changing generated by the zero mode ψ_0 relates R_n^q to R_{n-1}^q,

$$[Q, \psi R_n^q c] = -q R_{n-1}^q c \ , \tag{4.19}$$

so that the basic physical state with curvature $q \neq 0$ is $R_0^q = e^{-2\varphi} e^{q\pi}$ and the other R_n^q are picture changed versions. Note that the state $|R_0 c|^2$ can be interpreted as a double puncture operator in the sense that it forces the field γ and its derivative $\partial\gamma$ to vanish at a point.

The picture changing procedure generated by the ψ_0 mode can in fact be described in the $\beta\gamma$ system without reference to ψ by using an inverse picture changing operator $\oint X = \oint e^{-\varphi}\xi$. In particular, $\oint X P_n = P_{n-1}$, and $\oint X R_n = R_{n+1}$.

The basic puncture operator P is the picture changed version of an unphysical operator,

$$P_0 c = [Q, \xi (P_0 c)^2] \ , \tag{4.20}$$

where $(P_0 c)^2 = e^{-2\varphi} c \partial c$. In this sense, the puncture operator is locally exact but globally non-trivial. This relation is actually the first of a hierarchy of picture changing relations using the mode ξ_0,

$$(P_0 c)^n = [Q, \xi(P_0 c)^{n+1}] \ . \tag{4.21}$$

This has an interesting physical interpretation. The operator $|(P_0 c)^n|^2$ forces c, γ, and their first $n-1$ derivatives to vanish at a point. The relation (4.21) tells us that the order of vanishing for c and γ is redundant information as long as the order of vanishing is the same for both fields.

Another very interesting set of relations for the P^q_n and R^q_n that can be easily derived are

$$P^q_n c = -[Q, P^q_{n-1} c\partial c] \ , \ R^q_{n-1} c = -[Q, R^q_n c\partial c] \ , \tag{4.22}$$

for $n = 1, 2 \cdots$. These relations imply that states based on these 1-forms are BRST exact and hence decouple from the physical spectrum. However, this is not quite correct, because of the existence of contact terms in the operator algebra of these states[14]. The contact terms are due to an incompatibility of world sheet supersymmetry with a global holomorphic decomposition of the operator algebra[26]. The unusual dependence on the c ghost field of the 2-forms $\int |P^q_n \partial c|^2$ may be related to this; because of the ∂c dependence, these 2-forms are technically not part of the irreducible space of states for the sphere and the torus . Turning the argument around, equations (4.22) show that the only contribution to correlators of the P_n and R_n will be from the contact term algebra.

No such relations seem to exist between the S_n, although we have no proof that such a relation cannot exist. Rather, the S_n states obey another interesting recursion relation,

$$\begin{aligned} S_n &= [\oint_z Y_1] S_{n+1} c\partial c(z) \ , \\ Y_1 &= (\partial\chi\partial\phi + b)e^{\varphi}\partial\xi \ . \end{aligned} \tag{4.23}$$

The operator Y_1 can be interpreted as a screening operator; although there is no highest weight state that can be constructed from it, the 2-form $\int Y_1 \bar{Y}_1$ is in the BRST cohomology of the irreducible space of states. To see this, we first picture change the identity operator 1 (which is not a highest weight) with respect to the zero mode ξ_0,

$$[Q, \xi] = c\partial\xi - e^{\varphi}(\partial\chi\partial\phi + b) = \delta(\beta)[c\partial\beta - (\partial\chi\partial\phi + b)] \ . \tag{4.24}$$

The 1-form version $Y_0 = \partial\xi$ of this state is trivially obtained [8] by considering the action of a diffeomorphism, $\partial[Q, \xi] = [Q, \partial\xi]$. Then the 1-form Y_1 is obtained by subsequently picture changing Y_0,

$$[Q, \xi\partial\xi] = -2Y_1 + \partial(\cdots) \ . \tag{4.25}$$

The operator $\int |Y_0|^2$ plays the role of screening operator in reference [13].

