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## Sextupole reduction via chaos suppression at NSLS-II

Y. Li

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<b><i>Sextupole reduction via chaos suppression at NSLS-II</i></b>	

# Sextupole reduction via chaos suppression at NSLS-II

Yongjun Li ,\* Minghao Song, Yoshiteru Hidaka, Victor Smaluk, and Timur Shaftan  
Brookhaven National Laboratory, Upton, New York 11973, USA

We revisit the nonlinear lattice design approach for the National Synchrotron Light Source II (NSLS-II) storage ring. By suppressing chaos, we identify alternative sextupole configurations to the original design, which relied on the conventional approach of simultaneously minimizing Resonance Driving Terms (RDTs) and Amplitude-Dependent Detuning (ADD). These alternatives achieve comparable performance while requiring fewer sextupoles. A detailed comparison of two representative solutions is presented and supported by experimental validation. Our results indicate that dynamic aperture correlates more strongly with global chaos than with individual RDTs, and that the importance of minimizing ADD may have been overstated in earlier design approach.

## I. INTRODUCTION

When designing ring-based accelerators, their linear lattices are optimized to achieve key performance goals for users – such as luminosity for colliders and brightness for light sources. Subsequently, nonlinear magnets are incorporated to address operational requirements, including chromaticity compensation, injection efficiency, and beam lifetime etc. The presence of nonlinear magnets typically renders accelerators non-integrable Hamiltonian systems. As a result, exact solutions for particle motion are generally unattainable due to the lack of sufficient conserved quantities. However, long-term stable motions can be maintained within certain regions of the six-dimensional phase space based on the Kolmogorov-Arnold-Moser (KAM) theory [1]. Within these regions, quasi-periodic motions exist on approximate invariant tori, which are partially distorted to resist chaotic diffusion.

In accelerator physics, the stable region around the origin is referred to as the Dynamic Aperture (DA) [2]. Its projection onto the transverse plane at the injection point determines whether an injected beam can be successfully captured. The maximum allowable relative momentum deviation ( $\Delta P/P_0$ ) along the longitudinal direction, known as the Local Momentum Aperture (LMA), governs the probability of particle survival following intra-beam scattering. Thus, achieving sufficient DA and LMA is a fundamental requirement in lattice design.

It is generally believed that by controlling certain parameterized nonlinear performance indicators – such as Resonance Driving Terms (RDTs), Amplitude-Dependent Detuning (ADD), and chromaticity – the onset of chaotic motion can be delayed. Since nonlinear dynamics cannot be fully described analytically, numerical simulations remain indispensable for confirming long-term stability. However, long-term tracking is both computationally intensive and provides only limited physics insight into the underlying dynamics. The development of more efficient physics-guided lattice design approaches remains an active area of research. At NSLS-II, various

chaos indicators have been utilized as the optimization objectives in recent years. In this paper, we present a comprehensive comparison of two representative solutions obtained using different approaches: minimizing RDTs and suppressing chaos.

## II. RESONANCE DRIVING TERMS

In a storage ring, a single particle's motion is governed by a Hamiltonian  $H$  defined by its lattice.  $H$ , when expanded, has terms like:

$$H = H_2 + H_{n \geq 3},$$

where  $H_2$  represents all quadratic monomial terms to represent the linearized motion. The higher order terms  $H_{n \geq 3}$  are mainly generated by nonlinear magnetic fields such as sextupoles, octupoles etc. When performing a normal form analysis through the Lie algebraic expansion [3, 4], resonance terms can be formulated as perturbations to  $H_2$

$$h_{abcd,e} |abcd, e\rangle \stackrel{\text{def}}{=} h_{abcd,e} (r_x^+)^a (r_x^-)^b (r_y^+)^c (r_y^-)^d \delta^e,$$

where  $r_{x,y}^\pm = \sqrt{J_{x,y}} e^{\pm i\phi_{x,y}}$  are the eigen-modes (also referred to as the resonance basis) of the linear motion  $H_2$ ;  $J_{x,y} - \phi_{x,y}$  are linear action-angle variables;  $a, b, c, d, e$  are non-negative integers. When tune  $\nu_{x,y}$  satisfies a resonance condition  $(a - b)\nu_x + (c - d)\nu_y = n \geq 3$ , the corresponding coefficient  $h_{abcd,e}$  quantify how strongly the RDT  $|abcd, e\rangle$  is being driven. The geometric terms with  $e = 0$  are often related to on-momentum DA, and other  $e \neq 0$  chromatic terms can affect the off-momentum motion and determine the LMA eventually. Even linear tune might stay far away from a specific resonance, ADD terms, such as  $h_{2200,0}, h_{1111,0}, h_{0022,0}$ , can push the tune to match the resonant condition at certain amplitudes. Then particle motions is driven to be chaotic, even unbounded. The same mechanism also applies to off-energy particles with energy-dependent detuning.

