Fermi National Accelerator Laboratory

FERMILAB–PUB–89/241–T

November, 1989

Ward Identities and Characters of Kac–Moody Algebras

Chao–Shang Huang*

Fermi National Accelerator Laboratory

Theoretical Physics Department

P.O. Box 500, Batavia, IL 60510

and

De–Hai Zhang

Center of Theoretical Physics

CCAST (World Laboratory), Beijing, China

and

Institute of Theoretical Physics

Peking University, Beijing, China

Abstract

We derive partial differential equations for characters of Kac–Moody algebras by using Ward identities for energy–momentum tensors and currents on a torus and null vectors of current algebras. The solutions of the partial differential equations for affine algebras $A_\ell^{(1)}, D_\ell^{(1)}, E_\ell^{(1)} (\ell = 6,7,8)$ and $B_\ell^{(1)}$ are given.

*On leave of absence from Institute of Theoretical Physics, Academia Sinica, Beijing, China

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

I. Introduction

Two dimensional conformal field theories have been extensively studied[1], both for their purely mathematical interest and for their applications to critical phenomena and string theories. An important variety of conformal field theories is Wess–Zumino–Witten (WZQ) models.[2],[3] These models on higher genus Riemann surfaces provide an elegant example examining the interacting conformal field theories. As conformal field theories, WZW models possess the conformal symmetry that leads to existence of Ward identities for energy–momentum tensors and implies that the modes of the energy–momentum tensor form a Virasoro algebra. Furthermore, in WZW models there exists, in addition to the conformal symmetry, group or Kac–Mood symmetry that leads to existence of Ward identities for currents and implies that the modes of currents form a Kac–Moody algebra. One obtains mix Ward identities and combined Virasoro and Kac–Moody algebra by Sugawara construction. The Hilbert space of a theory is decomposed into a finite sum of irreducible representations of the algebra for a given level and the modular invariant one–loop partition function can be obtained from the characters of the Kac–Moody algebra.[3]

Ward identities on higher genus Riemann surfaces for both energy–momentum tensors and currents have been given.[4],[5] But in the current Ward identities existence of the terms which contain zero modes of currents in the correlation functions makes the Ward identities incomplete or unpowerful actually. In the case of the torus, Bernard[6] introduced expection values with an insertion of an element of the group G and derived the complete current Ward identities.

A new proof of the Weyl–Kac character formula has been given by Bernard in the reference [6]. He derived a heat equation by using the Ward identities and proved that the solution of the heat equation is just of the character given in Weyl–Kac character formula by means of the algebraic methods (using the affine Weyl group). Recently, Eguchi and Ooguri gave a different proof for group SU_2 and their proof is "complete physical" in the sense that only physical Ward identities and null vector fields are used.[7]

In this paper, we generalize their method to the general simple Lie groups. Our purpose is two–fold. One is to prove the Weyl–Kac character formula for general cases in the "complete physical" way. The other is to provide a new way to calculate string functions. The calculation of string functions is important in order to obtain the explicit expressions of the one–loop partition functions for the WZW theories. In many cases string functions can easily be found by using this method (analytical method). We derive partial differential equations satisfied by characters of affine Lie algebras in Section II. In Section III, we solve the partial differential equations for level one. In Section IV, we show how to calculate string functions by using the analytical method and give some concrete examples. Finally, some conclusions and discussions are given in the last section.

II. Partial Differential Equations for Characters of Kac–Moody Algebras

In the WZW model for simple Lie group G, the modes of currents form an untwisted affine algebra[3]

$$\begin{aligned}
\left[J^i_m, J^j_n\right] &= C^{ij}_a\, J^a_{m+n} + Km\, \delta_{m,-n}\delta^{ij} \\
\left[J^i_m, J^s_n\right] &= C^{is}_a\, J^a_{m+n} \\
\left[J^r_m, J^s_n\right] &= C^{rs}_a\, J^a_{m+n} + Km\, \delta_{m,-n}\, \delta^{r,-s}
\end{aligned} \tag{2.1}$$

where we have assumed the Cartan–Weyl bases for the finite algebra G of the group G (we denote the group and its algebra by a same letter G whenever no confusion arises) so that

$$\begin{aligned} C_a^{ij} &= 0, \quad C_a^{is} = \delta_{as}\alpha_i^{(s)}, \quad C_i^{rs} = \delta_{r,-s}\ \alpha_i^{(r)} \\ C_t^{rs} &= \begin{cases} \epsilon(r,s) & \text{if } \alpha^{(r)}+\alpha^{(s)}=\alpha^{(t)} \text{ is a root} \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{2.2}$$

The indices $i, j, etc. = 1, \cdots, \ell$ and the indices $r, s, etc. = \pm 1, \cdots, \pm N_G (N_G = (d_G - \ell)/2$ with $\ell = rank\ G$ and $d_G = dim.\ G)$ label the cartan subalgebra H of G and the coset G/H respectively, and indices a, b, etc. label the generators of G. The roots of G, correspondingly, are labeled as follows:

$$\begin{aligned} &\text{positive roots} \qquad \alpha^{(1)}, \cdots, \alpha^{(N_G)} \\ &\text{negative roots} \qquad \alpha^{(-1)}, \cdots, \alpha^{(-N_G)} \end{aligned} \tag{2.3}$$

so that $\alpha^{(-s)} = -\alpha^{(s)}$.

The advantage of using the Cartan–Weyl bases is that it leads to the one-to-one correspondence between the modes and the root system of the affine algebra $\hat{G}$[8] and, as one will see later on, it makes the derivation of the partical differential equations for characters easier.

