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Dynamics of E_6 chiral gauge theories

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We present exact nonperturbative vacuum solutions to chiral gauge theories based on the E_6 gauge group and several matter fermions in the fundamental **27**-dimensional representation. They are obtained when supersymmetric versions are perturbed by small supersymmetry breaking by anomaly mediation. The universality classes obtained are very different from what can be conjectured by the tumbling hypothesis. In particular, the case with three **27**s may have an unbroken SU(3) symmetry with massless composite fermions in **10** of SU(3). For this case, we employed numerical techniques to obtain the exact ground state.

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I. INTRODUCTION

Nature is chiral, namely, it distinguishes left from right. The best theory for fundamental physics today is the so-called Standard Model of particle physics, which is a chiral gauge theory. Yet surprisingly, even the definition of a chiral gauge theory is not completely clear [1]. For instance, a standard gauge-invariant regularization method, such as Pauli-Villars with massive fields with wrong statistics to cancel UV divergences, is not possible for chiral gauge theories because the gauge invariance forbids a mass term.[1] The dimensional regularization does not allow for a definition of the Levi-Civita tensor and hence one cannot define $\gamma_5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ as is required for a chiral projection. Lattice gauge theories have an intrinsic fermion doubling problem despite progress [3–11]. Higher-derivative regularization (see, e.g., [12,13]) is not practical in most contexts.

Nonperturbative dynamics of chiral gauge theories is even less clear given the lack of their precise definition and lattice simulations to date. The only known way for an "educated guess" is the tumbling hypothesis. It postulates that fermions form bilinear condensates in the most attractive channel (MAC) similar to QCD [14]. Yet by definition of the theory being chiral, such bilinear operators are not gauge invariant. Given Elitzur's theorem, gauge-invariant operators cannot have a nonvanishing expectation value [15]. Nonetheless, the hypothesis assumes exactly such a condensate that then breaks the gauge group by its own dynamics. By repeating this process, eventually the theory becomes vectorlike, and the chain ends with a QCD-like condensate. This hypothesis identifies an unbroken global symmetry as well as a spectrum of massless composite fermions that match the 't Hooft anomalies in the original UV theory. Even though the method is applicable to all chiral gauge theories, it lacks theoretical justifications or evidence from numerical simulations.

Recently, it was pointed out that chiral gauge theories augmented by supersymmetry (SUSY) and anomaly mediated supersymmetry breaking (AMSB) allow for exact solutions as long the size of supersymmetry breaking is small compared to the dynamical scale of the gauge theory $m \ll \Lambda$ [16]. This paper marked the first time that chiral gauge theories could be solved exactly. It is fascinating that the solution turned out to be very different from what was expected from the tumbling hypothesis. The challenge of this method is that supersymmetric gauge theories need to be studied individually for different gauge groups and particle contents, and it is much harder to draw general lessons compared to tumbling. It is imperative to accumulate a large number of cases to understand the general consequence of chiral gauge theories.

AMSB has the remarkable property of "UV insensitivity" which grants predictive power for strongly coupled theories in four dimensions [17,18]. Namely, the effects of

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[1]A potential exception to the rule is an infinite tower of Pauli–Villars fields [2].

broken supersymmetry can be precisely studied in the IR theory regardless of UV physics. Furthermore, the supersymmetric limit is continuously connected to nonsupersymmetric gauge theories via the supersymmetry breaking parameter m which decouples squarks and gauginos. These noteworthy features of AMSB allow us to simply draw from known nonperturbative results in the pure supersymmetric theory to compute exact vacuum solutions in nonsupersymmetric strongly coupled theories.

This program aims for a general tool to study strongly coupled chiral or nonchiral gauge theories by first studying their supersymmetric counterparts with small AMSB. Recently, this technique of perturbed AMSB has been applied to study a variety of chiral gauge theories [16,19–21]. The vacuum structure and mass spectrum for these cases have been solved exactly. The technique has also been applied to Seiberg's results in supersymmetric theories [22,23] to study the dynamics of QCD-like theories [24–26]. If we expect that the exact solutions for small SUSY breaking $m \ll \Lambda$ continuously connect to the non-SUSY limit $m \gg \Lambda$ where the theory is strongly coupled, we can uniquely identify the unbroken symmetries of the ground states and associated massless spectra of particles in non-SUSY gauge theories. In many cases, the near-SUSY limit $m \ll \Lambda$ allows for a weakly coupled description, while the non-SUSY limit $m \gg \Lambda$ is intrinsically strongly coupled and does not have well-controlled approximation methods.