Since each specific RDT can potentially excite a corresponding nonlinear resonance, it is desirable to minimize multiple RDTs simultaneously – particularly those

\* email: yli@bnl.gov

of lower orders. However, in practice, only a limited number of them can be effectively controlled due to the finite number of available tuning knobs. Even so, higher-order RDTs that remain uncontrolled may still have detrimental effects on the dynamic aperture. In other words, only minimizing low-order RDTs is a necessary but not sufficient condition for achieving the desired DA [5]. As a result, comprehensive tracking simulations remain indispensable for evaluating long-term stability.

### III. CHAOS INDICATORS

Besides the perturbation approaches, various chaos indicators [6, 7] – both experimental and numerical – have been adopted to detect and quantify chaotic behavior in accelerators. These CIs can distinguish regular and chaotic particle motion. Commonly used CIs include, though not limited to, Lyapunov exponents [8], Frequency Map Analysis (FMA) [9], forward-reversal integration [10, 11], entropies [12, 13], and others. Unlike approaches that target specific RDTs in the Hamiltonian, the CI methods aim to enhance the regularity of particle motion by suppressing global chaos, thereby enlarging the DA. Quantitative assessments of chaos are typically derived from particle trajectories in phase space, particularly through Poincaré maps or spectral analysis. At facilities such as NSLS-II and APS-U [14], delaying the onset of chaotic motion has been proven effective in optimizing their DAs.

### IV. EVOLUTION OF NONLINEAR LATTICE OPTIMIZATION AT NSLS-II

The NSLS-II, a third-generation light source, features an emittance of 2 nm and a double-bending achromat lattice. Its storage ring consists of 15 identical supercells, each composed of two mirror-symmetric single cells, as illustrated in Fig. 1. For nonlinear tuning, the machine employs six families of harmonic sextupoles and three families of chromatic sextupoles. The lattice design process, initiated in 2006 and finalized around 2012, focused on minimizing first- and second-order resonance driving terms (RDTs), i.e.,  $h_{a+b+c+d+e=3,4}$ , and on evaluating the DA and LMA through tracking simulations. Given the large number of RDTs, configurations with up to nine families of harmonic sextupoles were explored, but ultimately six were adopted. The final configuration was selected after rigorous robustness checks that accounted for engineering imperfections and insertion device integration.

In recent years, a different approach of using CIs has been explored. Rather than focusing on each individual resonance, we aim at controlling the overall nonlinear lattice performance characterized with various CIs [11–13, 15–17]. Among these, a representative solution obtained through suppressing fluctuations of Approximate

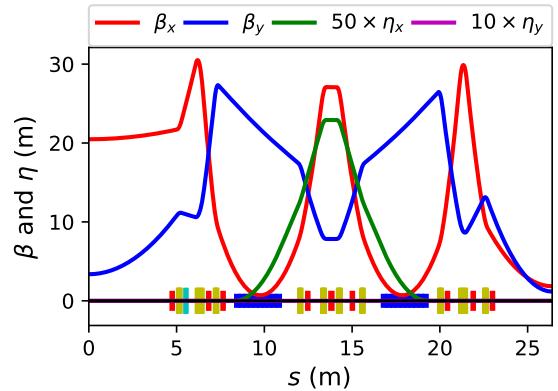


Figure 1. Magnet layout and linear optics of one cell in the NSLS-II ring. Sextupoles are represented as red blocks. Six families of harmonic sextupoles (in the non-dispersive sections) are used to compensate for the nonlinearity caused by the other three families of chromatic sextupoles.

Invariants (AI) [17] has been selected for comparison with our original design. This alternative offers comparable performance while requiring fewer sextupoles.

### V. PERFORMANCE COMPARISON OF NONLINEAR DYNAMICS

In the following, we present a detailed performance comparison between these two representative solutions. The original design is referred to as the Resonance Driving Term Minimization (RDTM) configuration, while the recently re-optimized solution is designated the Chaos Suppression (CS) configuration. Throughout this paper, unless otherwise specified, the properties of the RDTM configuration are shown in the left columns of figures and tables, and those of the CS configuration in the right columns.