The ground states (tachyon states) of the theory are the highest weight vectors of the integrable highest weight representations $\angle(\Lambda)$ (with the highest weights Λ) of the affine algebra $\hat{G}$,

$$\begin{aligned} J_n^a|\Lambda> &= 0, \quad n > 0 \\ J_0^i|\Lambda> &= \Lambda_i \quad |\Lambda>, \quad \Lambda_i\ \epsilon Z_+ \\ J_0^s|\Lambda> &= 0, \quad s > 0 \end{aligned} \tag{2.4}$$

and the Hilbert space of the theory decomposes into a finite sum of the integrable highest weight representations of the given level. The energy–momentum tensor is given by the Sugawara construction:

$$T(z) = \frac{1}{2(C_A + K)} : J_a(z) J^a(z) : \tag{2.5}$$

where C_A is the second Casimir of the adjoint representation.

In order to derive the partial differential equations for the character of $\angle(\Lambda)$ we need to have the current Ward identities[5,6]

$$\begin{aligned} &< J^a(z) J^{b_1}(w_1) \cdots J^{b_n}(w_n) > - < J^a(z) >< J^{b_1}(w_1) \cdots J^{b_n}(w_n) > \\ = &\sum_{i=1}^{n} K\, \partial_{w_i} G^{ab_i}(z, w_i) < J^{b_1}(w_1) \cdots J^{b_{i-1}}(w_{i-1}) J^{b_{i+1}}(w_{i+1}) \cdots J^{b_n}(w_n) > \\ + &\sum_{i=1}^{n} G^a_c\,(z, w_i) C^{cb}_d < J^{b_1}(w_1) \cdots J^{b_{i-1}}(w_{i-1}) J^d(w_i) J^{b_{i+1}}(w_{i+1}) \cdots J^{b_n}(w_n) > \\ + &\ \mathcal{L}_a < J^{b_1}(w_1) \cdots J^{b_n}(w_n) > \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} G(z, w) &= \frac{2\pi i}{u-1} + 2\pi i \sum_{n=1}^{\infty} \left(\frac{u^{-n}}{e^{-\gamma} - q^n} - \frac{u^n}{e^{\gamma} - q^n} \right) q^n , \\ u &= e^{2\pi i (z-w)} , \quad q = e^{2\pi i \tau} , \end{aligned} \tag{2.7}$$

$$\gamma = 2\pi i \xi^i t^i , \quad t^i \epsilon \text{ adjoint rep. of } G , \tag{2.8}$$

and $\mathcal{L}_a$ denotes the Lie derivative on the group manifold G along the left–invariant Killing vector $e^a_{\angle}$. In eq. (2.6), the correlation functions are defined by

$$< J^{a_1}(z_1) \cdots J^{a_n}(z_n) >= Z^{-1}(\tau, \xi)\, T_r \left(q^{\angle_0 - C_G/24} e^{\gamma} J^{a_1}(z_1) \cdots J^{a_n}(z_n) \right) , \tag{2.9}$$

$$Z(\tau, \xi) \equiv Z\left(\tau, \xi^1, \cdots, \xi^{\ell}\right) = T_r \left(q^{\angle_0 - C_G/24} e^{\gamma} \right) \tag{2.10}$$

with $C_G = \frac{Kd_G}{C_A+K}$.

We now derive the heat equation for characters $Z(\tau,\xi)$ (precisely speaking, $Z = q^{s_\Lambda} Ch_{\angle(\Lambda)}(\tau,\xi,0)$, where $S_\Lambda = \frac{C_\Lambda}{C_A+K} - \frac{C_G}{24}$ is the dimension of the field ϕ_Λ less the trace anomaly and $Ch_{\angle(\Lambda)}(\tau,\xi,t)$ is the character of the representation $\angle(\Lambda)$) by using eqs. (2.5 - 2.6) and the energy–momentum tensor Ward identity

$$2\pi i \ \frac{\partial Z(\tau,\xi)}{\partial \tau} =< T(\xi) > \quad . \tag{2.11}$$

Setting $n=1$ in eq. (2.6) and using eq. (2.2), we find

$$\begin{aligned} < J^a(z)J^i(w) > &= K\,\delta^{ai}\left(\tfrac{1}{(z-w)^2} - 2\eta_1\right) \\ &+ \ < J_o^a J_0^i > +0(z-w)\ , \end{aligned} \tag{2.12a}$$

$$\begin{aligned} < J^a(z)J^s(w) > &= \delta^{a,-s}\Big[K\left(\tfrac{1}{(z-w)^2} + \eta_1 - \tfrac{1}{2}\tfrac{1}{\theta_1}\tfrac{\partial^2\theta_1}{\partial x_s^2}\right) \\ &+ \ \left(-\tfrac{1}{z-w} + \tfrac{1}{\theta_1}\tfrac{\partial\theta_1}{\partial x_s}\right)\alpha_i^{(s)} \ < J_0^i > \Big] \\ &+ \qquad 0(z-w)\ , \end{aligned} \tag{2.12b}$$

where $\eta_1 = 2\pi i \frac{d\ell n\eta}{d\tau}(\eta(\tau) = q^{1/24}\prod_{n=1}(1-q^n)$ is the Dedekind function), $x_s = \alpha_i^{(s)}\xi^i, \theta_1 = \theta_1(\tau, x_s)$ is the standard elliptic theta function and we have used[6]

$$< J_0^a\phi_0\cdots\phi_n > - < J_0^a >< \phi_1\cdots\phi_n >= \mathcal{L}a < \phi_1\cdots\phi_n > \tag{2.13}$$

with ϕ_i being the field or the current. From eqs. (2.5), (2.11), (2.12), and $< J_0^i >= \frac{\partial \ell n Z}{\partial \xi^i}$, one obtains the following heat equation

$$\begin{aligned} 2\,(C_A+K)\,2\pi i\,\frac{\partial \ell n Z}{\partial\tau} &= (d_G - 3\ell)\,K\eta_1 - K\sum_{s=1}^{N_G}\frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial x_s^2} \\ &+ \ 2\sum_{s=1}^{N_G}\frac{\partial \ell n\theta_1}{\partial x_s}\,\alpha_i^{(s)}\,\frac{\partial \ell n Z}{\partial \xi^i} + \frac{1}{Z}\sum_{i=1}^{\ell}\frac{\partial^2 Z}{\partial \xi^{i2}}\,. \end{aligned} \tag{2.14}$$