The Bardeen-Cooper-Schrieffer (BCS) Bose-Einstein condensate (BEC) crossover, established both experimentally and theoretically in atomic, molecular, and optical (AMO) and condensed matter physics [27,28], provides an encouraging support for this program. The BCS state due to a weak attractive force in a Fermi gas continuously connects to the BEC of bosonic bound states of fermions formed by nonperturbative strong attractive force. Knowing that it is a crossover, we can learn a lot about the strong coupling limit (BEC) by studying the weak coupling limit (BCS).

However, it is not decisively known whether or not there is a phase transition when the SUSY breaking parameter is taken to be on the same order of the strong coupling scale $m \sim \Lambda$, but there are plausible arguments which suggest that there will not be a phase transition, implying a continuous connection to non-SUSY limits where $m \to \infty$. Ultimately, if we are to anticipate an advent of theoretical breakthroughs in lattice gauge theory in the near future, the nonperturbative results obtained by AMSB will have to be verified with lattice simulations.

In this paper, we study the dynamics of non-SUSY chiral gauge theories based on E_6 using the method of AMSB with the supersymmetry breaking parameter taken to be infinitesimal. The group structure of E_6 is quite nontrivial in comparison to the more commonly studied classical groups. However, it serves as an additional class of simple chiral gauge theories that is worthy of examination given that E_6 admits complex representations in which all of its representations are anomaly-free by construction. Therefore, we can naturally embed Weyl fermions in the 27-dimensional representation.

II. SUPERSYMMETRIC E_6 WITH 27'S

In the UV, we consider chiral superfields $\psi_i (i = 1, \ldots, N_F)$ that are in the 27-dimensional fundamental representation of E_6 and in the fundamental representation of the $SU(N_F)$ global symmetry. The **27** representation of E_6 can be expressed in terms of the maximal subgroup $SU(3) \times SU(3) \times SU(3)$ in which we find the following decomposition,

$$\mathbf{27} \to (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}) \oplus (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{3}). \tag{1}$$

The **27** can then be expressed as three 3×3 matrices. In the IR, below the strong coupling scale Λ, the theory can be described in terms of the gauge invariant polynomials, which are identified as coordinates of the moduli space. These invariants can be constructed using the fully symmetric, cubic invariant primitive tensor $d_{\mu\nu\lambda}$,[2]

$$S_{(ijk)} = d_{\mu\nu\lambda} \psi_i^\mu \psi_j^\nu \psi_k^\lambda, \tag{2}$$

$$T_{ijk;lmn} = D_{\mu\nu\lambda;\xi\rho\sigma} \psi_i^\mu \psi_j^\nu \psi_k^\lambda \psi_l^\xi \psi_m^\rho \psi_n^\sigma, \tag{3}$$

where the invariant T only appears for $N_F \geq 3$ and the rank 6 invariant tensor $D_{\mu\nu\lambda;\xi\rho\sigma}$ is formed from antisymmetric products of the cubic invariant

$$D_{\mu\nu\lambda;\xi\rho\sigma} = \frac{1}{36} d_{\alpha\beta\gamma} (d^{\mu\xi\alpha} d^{\nu\rho\beta} d^{\lambda\sigma\gamma} - d^{\nu\xi\alpha} d^{\mu\rho\beta} d^{\lambda\sigma\gamma} - d^{\nu\rho\alpha} d^{\mu\xi\beta} d^{\lambda\sigma\gamma} + \ldots). \tag{4}$$

Here, the Greek indices $(\mu, \nu, \lambda, \xi, \rho, \sigma = 1, \ldots, 27)$ are E_6 fundamental representation indices and the Latin indices $(i, j, k, l, m, n = 1, \ldots, N_F)$ are $SU(N_F)$ flavor indices. These gauge invariants polynomials can also be expressed in terms of Young tableaux,

$$S_{ijk} = \boxed{i \mid j \mid k}, \quad T_{ijk;lmn} = \begin{array}{|c|c|} \hline i & j \\ \hline k & l \\ \hline m & n \\ \hline \end{array} \tag{5}$$

In this work, we discuss cases $N_F = 1, 2, 3$. For $N_F = 4$, the moduli space is modified by quantum corrections and it is difficult to make concrete conclusions with AMSB. We therefore leave this case for future work. As each fundamental field develops a vacuum expectation value (VEV),

[2]The E_6 primitive tensor $d_{\mu\nu\lambda}$ is normalized so that every entry is either $0, \pm 1$.