#### A. Sextupole configurations

Sextupole strengths, defined as  $K_2 = \frac{1}{(B\rho)_0} \frac{\partial^2 B_y}{\partial x^2}$ , with a given beam rigidity  $(B\rho)_0$ , are provided for two configurations in Table I. Notably, the  $SH4$  family is entirely disabled in the CS configuration. While the RDTM scheme necessitates more sextupole knobs to control multiple objectives, the new approach aims at suppressing global chaos parameterized as a single objective with reduced knobs.

#### B. Poincaré map

In quasi-periodical nonlinear dynamical systems, Poincaré maps – constructed from the intersections of

Sext. name	RDTM	CS
	$K_2$ ( $\text{m}^{-3}$ )	$K_2$ ( $\text{m}^{-3}$ )
SH1	19.83	26.80
SH3	-5.86	-21.89
SH4	-15.82	0
SL1	-13.27	-2.10
SL2	35.68	30.80
SL3	-29.46	-23.00

Table I. Sextupole strengths for two configurations.  $SH_4$  is disabled in the CS configuration.

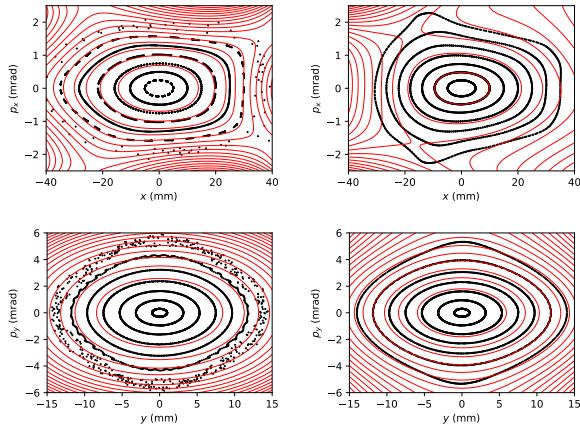


Figure 2. Poincaré maps observed at the injection point for the two configurations in the horizontal (top) and vertical (bottom) planes. Background red contour lines represent their 5<sup>th</sup>-order approximate invariant tori; black dots are simulated turn-by-turn trajectories.

multi-turn phase-space trajectories with a designated transverse cross-section – offer not only a visual representation but also a quantitative measure of chaos. In accelerator physics, such maps can be generated through either numerical simulations or experimental measurements [18]. Fig. 2 presents simulated transverse Poincaré maps for the two configurations under study. In the RDTM configuration, the horizontal Poincaré map exhibits a pentagonal shape with a tip oriented toward the inboard side, resulting in an asymmetric projected  $x - y$  aperture, as shown in Fig. 3. In contrast, the CS configuration demonstrates greater momentum acceptance, corresponding to a larger occupied phase-space volume. In the vertical plane, the RDTM configuration shows clear signs of chaotic motion at large amplitudes, a feature that is notably absent in the CS configuration.

### C. Dynamic aperture

In a narrow sense, the concept of DA in accelerator physics, is often referred to as its 2D projection in the transverse  $x - y$  plane at injection point. When an incident beam is injected, a sufficient aperture is required to accommodate it. Other reasons for the beam being

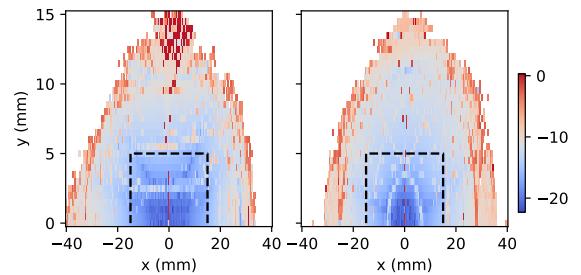


Figure 3. Projections of DA in the transverse  $x - y$  plane (on-momentum dynamic aperture) at the injection point. The black dashed boxes specify the desired dimension ( $\pm 15 \times \pm 5$  mm). The colors here and also in following Fig. 4 and 5 are the chaos measured with tune diffusion  $\log_{10} \sqrt{\Delta\nu_x^2 + \Delta\nu_y^2}$ .