The heat equation has been derived in ref. [6] where it possesses the different appearance. Dependence on the roots of an algebra in eq. (2.14) expresses dependence on the algebra explicitly, which is consequence of using the Cartan–Weyl bases.

It is clear that eq. (2.14) alone cannot completely determine the character Z. We need to have more equations in order to completely determine Z. We shall show that such equations follow from null vector states of the theory, which are due to the current symmetry, and the current Ward identities, eq. (2.16).

From the Kac–Moody algebra (2.1), one knows that for each given value of s and m, one has the following SU_2 algebra

$$\left[J_m^{-s},\, J_{-m}^{s}\right] = P_m^s, \tag{2.15a}$$

$$\left[P_m^s,\, J_{\mp m}^{\pm s}\right] = \mp J_{\mp m}^{\pm s} \tag{2.15b}$$

with

$$P_m^s = \frac{2}{|\alpha^{(s)}|^2}\left(Km - \alpha_i^{(s)} J_0^i\right) \tag{2.16}$$

For the highest weight state $|\Lambda>$, eq. (2.4) leads to that

$$P_m^s|\Lambda> = \frac{2}{|\alpha^{(s)}|^2}\,\left(Km - \left(\alpha^{(s)}, \Lambda\right)\right)|\Lambda> \equiv M_m^s(K,\Lambda)\,|\Lambda> \tag{2.17}$$

It is straightforward from the integrable property of the representation and eqs. (2.1) and (2.4) to prove that

$$|\chi_m^s(K,\Lambda)> = \left(J_{-m}^s\right)^{M_m^s(K,\Lambda)+1}|\Lambda> = 0 \tag{2.18}$$

is a null vector state. (The case of $m=1$ and $\alpha^{(s)}=$ the highest root of G is discussed in ref. [3].) For our purpose we take $m=1$ and $|\Lambda> = |0>$, singlet, so that

$$M_K^s \equiv M_1^s(K,0) = \frac{2K}{|\alpha^{(s)}|^2} \tag{2.19}$$

and we have null vector states

$$|\chi_1^s(K,0)> = \left(J_{-1}^s\right)^{M_K^s+1}|0> = 0,$$

or null vector fields

$$\chi_K^s = \left(J_{-1}^s\right)^{M_K^s+1} I = 0 \tag{2.20}$$

Now we would like to derive partial differential equations for characters of $\hat{G}$ by using null vector fields and current Ward identities. Let us consider the case of level one ($K = 1$) first. Then from eqs. (2.19) and (2.20) one has

1. Simply–laced algebras (i.e., A_ℓ, D_ℓ and E_ℓ)

$$\chi_1^s = \left(J_{-1}^s\right)^2 = 0 \tag{2.21}$$

2. B_ℓ, C_ℓ, and F_4

$$\chi_1^s = \left(J_{-1}^s\right)^2 = 0, \quad \text{for } \alpha^{(s)}\epsilon\Delta_\ell \tag{2.22}$$

$$\chi_1^s = \left(J_{-1}^s\right)^3 = 0, \quad \text{for } \alpha^{(s)}\epsilon\Delta_s \tag{2.23}$$

3. G_2

$$\chi_1^s = \left(J_{-1}^s\right)^2 = 0, \quad \text{for } \alpha^{(s)}\epsilon\Delta_\ell \tag{2.24}$$

$$\chi_1^s = \left(J_{-1}^s\right)^4 = 0, \quad \text{for } \alpha^{(s)}\epsilon\Delta_s \tag{2.25}$$

where $\Delta_\ell(\Delta_s)$ is the set of the long (short) roots of the algebras and the normalization of a root is the same as that in Kac's book,[9] i.e.

$$|\alpha|^2 = 2\kappa\ , \quad \text{for } \alpha\epsilon\Delta_\ell$$

$$|\alpha|^2 = 2\kappa/s\ , \quad s = \max_{a_{ij}\neq 0} \ a_{ji}/a_{ij}, \quad \text{for } \alpha\epsilon\Delta_s$$

where κ is the order of automorphism of the Dynkin diagram of G and a_{ij} is the element of the generalized Cartan matrix of the affine Lie algebra $\hat{G}$. (We discriminate affine and finite algebras with hat and without hat in this paper. For example, $\hat{\alpha}$ and

α denote a root of affine and finite algebras respectively. However, in the case that no confusion arises we shall omit hat for the sake of simplicity.)