TABLE I. Conjectured global symmetries and massless fermion content of the nonsupersymmetric chiral gauge theories. Representations of massless fermions are shown in Young tableaux of unbroken non-Abelian groups. The SUSY+ AMSB column shows the exact results valid for $m \ll \Lambda$ which extend continuously to $m \gg \Lambda$ if there is no phase transition at $m \sim \Lambda$. The Tumbling column shows the hypotheses predicted by a sequential self-breaking of the gauge group by fermion bilinear operators in the MAC.

UV theory		SUSY + AMSB		Tumbling	
N_F	G_{global}	H_{global}	$m_f = 0$	H_{global}	$m_f = 0$
		$\mathrm{SU}(N_C) + A(\begin{smallmatrix}\Box\\\Box\end{smallmatrix}) + (N_C - 4 + N_F)\tilde{F}(\bar{\Box}) + N_F F(\Box)$, N_C =Odd [16,21]			
0	$\mathrm{SU}(N_C-4)\times \mathrm{U}(1)$	$\mathrm{Sp}(N_C-5)\times \mathrm{U}(1)$	$\Box + 1$	$\mathrm{SU}(N_C-4)\times \mathrm{U}(1)$	$\Box\Box$
1	$\mathrm{SU}(N_C-3)\times \mathrm{U}(1)^2$	$\mathrm{Sp}(N_C-3)\times \mathrm{U}(1)$	$\Box$	$\mathrm{SU}(N_C-4)\times \mathrm{U}(1)$	$\Box\Box$
2	$\mathrm{SU}(N_C-2)\times \mathrm{SU}(2)\times \mathrm{U}(1)^2$	$\mathrm{Sp}(N_C-3)\times \mathrm{U}(1)^2$	$\Box$	$\mathrm{SU}(N_C-4)\times \mathrm{SU}(2)\times \mathrm{U}(1)$	$(\Box\Box, 1)$
		$\mathrm{SU}(N_C) + A(\begin{smallmatrix}\Box\\\Box\end{smallmatrix}) + (N_C - 4 + N_F)\tilde{F}(\bar{\Box}) + N_F F(\Box)$, N_C =Even [16,21]			
0	$\mathrm{SU}(N_C-4)\times \mathrm{U}(1)$	$\mathrm{Sp}(N_C-4)$	None	$\mathrm{SU}(N_C-4)\times \mathrm{U}(1)$	$\Box\Box$
1	$\mathrm{SU}(N_C-3)\times \mathrm{U}(1)^2$	$\mathrm{Sp}(N_C-4)\times \mathrm{U}(1)$	None	$\mathrm{SU}(N_C-4)\times \mathrm{U}(1)$	$\Box\Box$
2	$\mathrm{SU}(N_C-2)\times \mathrm{SU}(2)\times \mathrm{U}(1)^2$	$\mathrm{Sp}(N_C-4)\times \mathrm{SU}(2)\times \mathrm{U}(1)$	None	$\mathrm{SU}(N_C-4)\times \mathrm{SU}(2)\times \mathrm{U}(1)$	$(\Box\Box, 1)$
		$\mathrm{SU}(N_C) + S(\Box\Box) + (N_C + 4)\tilde{F}(\bar{\Box})$ [19]			
0	$\mathrm{SU}(N_C+4)\times \mathrm{U}(1)$	$\mathrm{SO}(N_C+4)$	None	$\mathrm{SU}(N_C+4)\times \mathrm{U}(1)$	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$
		$\mathrm{SO}(10) + N_F\psi(\mathbf{16})$ [20]			
1	None	None	None	None	None
2	SU(2)	SU(2)	None	SO(2)	None
3	SU(3)	SO(3)	None	SO(3)	None
		$E_6 + N_F\psi(\mathbf{27})$ [This Work]			
1	None	None	None	None	None
2	SU(2)	None	None	SO(2)	None
3	SU(3)	SU(3)	$\Box\Box\Box$	SO(3)	None
		$\mathrm{U}(1)\times \mathrm{U}(1)$	S_{iii} $(i = 1, 2, 3)$		
		None	None		