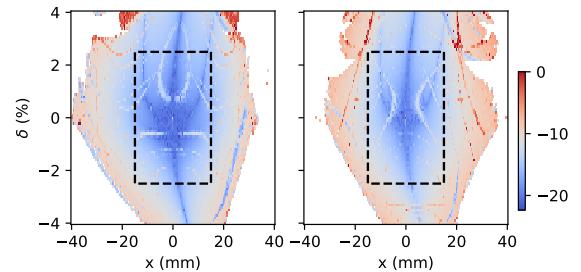


Figure 4. Projection of DAs in the  $x - \delta$  plane. The black dashed boxes define the desired dimension ( $\pm 15$  mm  $\times$   $\pm 2.5\%$ ).

off-axis can be finite bunch size, energy spread of the injected beam, magnet misalignment and other imperfections. The DAs for both configurations are sufficient as shown in Fig. 3, in which the desired dimensions are denoted as the black dashed boxes. The colors represent tune diffusion [19] as explained later.

The 2D projection in the horizontal and momentum  $x - \delta$  plane is often used to observe the maximum allowed aperture for the off-axis and off-energy beam at injection point. Such comparison is shown in Fig. 4.

### D. Frequency map analysis and amplitude-dependent detuning

Frequency map analysis (FMA) projects the amplitude dependent tune footprint colored with its diffusion [19]. Here tune diffusion is defined as the fluctuation of tune as observed with different segments of the same trajectory. It represents how sensitive betatron tune is to initial condition, and particularly identifies tune smear in the vicinity of resonances. Tune diffusion is also a kind of chaos indicator extracted through precise spectral analyses [9].

The FMAs for the two configurations as illustrated in Fig. 5 have distinct patterns. The linear ADD in the RDTM configuration is well minimized and the tune footprint is confined within a small range. While in the CS

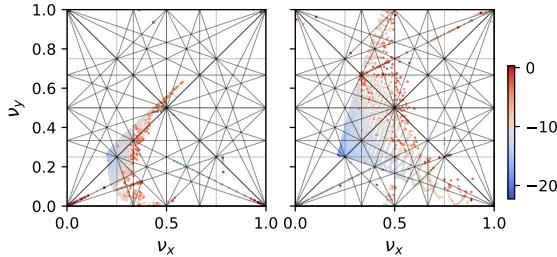


Figure 5. Frequency Map Analyses (FMA) in the tune space, also referred to as tune footprint, for two configurations.

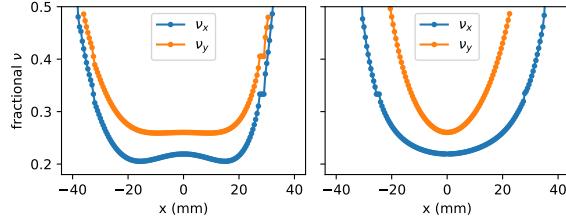


Figure 6. Amplitude-Dependent Detuning (ADD) in the horizontal plane ( $y = 0$ ) for two configurations. A larger ADD is observed in the CS configuration without degrading the DA.

configuration, well-controlled fluctuations in the approximate invariants allow the tune to cross low order resonance, even  $\nu_x = \frac{1}{3}$  safely. A detailed ADD information in the horizontal plane is also illustrated in Fig. 6.

From the above comparisons of the tune footprint in FMA and ADD, we conclude that chaos suppression delays the transition from regular to chaotic motion, causing the lattice to behave more like a near-integrable system. This enhances its resistance to chaos and allows safer resonance crossing. Similar resonance-crossing behavior has been observed in Integrable Optics Test Accelerator (IOTA) machines [20–22]. Traditionally, enlarging the dynamic aperture has relied on delaying resonance crossing and narrowing the resonance stop-band width down simultaneously. This approach therefore places excessive emphasis on minimizing amplitude-dependent detuning. In contrast, the CS configuration suggests an alternative approach – optimizing the lattice to be as close as possible to an integrable system, potentially granting it similar resonance-crossing robustness as observed in integrable systems.

### E. Longitudinal variation of RDTs

Previous studies [23] have shown that enlarging dynamic aperture can be achieved when RDTs are periodically well canceled. Further investigations [24, 25] revealed that even when perfect periodic cancellation is unattainable, minimizing their variations along the longitudinal direction  $s$  is helpful. Therefore, it is of in-

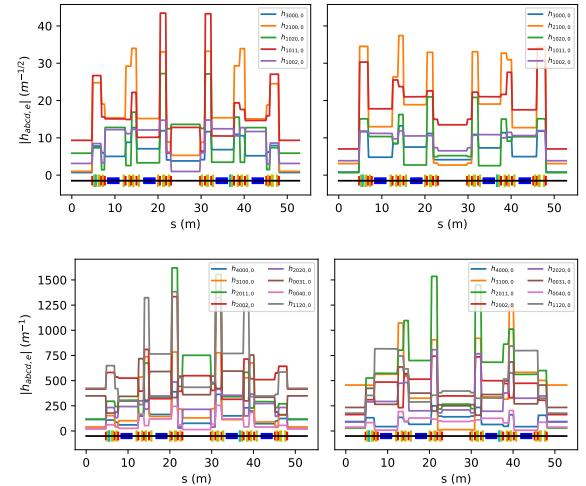


Figure 7. Variation of the amplitudes of the third-order (top) and forth-order (bottom) RDTs along the longitudinal direction for two configurations.