We use the global G symmetry in order to obtain partial differential equations from null vectors

$$g\chi_1^s g^{-1} = 0 \ , \ \ g\epsilon G \ . \tag{2.26}$$

A little calculation shows

$$2\alpha_i^{(s)}\alpha_j^{(s)} < J_{-1}^i J_{-1}^j > -|\alpha^{(s)}|^2 < J_{-1}^s J_{-1}^{-s} + J_{-1}^{-s} J_{-1}^s >= 0 \text{ for } \chi_1^s = (J_{-1}^s)^2 \tag{2.27}$$

$$\begin{aligned} &|\alpha^{(s)}|^2\alpha_i^{(s)} < J_{-1}^i \left(J_{-1}^s J_{-1}^{-s} + J_{-1}^{-s} J_{-1}^s\right) > + \left(J_{-1}^s J_{-1}^{-s} + J_{-1}^{-s} J_{-1}^s\right) J_{-1}^i + J_{-1}^s J_{-1}^i J_{-1}^{-s} \\ &+ J_{-1}^{-s} J_{-1}^i J_{-1}^s > -2\alpha_i^{(s)}\alpha_j^{(s)}\alpha_k^{(s)} < J_{-1}^i J_{-1}^j J_{-1}^k >= 0 \text{ for } \chi_1^s = (J_{-1}^s)^3 \ . \end{aligned} \tag{2.28}$$

Recall that

$$< J_{-1}^a J_{-1}^b J_{-1}^c(z) >= \oint_{w,v,z} dy(y-z)^{-1} \oint_{v,z} dw(w-z)^{-1} \oint_z dv(v-z)^{-1} < J^a(y) J^b(w) J^c(v) > \tag{2.29}$$

By using the current Ward identities (2.6) and eqs. (2.27), (2.28) and (2.29), we obtain the following partial differential equations for characters:

1. Simply–laced algebras

$$D_s^{(2)}\, Z = 0 \ , \qquad s = 1, \cdots, N_G \tag{2.30}$$

$$D_s^{(2)} \equiv \frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial x_s^2} - 6\eta_1 - 2\,\frac{\partial \ell n\theta_1}{\partial x_s}\,\alpha_i^{(s)}\,\frac{\partial}{\partial \xi^i} + \alpha_i^{(s)}\alpha_j^{(s)}\,\frac{\partial^2}{\partial\xi^i\partial\xi^j} \tag{2.31}$$

2. B_ℓ, C_ℓ, F_4

$$D_s^{(2)} Z = 0 \ , \quad \text{for } \ \alpha^{(s)}\epsilon\Delta_\ell \tag{2.32a}$$

$$D_s^{(3)} Z = 0 \ , \quad \text{for } \ \alpha^{(s)}\epsilon\Delta_s \tag{2.32b}$$

$$D_s^{(3)} \equiv \frac{1}{2}\frac{1}{\theta_1}\frac{\partial^3\theta_1}{\partial x_s^3} - \frac{3}{2}\frac{\partial \ell n\theta_1}{\partial x_s}\frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial x_s^2} + 6\eta_1\frac{\partial \ell n\theta_1}{\partial x_s}$$
$$+ \; 3\left[\left(\frac{\partial \ell n\theta_1}{\partial x_s}\right)^2 - 2\eta_1\right]\alpha_i^{(s)}\frac{\partial}{\partial\xi^i} - 3\frac{\partial \ell n\theta_1}{\partial x_s}\alpha_i^{(s)}\alpha_j^{(s)}\frac{\partial^2}{\partial\xi^i\partial\xi^j}$$
$$+ \; \alpha_i^{(s)}\alpha_j^{(s)}\alpha_k^{(s)}\frac{\partial^3}{\partial\xi^i\partial\xi^i\partial\xi^k} \tag{2.33}$$

It is easy to see from the above results that the number of the modes contained in a null vector is equal to the order of the partial differential equation derived from the null vector and this fact is essentially due to the definition of the action of the zero modes J_0^a. For $G = G_2$ forth order partial differential equations will similarly be derived and we shall not further discuss G_2 hereafter for the sake of simplicity.

The same procedures can be applied to cases of higher level and the partial differential equations of the higher order will be obtained. For instance, at level $K = 2$, we have the partial differential equations of the third order for simply–laced algebras and those of the third and fifth order for B_ℓ, C_ℓ and F_4. The reason of increase of the order with increasing K is that the number of the modes (J_{-1}^s) contained in null vectors increases with increasing K (see eqs. (2.19, (2.20).

III. Solutions of Partial Differential Equations for $K = 1$

The partial differential equations (2.30) for simply–laced algebras or (2.32) for B_ℓ, C_ℓ and F_4 are not independent because the number of the equations is $N_G = (d_G - \ell)/2$, the number of the positive roots of the algebra G (it is easy to see that the substitution $\alpha^{(s)} \to \alpha^{(-s)}$ leads to the same equation), and it exceeds the number of variables, ℓ, except for the case of A_1. The number of independent partial differential equations depends on the structure of $\hat{G}$. Given $\hat{G}$, one can find out the independent ones from the partial differential equations.

We now combine the heat equation and the set of the partial differential equations originated from null vectors and find the solutions of these equations for simply–laced algebras and B_ℓ.

1. Simply–laced algebras

For $K = 1$, we have N_G partial differential equations (2.30). Summing these N_G equations, one obtains

$$\sum_{s=1}^{N_G} \left[\frac{1}{\theta_1} \frac{\partial^2 \theta_1}{\partial x_s^2} - 2 \frac{\partial \ell n \theta_1}{\partial x_s} \alpha_i^{(s)} \frac{1}{Z} \frac{\partial Z}{\partial \xi^i} \right]$$
$$-3 (d_G - \ell) \eta_1 + C_A \frac{1}{Z} \sum_{i=1}^{\ell} \frac{\partial^2 Z}{\partial \xi^{i^2}} = 0 \tag{3.1}$$

where we have used

$$\sum_{s=1}^{N_G} \alpha_i^{(s)} \alpha_j^{(s)} = C_A \delta_{ij} \tag{3.2}$$

Taking $K = 1$ in eq. (2.14), one has

$$\begin{aligned} (C_A + 1) 4\pi i \frac{\partial \ell n Z}{\partial \tau} &= (d_G - 3\ell) \eta_1 - \sum_{s=1}^{N_G} \left[\frac{1}{\theta_1} \frac{\partial^2 \theta_1}{\partial x_s^2} - 2 \frac{\partial \ell n \theta_1}{\partial x_s} \alpha_i^{(s)} \frac{1}{Z} \frac{\partial Z}{\partial \xi^i} \right] \\ &+ \frac{1}{Z} \sum_{i=1}^{\ell} \frac{\partial^2 Z}{\partial \xi^{i^2}} \end{aligned} \tag{3.3}$$

