# GENERATION OF TIME HISTORIES WITH A SPECIFIED AUTO SPECTRAL DENSITY AND PROBABILITY DENSITY FUNCTION

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## INTRODUCTION

It is recognized that some dynamic and noise environments are characterized by time histories which are not Gaussian. An example is high intensity acoustic noise. Another example is some transportation vibration. A better simulation of these environments can be generated if a zero mean non-Gaussian time history can be reproduced with a specified auto (or power) spectral density (ASD or PSD) and a specified probability density function (pdf). After the required time history is synthesized, the waveform can be used for simulation purposes. For example, modern waveform reproduction techniques can be used to reproduce the waveform on electrodynamic or electrohydraulic shakers. Or the waveforms can be used in digital simulations. A method is presented for the generation of realizations of zero mean non-Gaussian random time histories with a specified ASD, and pdf.

First a Gaussian time history with the specified auto (or power) spectral density (ASD) is generated. A monotonic nonlinear function relating the Gaussian waveform to the desired realization is then established based on the Cumulative Distribution Function (CDF) of the desired waveform and the known CDF of a Gaussian waveform. The established function is used to transform the Gaussian waveform to a realization of the desired waveform. Since the transformation preserves the zero-crossings and peaks of the original Gaussian waveform, and does not introduce any substantial discontinuities, the ASD is not substantially changed.

Several methods are available to generate a realization of a Gaussian distributed waveform with a known ASD. The method of Smallwood and Paez (1993) is an example. However, the generation of random noise with a specified ASD but with a non-Gaussian distribution is less well known.

*This work was supported by the United States Department of Energy under Contract DE-AC04-94AL85000.

## THEORY

If the pdf of the desired waveform is known or can be estimated, the resulting CDF can be used with the known distribution of a Gaussian waveform to establish a transformation function. This function relates a variable with a Gaussian distribution to a waveform with the desired distribution. To arrive at the required function, the formula for the change in variables using the cumulative distribution function will be used (Wirsching, Paez, and Ortiz, 1995).

$$F_Y(y) = F_X(g^{-1}(y)) \tag{1}$$

where

$$y = g(x) \tag{2}$$

is a monotonically increasing function. The inverse is denoted as

$$x = g^{-1}(y) \tag{3}$$

The procedure for finding $g(x)$ from the known functions $F_Y(y)$ and $F_X(x)$ is best illustrated graphically in Fig. 1. $F_Y(y)$ and $F_X(x)$ will be restricted to functions which result in a monotonically increasing function $g(x)$. It is seen from Fig. 1 that a point in the $x$-$y$ plane $(x_1, y_1)$ for which $F_X(x_1) = F_Y(y_1)$ is a point on the function $y = g(x)$. If both $F_Y(y)$ and C are known, $y = g(x)$ can be constructed. Usually for experimental data the pdf is first estimated and the CDF is estimated by integrating the pdf. In this development $F_Y(y)$ is the target non-Gaussian distribution, and $F_Y(y)$ is a Gaussian distribution.

A sampled realization of a waveform with a Gaussian distribution and with the specified spectral density will then be generated. Each sample resulting realization will be transformed using the previously derived function, $y = g(x)$. If the function is "smooth" and monotonically increasing, the transformation will preserve all the zero crossings, minimums and maximums of the original waveform. Since most of the spectral information is contained in the zero crossings (Bendat and Piersol, 1986, Sec. 12.6.4), the spectrum will not usually be substantially changed. Some harmonic distortion will be introduced by the transformation. The spectrum of the distorted waveform can be estimated and