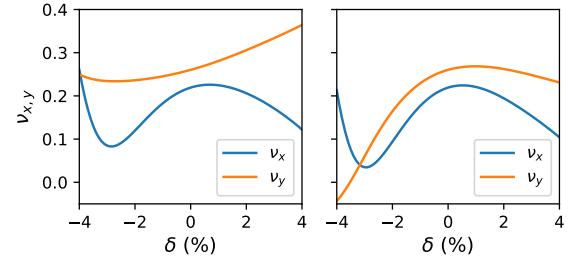


Figure 8. Chromaticity (energy-dependent detuning) for two configurations..

terest to compare the RDTs and their variations in the two configurations. Although RDTs are completely ignored during chaos suppression, we observe that, under chaos suppression, most third- and fourth-order RDTs – as well as their variations – are passively well minimized, as illustrated in Fig. 7.

### F. Chromaticity and local momentum aperture

Chromaticity describes how the betatron tune varies with beam energy deviation. Although harmonic sextupoles do not affect the linear chromaticity, they influence higher-order terms, which can drive off-momentum particles into resonance and ultimately limit the Local Momentum Aperture (LMA). Controlling energy-dependent detuning is therefore also essential. A comparison of the two configurations is shown in Fig. 8. Within the required range of  $|\delta| < 2.5\%$ , tune variation is well confined in both cases.

With a given chromaticity for the entire ring, the stability of off-momentum particles remains a localized

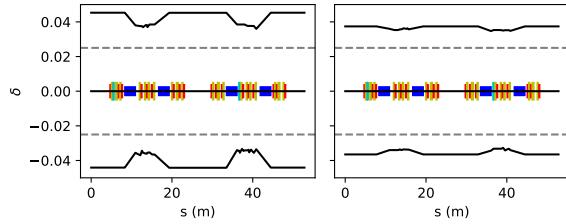


Figure 9. Local momentum aperture within one supercell for two configurations. The gray dashed lines represent the desired momentum aperture at  $\delta = \pm 2.5\%$ .

property. At a given longitudinal position, the LMA defines the maximum relative energy deviation a particle can sustain while remaining stable [26]. In high-density bunched beams, the Touschek effect leads to intra-beam scattering, whereby particles exchange transverse momentum for longitudinal momentum. To ensure adequate beam lifetime, sufficient LMA is required for the majority of scattered particles to survive. Fig. 9 shows the comparison of LMA within one cell. Although the CS configuration exhibits a smaller LMA than the RDTM configuration – particularly in the non-dispersive sections – the two are comparable at the dipole locations, where the most severe scattering occurs. A detailed Touschek lifetime calculation further confirms that both configurations remain comparable and satisfy the requirement of  $|\delta| \geq 2.5\%$ , corresponding to a beam lifetime of approximately three hours.

#### G. Sensitivity to errors

The performance of the dynamic aperture unavoidably degrades in the presence of magnetic field imperfections – such as undesired higher-order multipole components, and physical aperture constraints, particularly tight vertical gaps of insertion devices in light source rings. Such gaps – typically on the order of a few millimeters – can cause injected beam scraping due to nonlinear coupling. A robust lattice design is essential to ensure tolerance against realistic imperfections. Fig. 10 illustrates the shrunk DAs – due to specified multipole errors and restricted physical apertures – are also comparable.