Substituting eq. (3.1) into eq. (3.3), we obtain

$$(C_A + 1) 4\pi i \frac{\partial \ell n Z}{\partial \tau} = -2 d_G \eta_1 + (C_A + 1) \frac{1}{Z} \sum_{i=1}^{\ell} \frac{\partial^2 Z}{\partial \xi^{i^2}} \tag{3.4}$$

Note that for simply–laced algebras,

$$d_G = \ell (C_A + 1) \tag{3.5}$$

From eqs. (3.4) and (3.5), we have finally

$$4\pi i \, \frac{\partial(\eta^{\ell} Z)}{\partial \tau} = \sum_{i=1}^{\ell} \frac{\partial^2 \left(\eta^{\ell} Z\right)}{\partial \xi^{i^2}} \tag{3.6}$$

Thus we obtain the following solutions:

$$Z = \frac{\theta_{\Lambda}(\tau, \xi, 0)}{\eta^{\ell}(\tau)} \tag{3.7}$$

where

$$\theta_{\Lambda}\left(\tau, \xi, t\right) = e^{-2\pi i t} \sum_{\alpha \epsilon M+\Lambda} e^{\pi i \tau |\alpha|^2 - 2\pi i (\alpha_1 \xi^1 + \cdots + \alpha_{\ell} \xi^{\ell})} \tag{3.8}$$

is the classical theta function of degree 1 (with characteristic Λ).[9] These are exactly the characters for $G = A_{\ell}^{(1)}, D_{\ell}^{(1)}$ and $E_{\ell}^{(1)}$ which have been given by Kac.[9]

2. B_{ℓ}

Because $\alpha_i^{(s)} = \delta_{si}$ for $\alpha^{(s)} \epsilon \Delta_s$, the eqs. (2.32b) decouple, i.e., one has the ℓ independent equations each of which contains only the partial differentials for one variable:

$$\begin{aligned} &\left(\frac{1}{2} \frac{1}{\theta_1} \frac{\partial^2 \theta_1}{\partial \xi^{i^2}} - \frac{3}{2} \frac{\partial \ell n \theta_1}{\partial \xi^i} \frac{1}{\theta_1} \frac{\partial^2 \theta_1}{\partial \xi^{i^2}} \; + \; 6\eta_1 \frac{\partial \ell n \theta_1}{\partial \xi^i}\right) Z \\ +3 &\left[\left(\frac{\partial \ell n \theta_1}{\partial \xi^i}\right)^2 - 2\eta_1\right] \frac{\partial Z}{\partial \xi^i} - 3 \frac{\partial \ell n \theta_1}{\partial \xi^i} \frac{\partial^2 Z}{\partial \xi^{i^2}} \; + \; \frac{\partial^3 Z}{\partial \xi^{i^3}} = 0 \\ &i \; = \; 1, \cdots, \ell \, . \end{aligned} \tag{3.9}$$

(Note that now $x_s = \alpha_i^{(s)} \xi^i = \xi^s, s = 1, \cdots \ell$.) Therefore the solutions assume the factorized form

$$Z\left(\tau, \xi\right) = C\left(\tau\right) \prod_{i=1}^{\ell} f_i\left(\tau, \xi^i\right) \tag{3.10}$$

Substituting eq. (3.10) into eq. (3.9), we obtain the ℓ same third differential equations for f_i and the solutions of the equation are

$$f_i\left(\tau,\xi^i\right)=\theta_m\left(\tau,\xi^i\right)\ ,\qquad m=2,3,4 \tag{3.11}$$

where θ_m is the usual theta function on torus.[10] The unknown function $C(\tau)$ of eq. (3.10) can be determined by the heat equation

$$\begin{aligned}(C_A+1)\,4\pi i\,\frac{\partial \ell n Z}{\partial\tau} &= \sum_{i=1}^{\ell}\Big[-4\pi i\frac{\partial \ell n\theta_1(\tau,\xi^i)}{\partial\tau}-2\,(C_A-1)\,\eta_1\\ &+ 2\,\frac{\partial \ell n\theta_1(\tau,\xi^i)}{\partial\xi^i}\,\frac{\partial \ell n Z}{\partial\xi^i}+C_A\,\frac{1}{Z}\,\frac{\partial^2 Z}{\partial\xi^{i^2}}\Big]\ ,\end{aligned} \tag{3.12}$$

where we have used

$$\frac{\partial^2\theta_1(\tau,\zeta)}{\partial\zeta^2}=4\pi i\,\frac{\partial\theta_1(\tau,\zeta)}{\partial\tau} \tag{3.13}$$

From eqs. (3.10), (3.11) and (3.12) we finally find the solutions as follow:

$$Z_m=C_m(\tau)\,\prod_{i=1}^{\ell}\,\theta_m(\tau,\xi^i)\ ,\quad m=2,3,4 \tag{3.14}$$

with

$$\begin{aligned}C_2(\tau) &= \eta^{-(\ell+1)}(\tau)\eta(2\tau)\\ C_3(\tau) &= \eta^{-1}(\tau/2)\eta^{2-\ell}(\tau)\eta^{-1}(2\tau)\\ \text{and}\ C_4(\tau) &= \eta^{-(\ell+1)}(\tau)\eta(\tau/2)\ .\end{aligned} \tag{3.15}$$