Physical States of the $\beta\gamma$ and Weyl Sectors

Consider the construction of possible physical states which include dependence on the fields ϕ, π, χ, ψ, i.e. the Weyl sector of the theory. Recall that the Weyl sector is generated by constraining the curvature R in order to fix ϕ. This is the simplest gauge constraint which respects background general covariance and leaves the zero mode ϕ_0 unfixed. Since some information about the Weyl mode remains even after integration over the Lagrange multiplier field π, there is hope that new non-trivial states can be constructed by using the fields in this sector.

A similar set of fields and BRST symmetries arises if one introduces a Lorentz mode κ into the pure gravity theory by using zwei-beins as the basic gravitational degrees of freedom, and uses the spin connection as the gauge fixing constraint. In this case however, it seems that one can also gauge fix the Lorentz mode to $\kappa = 0$ directly and obtain a Lorentz sector with no propagating fields. Therefore, any non-trivial dynamics which arises from the Weyl sector will have to be intimately related to the existance of a non-vanishing cosmological constant, which prevents us from eliminating the zero mode ϕ_0.

The field ϕ is almost completely redundant, and its BRST transformation property $\delta_b\phi = \psi + \cdots$ makes it unlikely that a BRST invariant state independent of ψ_0 can be constructed. (One exception of course, is the area operator e^{ϕ_0}.) We will therefore focus on using the other three fields χ, π and ψ of the Weyl sector to construct BRST covariant holomorphic 1-forms.

With the χ, π system, we can construct screening operators with non-vanishing curvature. Consider the holomorphic 1-form

$$U^q = e^{q\pi}\partial\chi \ . \tag{4.26}$$

Under BRST variations it transforms as a total derivative, and we can use it to construct a 2-form physical state $\mathcal{U}^{(2)q\bar{q}} = \int U^q \bar{U}^{\bar{q}}$. For q and $\bar{q}$ non-vanishing, this is a well defined on-shell operator . BRST invariance of these operators relies on the supersymmetry variation of $[Q_s, \chi] = \pi$. It does not seem possible to construct new BRST invariant 1-forms by multiplying the U^q by dimension zero primaries of other fields in the theory. Nor does it seem that these operators correspond to a BRST invariant physical state, since the 0-form version of these fields is not a c and γ highest weight. Nevertheless, they may serve a useful role in the calculation of correlation functions by cancelling fermion charge.

We are left to discuss the field ψ, which transforms under BRST as a dimension 0 primary field plus additional terms proportional to γ or $\partial\gamma$. Define the operator

$$O_\psi^n = \partial\psi \cdots \partial^n\psi \ . \tag{4.27}$$

It is a primary field with conformal dimension $\frac{1}{2}n(n+1)$ and is independent of the zero mode ψ_0. Now define the field $W(n,a,b) = O_\psi^n V_{a,b}$, where $V_{a,b}$ is given by equation (4.17). To be the 1-form component of a physical state, it must be conformal dimension one, and have a non-singular operator product with the $\gamma\partial\chi\partial\phi$ term in the supersymmetry charge $\tilde{Q}_s$,

$$\begin{aligned} n(n+1) + a(a+1) &= b(b+1) \ , \\ a - b &\geq n + 1 \ . \end{aligned} \tag{4.28}$$

To satisfy the second constraint, introduce an integer $p = 1, 2, \cdots$ such that $a = b + n + p$. The conformal dimension constraint can be solved in terms of p,

$$b = -(n+1) - \frac{p(p-1)}{2(n+p)} \ . \tag{4.29}$$

If b is non-integral, then it is not clear how to impose the independence of the zero mode ξ_0 in the path integral. So define the positive integer m,

$$m = \frac{p(p-1)}{2(n+p)} \ , \tag{4.30}$$

so that $b = -(n+1) - m$ and $a = p - m - 1$. For a given m, there are an infinite number of solutions to the constraints (4.28). Define $W_{m,p}$ to be the holomorphic 1-form component of the resulting physical states. The simplest cases are $m = 1$, for which $n = \frac{1}{2}p(p-3)$ with $p = 4, 5, \cdots$,

$$W_{1,p} = O_\psi^{\frac{1}{2}p(p-3)} e^{i(p-2)\chi} e^{-(\frac{1}{2}p(p-3)+3)\varphi} \ . \tag{4.31}$$

The $m = 2, 3, 4$ series start at $(n,p) = (6,8), (3,9), (14,16)$ respectively. Each holomorphic 1-form $W_{m,p}$ can be multiplied by $e^{q\pi}$ to create 1-forms $W_{m,p}^q$ which carry non-vanishing curvature.