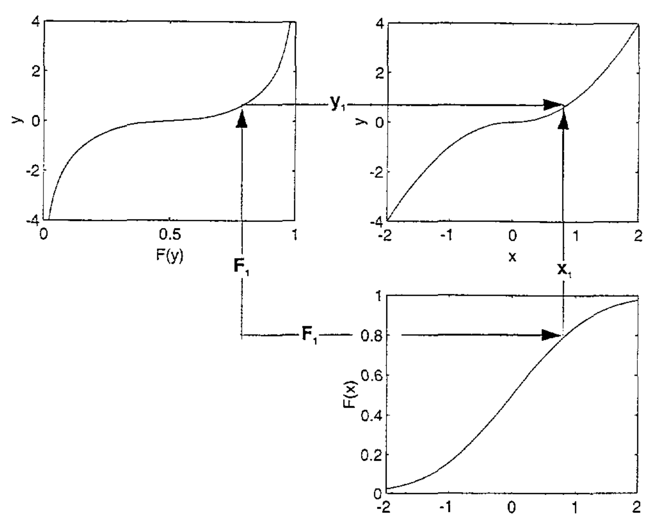


*Figure 1 Generation of the transformation function, $y = g(x)$*

an error spectrum generated. The error spectrum can be subtracted from the original spectrum and a new realization of the Gaussian waveform can be generated from the corrected spectrum in an iterative fashion. Often this iteration is not required as the distorted waveform spectrum is near enough to the target spectrum to be useful without correction.

If experimental data are used to generate the probability density of the target spectrum, a procedure which results in a smooth continuous estimate of the probability density is needed. A simple histogram will not usually be sufficient. The method used in this paper is discussed by Silverman (1986). The actual probability density estimation routine was supplied by Norm Hunter of Los Alamos National Laboratories. The resulting probability density was integrated using a simple trapezoidal rule to estimate the cumulative distribution function. If the target distribution is known as one of the classical distributions the probability density and cumulative distribution can often be evaluated analytically. The probability density function and the cumulative distribution function of the Gaussian distribution were generated using standard functions in MATLAB®.

## PROCEDURE FOR GENERATING A NON-GAUSSIAN WAVEFORM WITH A SPECIFIED SPECTRAL DENSITY

The procedure for generating a realization of a non Gaussian waveform with specified spectral density can now be outlined.

1) Determine the CDF of the desired waveform, $F_Y$.

2) Determine the ASD of the desired waveform, $G_{dd}$.

3) Determine the transformation, $y = g(x)$, from $F_Y(y)$ and the known distribution of a Gaussian waveform, $F_X(x)$. $F_X(x)$ is scaled for the desired variance, determined by the area under the desired spectrum, $G_{dd}$. Check to make sure $y$ is monotonically increasing. If the function is not monotonically increasing a solution will not exist using this technique.

4) Set the spectrum of a trial Gaussian waveform as $G_{xx} = G_{dd}$.

5) Generate a realization, $\{x_i\}$, of a Gaussian waveform with the spectrum, $G_{xx}$. The method of Smallwood and Paez (1993) can be used.

6) Transform the realization, $\{x_i\}$ into a realization $\{y_i\}$ using the transformation $y = g(x)$.

7) Estimate the spectrum of $\{y_i\}$, $G_{yy}$.

8) Determine the error spectrum $G_{ee} = G_{yy} - G_{dd}$

9) If at each frequency, error spectrum is within tolerance, or if $G_{xx}$ is zero, the procedure is finished.

10) Otherwise, subtract the error spectrum from $G_{xx}$, $G_{xx} = G_{xx} - \gamma G_{ee}$. $G_{xx}$ cannot be less than zero. $\gamma$ is a convergence parameter sometimes needed because only an estimate of the error spectrum is available.

Repeat steps 5-9. Once convergence has been achieved, additional realizations can be generated using just steps 5 and 6. If the spectrum of the Gaussian process is reduced to zero and the spectrum of the transformed realization is still too large at some frequencies, a solution will not exist using this technique. This implies the harmonic distortion introduced by the transformation exceeds the desired spectrum at those frequencies.

## EXAMPLE

For the example, the target probability density is given Fig. 2. This probability density was generated by superimposing three Gaussian distributions. By superimposing three Gaussian distributions, distributions with a large variety of skewness and kurtosis can be achieved. Generally this method will produce distributions with kurtosis between one and three and positive or zero values of skewness. If negative skewness is desired, a realization with a positive skewness can be inverted. The mean of the distribution with a negative mean is *-b* (see Fig. 2). *b-a* and *b+a* are the means of two distributions with positive means. The area of the distribution with mean *-b* is 1/2 and the areas of the distributions with means *b-a* and *b+a* are 1/4. The root mean square (the standard deviation) of each distribution is *s*. The distribution of the random variable, *y*, is defined as the superposition of the three Gaussian distributions. The first four moments of *y* are given by,

$$E[y] = .25(b-a) + .25(b+a) - .5b = 0 \quad (4)$$

$$E[y^2] = s^2 + b^2 + .5a^2 \quad (5)$$

$$E[y^3] = 1.5ba^2 \quad (6)$$

$$E[y^4] = 3s^4 + b^4 + 3b^2a^2 + .5a^4 \quad (7)$$

Equation 4 is satisfied for any combination of *a* and *b*. For many specified values of moments of *y*, equations 5-7 can be solved for the three unknowns *a*, *b*, and *s*. Knowing the values *a*, *b*, and *s* the probability density function of *y* can be determined, and hence the function $y = g(x)$ can be determined.