By linearly combining, the solutions can be written in a familar form, i.e.

$$\begin{aligned}Z_{\Lambda_\ell} &= C^{\Lambda_\ell}_{\Lambda_\ell}(\tau)\theta_{\Lambda_\ell}(\tau,\xi)\\ Z_{\Lambda_0} &= C^{\Lambda_0}_{\Lambda_0}(\tau)\theta_{\Lambda_0}(\tau,\xi)+C^{\Lambda_0}_{\Lambda_1}(\tau)\theta_{\Lambda_1}(\tau,\xi)\\ Z_{\Lambda_1} &= C^{\Lambda_1}_{\Lambda_0}\Theta_{\Lambda_0}+C^{\Lambda_1}_{\Lambda_1}\theta_{\Lambda_1}\ ,\end{aligned} \tag{3.16}$$

$$C^{\Lambda_1}_{\Lambda_1} - C^{\Lambda_1}_{\Lambda_0} = C_3, \quad C^{\Lambda_1}_{\Lambda_1} + C^{\Lambda_1}_{\Lambda_0} = C_4 , \quad C^{\Lambda_\ell}_{\Lambda_\ell} = C_2 ,$$
$$C^{\Lambda_0}_{\Lambda_0} = C^{\Lambda_1}_{\Lambda_1} , \quad C^{\Lambda_1}_{\Lambda_0} = C^{\Lambda_0}_{\Lambda_1} , \tag{3.17}$$

where Λ_ℓ, Λ_0 and Λ_1 are all integrable representations of level one of $\hat{B}_\ell$. Eq. (3.16) is exactly the Weyl–Kac character formula at $K = 1$ for $\hat{B}_\ell$.

IV. Calculations of String Functions

The algebraic methods and some results of calculating string functions have been given by Kac and Peterson.[11] Here we show by means of some examples how to calculate string functions by using the analytical methods (i.e., using the partial differential equations for characters).

Now our start point is the Weyl–Kac character formula

$$Z_\Lambda(\tau, \xi, 0) = \sum_{\lambda \epsilon P_+^K} C^\Lambda_\lambda(\tau)\, \theta_\lambda(\tau, \xi) , \tag{4.1}$$

where θ_λ is the classical theta function of degree K (see eq. (3.8) for $K = 1$)[9] and our purpose is to find out the unknown string functions $C^\Lambda_\lambda(\tau)$. Let us denote the eq. (2.14) by

$$(C_A + K)\, 4\pi i\, \frac{\partial Z}{\partial \tau} = DZ , \tag{4.2}$$

$$D \equiv \sum_{i=1}^{\ell} \frac{\partial^2}{\partial \xi^{i2}} + 2 \sum_{s=1}^{N_G} \frac{\partial \ell n \theta_1}{\partial x_s}\, \alpha_i^{(s)}\, \frac{\partial}{\partial \xi^i} + (d_G - 3\ell)\, K \eta_1 - K \sum_{s=1}^{N_G} \frac{1}{\theta_1}\, \frac{\partial^2 \theta_1}{\partial x_s^2} . \tag{4.3}$$

Substituting $Z = C(\tau) f(\tau, \xi)$ into eq. (4.2), one has

$$(C_A + K)\, 4\pi i\, \frac{d \ell n C}{d\tau} = \frac{1}{f} \left(D - (C_A + K)\, 4\pi i\, \frac{\partial}{\partial \tau} \right) f . \tag{4.4}$$

Because the left hand of eq. (4.4) is independent of variables $\xi^1, \cdots, \xi^\ell$, the right hand of eq. (4.5) should also be so. Therefore, we can calculate it at any values of $\xi^1, \cdots, \xi^\ell$ and the easy way is calculating it at $\xi^1 = \cdots = \xi^\ell = 0$ (we shall write $\xi = 0$ instead of $\xi^1 = \cdots = \xi^\ell = 0$ for the sake of simplicity). Thus we have

$$\begin{aligned} Df\,|_{\xi=0} &= \sum_{i=1}^{\ell} \frac{\partial^2 f}{\partial \xi^{i^2}}\,|_{\xi=0} + 2\,\frac{\partial \ell n \theta_1(\tau,x)}{\partial x}\,|_{x=0}\, \frac{\partial f}{\partial \xi^i}\,|_{\xi=0}\; P_i^G \\ &+ \left[(d_G - 3\ell)\,K\eta_1 - K4\pi i\,\frac{\partial \ell n \theta_1(\tau,0)}{\partial \tau}\,N_G \right] f(\tau,0)\ , \end{aligned} \tag{4.5}$$

where

$$P_i^G = \sum_{s=1}^{N_G} \alpha_i^{(s)}\ . \tag{4.6}$$

We now turn to some examples.