#### H. Correlation with dynamic aperture

The ultimate goal of nonlinear lattice design is to achieve sufficient DA and LMA. However, DA simulations are computationally intensive and therefore impractical for large-scale ring design, while also providing limited physics-based insight into the underlying particle dynamics. A more effective approach is to employ computationally efficient approaches – such as using RDTs or CIs – to filter out unsuitable candidates, followed by de-

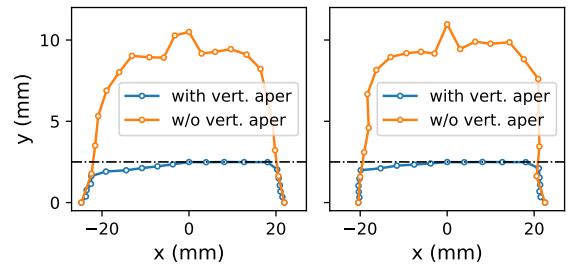


Figure 10. Dynamic apertures in the presence of higher-order multipole errors and physical apertures for two configurations. The dashed black lines denote in-vacuum undulators vertical gaps at  $y = \pm 2.5$  mm.

tailed tracking simulations applied selectively to the most promising configurations. Within this framework, understanding the correlation between optimization objectives and DA is critical for achieving robust solutions. Ref. [5] demonstrated that minimizing RDTs is a necessary but insufficient condition for obtaining a large DA; although some correlation exists, it is relatively weak. Interestingly, by applying a chaos suppression approach, we observed a stronger correlation, which led to high-quality solutions even when using fewer sextupole families.

To further explore this, a numerical experiment was conducted to compare the correlation between DA and optimization objectives based on RDTs and CIs. In this study, the optimization objectives were either RDTs or CIs, and the control variables were the same five families of sextupoles. A population-based multi-objective genetic algorithm was used to evolve candidate solutions across generations. The results reveal that, when chaos suppression is the objective, a strong correlation emerges between DA and CI as shown in Fig. 11. The strong correlation not only accelerates convergence but also produces significantly better candidate solutions. Furthermore, this experiment clarifies why, in our early design stage, a six-family sextupole scheme was required when optimization targeted RDTs: if with only five families, the optimal solutions failed to provide the desired DA.

## I. Experimental validation

The comparison was validated by measuring the dynamic apertures of two configurations at the NSLS-II ring. Experimentally, DAs can be determined by observing stored beam loss following excitation with a pulsed magnet – commonly referred to as a pinger magnet. In this measurement, a short train of stored beam bunches was subjected to progressively increasing kick voltages, as illustrated in Fig. 12. The resulting beam loss patterns exhibit strong similarity, validating the comparable dynamic aperture performance observed in prior simulations. Additionally, the beam lifetime remains similar under identical bunch filling patterns and beam current

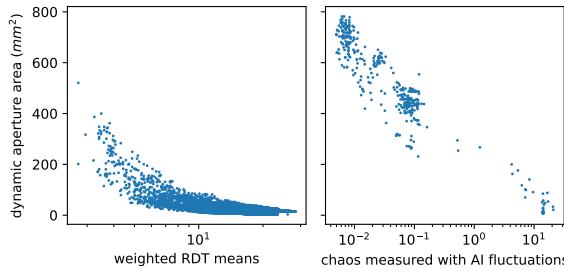


Figure 11. Correlation of dynamic aperture areas with different optimization objectives, obtained from the optimization data pool. In the left subplot, because the magnitudes of the RDTs increase dramatically with order, each RDT was weighted prior to summation to establish the correlation. A stronger correlation is observed when CI was the optimization target.

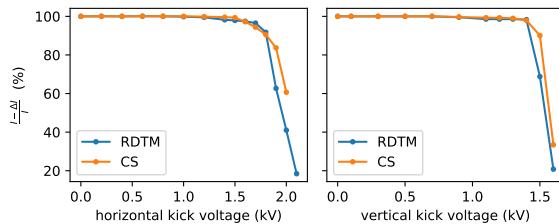


Figure 12. Comparison of measured dynamic apertures in the horizontal (left) and vertical (right) planes. The beam loss pattern subjected to progressively increasing kicks for the RDTM configuration is shown in blue line-dots, while that for the CS configuration is shown in orange.

conditions, which confirms that their LMAs are comparable as well.

## VI. SUMMARY

By suppressing chaotic behavior, we obtained and experimentally validated an alternative nonlinear lattice scheme for the NSLS-II ring. This chaos-suppression approach enhances the beam's resistance to chaos and enables safe resonance crossings, while requiring fewer sextupoles than the conventional design. In contrast, the traditional approach emphasizes minimizing amplitude-dependent detuning, an objective that our results suggest is less critical than previously assumed.

Another method for DA optimization, though not the focus of this paper, is directly based on tracking [27]. Although this approach could also yield high-quality solutions when sufficient computational resources are available, it provides limited physical insight. While such tracking-based studies have also been conducted at NSLS-II [5], the emphasis of this paper is on physics-guided approaches that elucidate the underlying mechanisms of nonlinear dynamics.

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