For this example the desired spectrum is defined by the dashed line in Fig. 3c. The spectrum is defined from 40 to 1020 Hz. The first moment of *y*, the mean, is zero. The desired spectrum defines the second

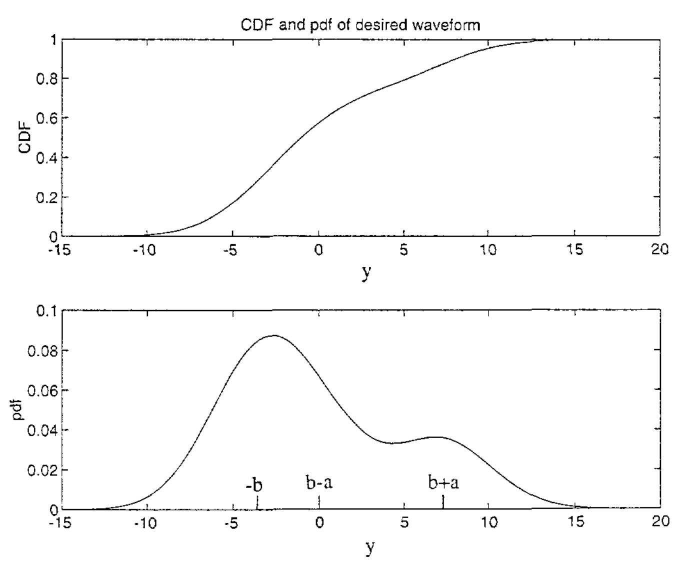


*Figure 2 Target probability density function*

moment. The standard deviation of $y$ is 5.34 for the given spectrum. Let the normalized third moment, $E[x^3]/E^{3/2}[x^2]$, be 1/2, and the normalized fourth moment, $E[x^4]/E^2[x^2]$, be 2.5. This results in the values of 3.66, 3.66, and 2.90 for $a$, $b$, and $s$ respectively. These values result in the probability density given in Fig. 2. The resulting function $y = g(x)$ is given in Fig. 4. The function resembles the force deflection curve of a non-symmetrical softening spring. A realization of the Gaussian waveform, $x$, and the distorted waveform, $y$, are shown in Figs. 3a and 3b. The estimated spectrum of $y$ is shown as the solid line of Fig. 3c. As can be seen the spectrum of $y$ is very close to the desired spectrum without any iteration. Some evidence of high frequency distortion is seen above 1 kHz.

## CONCLUSIONS

The procedure provides a convenient way to generate non Gaussian waveforms with a particular spectral density. The procedure also illustrates how a nonlinear gain applied to a Gaussian process can lead to a non Gaussian process. The procedure also gives insight into the meaning of a spectral density of a non Gaussian waveform. The insight confirms that most of the spectral information is contained in the zero crossings and locations of the maximum and minimums of the waveform. The probability distribution influences the spectral information in a minor way.