States constructed out of these 1-forms are BRST invariant, but are they BRST exact (and therefore spurious)? We now consider whether the Virasoro highest weight field $W(n,a,b)e^{q\pi}$ is the supersymmetry variation of another Virasoro highest weight field. Examination of terms in $\tilde{Q}_s$ and charge conservation of the bosonic vertex operators implies that the field can be the variation of either $W(n+1,a-1,b-1)e^{q\pi}$, or $W(n,a-1,b-1)\partial c$ for $q=0$. In the first case, the OPE of $\gamma\partial\chi\partial\phi$ with $W(n+1,a-1,b-1)$ is non-singular, for $p \geq 3$. In the second case, the relavant OPEs are also non-singular. The conclusion is that the states constructed from the 1-forms $W^q_{m,p}$ may be in the cohomology; they are not BRST exact in an obvious fashion.

5. Discussion

The pure 2-D gravity theory has an additional symmetry which alters the quantization procedure relative to the system of gravity plus matter. The impact of this additional symmetry in the quantization procedure is quite significant, since it seems to imply that the gauge fixed path integral has no conformal anomaly. The resulting conformal field theory for pure gravity has essentially the same field content as 2-D gravity coupled to a $c=-2$ matter system[13]. However, there are a number of important differences between the two models. The pure gravity theory has a different (larger) BRST symmetry (due to a larger set of constraints), which has a smaller cohomology of physical states than the gravity plus $c=-2$ system. Also, the Weyl sector of the theory, which leads to states in the cohomology, is not part of the gravity plus $c=-2$ system. Recall that we were led to this system in the gauge fixing procedure by fixing the curvature R. This constraint leaves the zero mode ϕ_0 of the Weyl mode unfixed, as required.

The moduli space of the pure gravity theory is much smaller than ordinary string theory moduli space. Only one point in string moduli space is required to represent the gauge orbit of the pure gravity theory, although one still has to integrate over the area of each surface. It is not clear whether it is necessary to include the compactification boundary points in the formal definition of moduli space. If we choose to do so, then each topology of boundary requires a representative in the gauge fixed path integral. At the very least, including the compactification boundary will change the relative weights that each genus contributes to the path integral.

We have not attempted to calculate correlation functions of the states in the BRST cohomology that have been found. The correlation functions of some of the states represented by P_n^q in this paper were calculated in references [13][14] in different ways. As discussed in section 4, contact terms can be the only contribution to correlators of the operators. The P_n^q states for $q \neq 0$ are picture changed versions of the simplest primary puncture operator P, and do not describe additional information about the system. Note that the $q = -1$ cases correspond to the curvature insertions required to describe the path integral on the boundary of the compactification of moduli space.

The P_n^0 states are the most likely candidates for the Φ^{2n} type operators in the matrix model approach[13][14]. With this identification, the states generated by R_n and S_n should have a similar type of interpretation. There is no ghost number cancellation argument which prevents these states from contributing to non-vanishing correlation functions. A precise determination of these correlation functions is clearly required for a better understanding of their role in the theory. Note that it is possible that the extra states could represent a redundant parametrization of the P_n physical states. Another central issue to resolve is whether the 1-forms $W_{m,p}$ generate new primaries, or states redundant with the P_n.

There is in principle no reason not to use left-right asymmetric combinations of holomorphic and antiholomorphic operators as physical states. Each sector could then satisfy a different set of selection rules for non-vanishing correlators, because of differing cancellation of bosonic charges. Correlators in the matrix model approach are derived by varying couplings in the partition function which correspond to different potentials. The specific heat of the partition function satisfies a differential equation (the string equation) for each critical potential. The asymmetric matrix models[27] are specified by two independent string equations. The left-right asymmetric vertex operators might therefore generate deformations of the pure gravity theory that correspond to the asymmetric models.