a. $B_\ell^{(1)}$

The spinor representation of level one of $B_\ell^{(1)}$ is labeled by the highest weight $\hat{\Lambda}_\ell = (0, \Lambda_\ell)$ with

$$\Lambda_\ell = \left(0^{\ell-1}1\right) = \sum_{j=1}^{\ell} \frac{j}{2}\,\alpha^{(j)}\ , \tag{4.7}$$

where $\alpha^{(j)}$ is the simple root of B_ℓ. From eq. (3.8) one has

$$\theta_{\Lambda_\ell} = \prod_{j=1}^{\ell} \theta_2(\tau, \xi^j) \tag{4.8}$$

Taking $K = 1$ and $f = \theta_{\Lambda_\ell}$ in eqs. (4.4) and (4.5), a straightforward calculation shows

$$\begin{aligned} \text{the right hand of eq. (4.4)} &= 4\pi i \Big[-2\ell(\ell+1)\frac{d\ell n\eta(\tau)}{d\tau} \\ &+ 2\ell\,\frac{d\ell n\eta(2\tau)}{d\tau} \Big]\ . \end{aligned} \tag{4.9}$$

Thus we find

$$C(\tau) = \eta^{-(\ell+1)}(\tau)\eta(2\tau), \quad ;\text{up to a multiplicable constant} . \tag{4.10}$$

b. $D_\ell^{(1)}$

Consider the fundamental representation $\angle\left(\hat{\Lambda}_1\right)$ of $K = 1$ of $D_\ell^{(1)}$ with $\hat{\Lambda}_1 = (010^{\ell-1}) = (0, \Lambda_1), \Lambda_1 = (10^{\ell-1}) = e_1$. From eq. (3.8) one has

$$\theta_{\Lambda_1} = \frac{1}{2}\left(\prod_{i=1}^{\ell}\theta_3\left(\tau,\xi^i\right) - \prod_{i=1}^{\ell}\theta_4\left(\tau,\xi^i\right)\right) . \tag{4.11}$$

The calculation will be very complicated if one substitutes directly $Z_{\Lambda_1} = C_{\Lambda_1}^{\Lambda_1}\theta_{\Lambda_1}$ into eq. (4.2) since eq. (4.11 is of a non-factorized expression. Fortunately, because the operator D is a linear operator we can use the principle of superposition for solutions. Recall that

$$\begin{aligned} Z_{\Lambda_0} &= C_{\Lambda_0}^{\Lambda_0}(\tau)\theta_{\Lambda_0} , \quad \hat{\Lambda}_0 = (1, 0^\ell), \ \Lambda_0 = (0^\ell) \\ \theta_{\Lambda_0} &= \frac{1}{2}\left(\prod_{i=1}^{\ell}\theta_3(\tau,\xi^i) + \prod_{i=1}^{\ell}\theta_4(\tau,\xi^i)\right) \end{aligned} \tag{4.12}$$

and $C_{\Lambda_0}^{\Lambda_0} = C_{\Lambda_1}^{\Lambda_1}$. From eqs. (4.11) and (4.12), one has

$$Z_{\Lambda_1} + Z_{\Lambda_0} = C_{\Lambda_1}^{\Lambda_1}(\tau)\prod_{i=1}^{\ell}\theta_3(\tau,\xi^i) . \tag{4.13}$$

Substituting eq. (4.13) into eq. (4.2), one can easily find

$$C_{\Lambda_1}^{\Lambda_1} = \eta^{-\ell}(\tau) , \tag{4.14}$$

as expected.

c. $C_3^{(1)}$

There are $\ell+1$ representations of level one of $C_\ell^{(1)}$ and the calculation of string functions for $C_\ell^{(1)}$ is still an open problem.[9] Here we report the results for $\ell=3$ and the calculation for arbitrary ℓ is in progress.

The highest weights of the $K=1$ representations for $C_3^{(1)}$ are $\hat\Lambda_0,\hat\Lambda_1,\hat\Lambda_2$ and $\hat\Lambda_3$. As an example, let us focus our attention to $\angle(\hat\Lambda_3)$. According to the Weyl–Kac character formula (4.1), one has

$$Z_{\Lambda_3} = C^{\Lambda_3}_{\Lambda_1}\theta_{\Lambda_1} + C^{\Lambda_3}_{\Lambda_3}\theta_{\Lambda_3} \tag{4.15a}$$

$$= C^{\Lambda_3}_{\Lambda_1}\left(\theta_{\Lambda_1} + R(\tau)\theta_{\Lambda_3}\right) , \tag{4.15b}$$

where θ_{Λ_1} and θ_{Λ_3} can be expressed as

$$\theta_{\Lambda_1} = \theta_e\left(\frac{\tau}{2},\xi^2/\sqrt{2}\right)\left(\theta_0\left(\frac{\tau}{2},\xi^1/\sqrt{2}\right)\theta_e\left(\frac{\tau}{2},\xi^3/\sqrt{2}\right)+\theta_e\left(\frac{\tau}{2},\xi^1/\sqrt{2}\right)\theta_0\left(\frac{\tau}{2},\xi^3/\sqrt{2}\right)\right) \tag{4.16}$$

$$\theta_{\Lambda_3} = \theta_0\left(\frac{\tau}{2},\xi^2/\sqrt{2}\right)\left(\theta_e\left(\frac{\tau}{2},\xi^1/\sqrt{2}\right)\theta_e\left(\frac{\tau}{2},\xi^3/\sqrt{2}\right)+\theta_0\left(\frac{\tau}{2},\xi^1/\sqrt{2}\right)\theta_0\left(\frac{\tau}{2},\xi^3/\sqrt{2}\right)\right) \tag{4.17}$$

with

$$\theta_e = \frac{1}{2}(\theta_3+\theta_4) \tag{4.18a}$$

$$\theta_0 = \frac{1}{2}(\theta_3-\theta_4) \qquad 4.18b)$$

From eqs. (2.32b) and (4.15b), we obtain

$$R(\tau) = -\frac{D_s^{(3)}\theta_{\Lambda_1}}{D_s^{(3)}\theta_{\Lambda_3}} . \tag{4.19}$$