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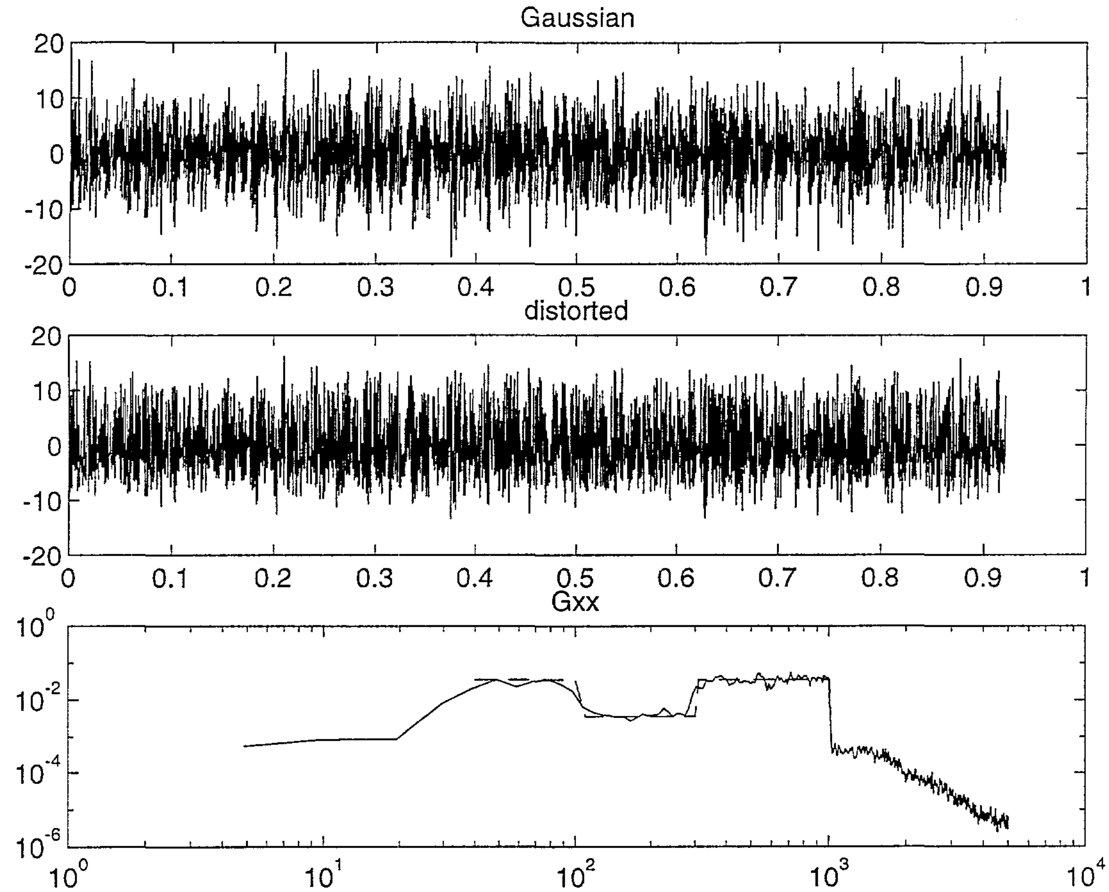


*Figure 3 Realization of random waveform, (a) undistorted waveform, (b) distorted waveform, (c) target and estimated power spectral density of distorted waveform*

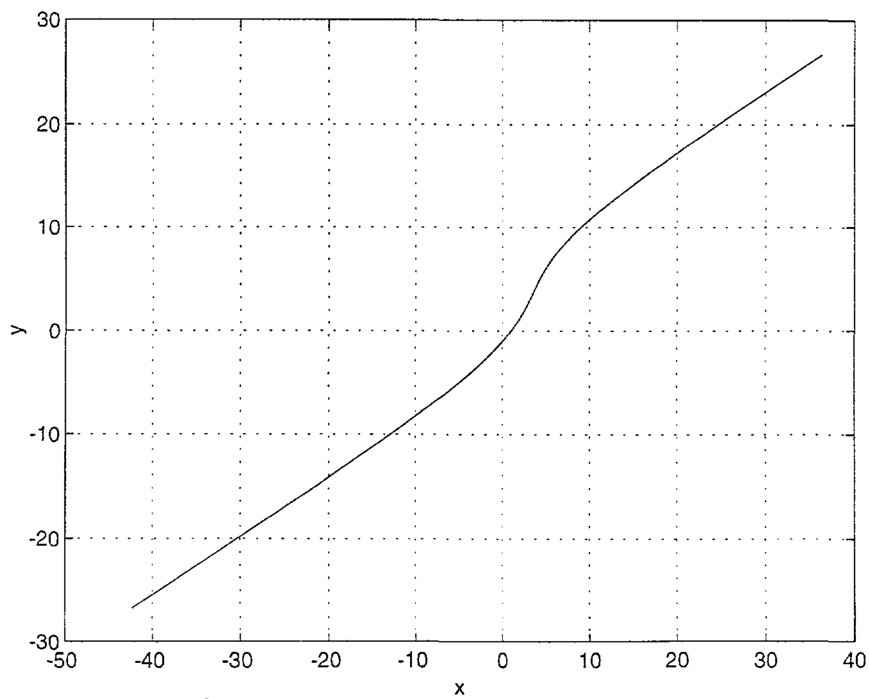


*Figure 4 Transformation function, $y = g(x)$*