Acknowledgements

I would like to thank W. Bardeen, Z. Bern, H. Dykstra, O. Hernandez, D.Kosower, J. Lykken, E. Martinec, and T. Morris for useful discussions and encouragement. I am particularly grateful for discussions with S. Chaudhuri which led to the development of the work in section 2.

References

[1] M. Douglas and S. Shenker, Nucl. Phys. B335 (1990), 635,
D. Gross and A.A. Migdal, Phys. Rev. Lett. 64 (1990) 717,
E. Brezin and V. Kazakov, Phys. Lett. 236B (1990) 144.

[2] M. Douglas, Rutgers Preprint RU-89-51 (1989),
E. Brezin, M. Douglas, V. Kazakov, S. Shenker, Rutgers Preprint RU-89-47 (1989),
D. Gross and A. Migdal, Princeton Preprint PUPT-1156 (1989) .

[3] D. Gross and A. Migdal, Nucl. Phys. B340 (1990) 333.

[4] T. Banks, M. Douglas, N. Seiberg, S. Shenker, Phys. Lett. 238B (1990) 279.

[5] A.M. Polyakov, Phys. Lett. 103B (1981) 287.

[6] A.M. Polyakov, Mod. Phys. Lett. A2 (1987) 893,
V. Knizhnik, A. M. Polyakov, A.B. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819.

[7] J. Distler, H. Kawai, Nucl Phys. B321 (1989) 509,
F. David, Mod. Phys. Lett. A3 (1988) 1651.

[8] E. Witten , Commun. Math. Phys. 188 (1988) 411.

[9] E. Witten, Nucl. Phys. B340 (1990) 281.

[10] R. Dijkgraaf and E. Witten, Princeton Preprint PUPT-1166 (1990).

[11] J. Labastida, M. Pernici, E. Witten, Nucl. Phys. B310 (1988) 611.

[12] D. Montano and J. Sonnenschein, Nucl. Phys. B313 (1989) 258.

[13] J. Distler, Nucl. Phys. B342 (1990) 523.

[14] E. Verlinde and H. Verlinde, Preprint IASSNS-HEP-90/40.

[15] I. Batalin and G. Vilkovisky, Phys. Rev. D28, (1983) 2567.

[16] J. Polchinski, Texas Preprint UTTG-39-90 (1990).

[17] E. D'Hoker and D. Phong, Rev. Mod. Phys. 60 (1988) 917.

[18] N. Mavromatos and J. Miramontes, Mod. Phys. Lett. A4 (1989) 1847,
E. D'Hoker and P. Kurzepa, UCLA Preprint UCLA/90/TEP/15 .

[19] E. D'Hoker and D. Phong, Phys. Rev. Lett. 56 (1986) 912 .

[20] D. Friedan, E. Martinec, S. Shenker, Nucl. Phys. B271 (1986) 93.

[21] D. Mumford, "Towards an Enumerative Geometry of the Moduli Space of Curves", in *Arithmetic and Geometry* (Birkhauser).

[22] E. Verlinde and H. Verlinde, Phys. Lett. 192B (1987) 95.

[23] R. Dijkgraaf, H. Verlinde, and E. Verlinde, Princeton preprint PUPT-1184 1990.

[24] M. Fukuma, H. Kawai and R. Nakayama, Tokyo preprint UT-562 1990 .

[25] Vl. Dotsenko and V. Fateev Nucl. Phys. B 240 (1984) 312; Nucl. Phys. B999 (1984) 999 .

[26] M. Green and N. Seiberg, Nucl. Phys. B299 (1988)559.

[27] S. Chaudhuri, J. Lykken, T. Morris, FERMILAB-PUB-90/168 (1990), C. Bachas and P. Petropoulos, CERN preprint CERN-TH.5714/90 (1990), P. Petropoulos, Ecole Polytechnique Preprint CPTH-A993.0890 (1990).