By using eqs. (2.33), (4.16), (4.17) and (4.19), it is straightforward to derive that

$$R(\tau) = \frac{\theta_e(0)\left(\theta_0''(0)\,\theta_e(0) - 5\theta_e''(0)\theta_0(0)\right)}{\theta_0''(0)\left(2\theta_0^2(0)+3\theta_e^2(0)\right) - \theta_e''(0)\theta_e(0)\theta_0(0)} \tag{4.20}$$

where we write $\theta_e(0), \theta_e''(0)$, etc. instead of $\theta_e(\tau, 0)$, $\frac{\partial^2\theta_e(\tau,\zeta)}{\partial\zeta^2}\mid_{\zeta=0}$, etc. for the sake of simplicity. Setting $K = 1$ and $f = \theta_{\Lambda_1} + R(\tau)\theta_{\Lambda_3}$ in eqs. (4.4) and (4.5), we find by the straightforward calculation

$$C_{\Lambda_1}^{\Lambda_3} = \eta^{-21/5}\phi^{2/5} \exp\left\{-\frac{7}{5}\int \frac{dR}{d\tau}\frac{\theta_{\Lambda_3}(0)}{\phi}d\tau\right\}, \tag{4.21}$$

where

$$\phi = \theta_{\Lambda_1}(0) + R\theta_{\Lambda_3}(0) = (2 + R)\,\theta_e^2(0)\theta_0(0) + R\theta_0^3(0)\,. \tag{4.22}$$

Finally, according to the definition of R (see eq. (4.15)), we have

$$C_{\Lambda_3}^{\Lambda_3} = RC_{\Lambda_1}^{\Lambda_3} \tag{4.23}$$

V. Discussions

In summary, we have derived the partial differential equations for characters of Kac–Moody algebras by using the current and conformal Ward identities, Sugawara construction and the null vectors of Kac–Moody algebras. For level one representations of $A_\ell^{(1)}, D_\ell^{(1)}, E_\ell^{(1)}$ and $B_\ell^{(1)}$, we have given the solutions of the partial differential equations and consequently gave a complete "physical" proof of Weyl–Kac character formula in these cases. The partial differential equations for other affine algebras and higher levels can also be solved and we leave it in the future.

The number of the partial differential equations, in general, is larger than that of the independent variables. We know from the origin of these equations that they all are consistent with each other. But, unfortunately, there is no general way by which one can determine the number of the independent partial differential equations. The number depends on the specific algebra. Given an algebra, one can find out the independent ones from eq. (2.30) or (2.32) and solve them so that the all independent solutions can be found.

For higher level, we need to solve the partial differential equations of higher orders. Although there is no principled difficulty to solve them, the concrete solving will be technically complicated. Alternatively, we can use the method of decomposing of a direct product representation and obtain the characters of higher levels from those of the lower ones, as discussed in ref. [7] for $\hat{G} = A_1^{(1)}$. Nevertheless, this makes the proof be not completely analytic again.

A generalization to semi-simple algebras and the twisted affine algebras is straightforward. By using the super current and super conformal Ward identities on the supertorus[12], one can also generalize the methods of this paper to super characters. As for generalization to higher genus ($g > 1$) Riemann surfaces, there is much work to do.

VI. Acknowledgement

C. S. Huang would like to thank W. A. Bardeen and the Theory Group of Fermilab for the warm hospitality. He acknowledges also W. A. Bardeen, P. A. Griffin and H. Sato for discussions, and L. Deringer for typewriting the manuscript. This work is supported in part by the National Natural Science Foundation of China.

References

[1] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. **B241**, (1984) 333; D. Friedan, Z. Qiu and S.H. Shenker, Phys. Rev. Lett. **52**, (1984)

1575; P. Goddard, A. Kent and D. Olive, Phys. Lett. **152B**, (1985) 88; V.G. Kac and T. Todorov, Comm. Math. Phys. **102**, (1985) 337; D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. **B271**, (1986) 93; D. Friedan and S. Shenker, Nucl. Phys. **B281**, (1987) 509; E. Verlinde, Nucl. Phys. **B300**, (1988) 360.

[2] J. Wess and B. Zumino, Phys. Lett. **37B**, (1971) 95; S.P. Novikov, Sov. Math. Dokl. **24**, (1981) 222; E. Witten, Comm. Math. Phys. **92**, (1984) 455; A.M. Polyakov and P.B. Wiegman, Phys. Lett. **141B**, (1983) 223.

[3] D. Gepner and E. Witten, Nucl. Phys. **B278**, (1986) 493.

[4] H. Sonoda, Nucl. Phys. **B281**, (1987) 546.

[5] T. Eguchi and H. Ooguri, Nucl. Phys. **B282**, (1987) 308.

[6] D. Bernard, Nucl. Phys. **B303**, (1988) 77.

[7] T. Eguchi and H. Ooguri, Nucl. Phys. **B313**, (1989) 492.

[8] P. Goddard and D. Olive, Int. Journ. Mod. Phys. **A1**, (1986) 303.

[9] V.G. Kac, "Infinite Dimensional Lie Algebras," Birkhäuser, Boston, 1983.

[10] E.T. Whittaker and G.N. Watson, "A Course of Modern Analysis," Cambridge University Press, 1927.

[11] V.G. Kac and D.H. Peterson, Adv. in Math. **53**, (1984) 125.

[12] J. Grundberg and R. Nakayama, Nucl. Phys. **B306** (1988) 497; K.-Ji Hamada and M. Takao, Peprint KEK-TH-209 (1988); N. Kawamato and Y. Saeki, Preprint KUNS 970 (1989); C.S. Huang, K.W. Xu and Z.Y. Zhao, Preprint IC/292/89.