the generic symmetry breaking pattern is then $E_6 \to F_4 \to \mathrm{SO}(8) \to \mathrm{SU}(3) \to 1$. In each case, we use gauge symmetry to rotate the fields to leave only the first 3×3 matrix nonzero, so that the D-flat configurations reside in the $(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$ part of the representation.

The result is summarized in Table I, together with other cases that have already been studied [16,19–21]. In most cases, the conjectured symmetry breaking pattern based on the tumbling hypothesis is not reproduced by the analysis based on SUSY and AMSB. Details follow in the next sections.

For the E_6 theories with N_F **27**'s, the tumbling hypothesis suggests $\mathrm{SU}(N_F)$ is broken to $\mathrm{SO}(N_F)$ with no massless fermions. The MAC is $(\psi^{\{i,}\psi^{j\}})(\mathbf{27}^*)$ whose VEV breaks E_6 to SO(10). Out of $\psi(\mathbf{27})$, the singlet pair condenses and $\mathbf{10} + \mathbf{16}$ remain. The next MAC is $(\psi^{\{i,}(\mathbf{10})\psi^{j\}}(\mathbf{10}))(\mathbf{1})$ which condenses without further breaking of the gauge group. Finally $(\psi^{\{i,}(\mathbf{16})\psi^{j\}}(\mathbf{16}))(\mathbf{10})$ condenses and breaks SO(10) to SO(9). The 16 is then a real representation and the SO(9) theory is vectorlike. The theory is gapped and there are no massless fermions.

What we find in this work is very different from what tumbling suggests. For $N_F = 1$, there is no global symmetry to speak of. For $N_F = 2$, tumbling suggests $\mathrm{SU}(2) \to \mathrm{SO}(2)$, while SUSY + AMSB suggests no global symmetry left. The case $N_F = 3$ is interesting. We found three local minima, each with different symmetry breaking patterns. Any one of them could persist to the non-SUSY limit $m \to \infty$. In particular, the case $N_F = 3$ with unbroken SU(3) with massless composite fermions in the $S_{ijk} = \boxed{i \,|\, j \,|\, k}$ representation was totally unanticipated.

III. ANOMALY MEDIATION

With AMSB, we infinitesimally break $\mathcal{N} = 1$ supersymmetry. All supersymmetry breaking is parametrized in terms of the Weyl compensator superfield $\Phi = 1 + \theta^2 m$ where θ is the usual Grassmann coordinate in superspace. All dimensionful fields in the supersymmetric Lagrangian are multiplied by the Weyl compensator, producing a new Lagrangian,

$$\mathcal{L} = \int d^4\theta \Phi^* \Phi K + \int d^2\theta \Phi^3 W + \text{H.c.}, \tag{6}$$

where K and W are the Kähler potential and superpotential, respectively. In a conformal theory, this compensator field could be removed by a simple rescaling of all fields by

$\psi \to \Phi^{-1}\psi$, leaving supersymmetry unbroken. However, once a mass scale is introduced in the theory, the Weyl compensator cannot be fully removed by a rescaling. This introduces a new term proportional to the SUSY breaking scale m into the tree-level Lagrangian,

$$\mathcal{L}_{\text{tree}} = m\left(\psi_i^\mu \frac{\partial W}{\partial \psi_i^\mu} - 3W\right) + \text{H.c.}, \tag{7}$$

in addition to the normal F-term and D-term scalar potentials.

It is often useful to require that the theory live along the D-flat directions, or equivalently at points in the moduli space where the D-term potential vanishes. This occurs when the following condition is satisfied:

$$D^a = \sum_i \phi_i^\dagger T^a \phi_i = 0. \tag{8}$$

With the addition of the new AMSB contribution, the potential now has a nonsupersymmetric vacuum and the squarks and gauginos gain nonzero VEVs.

IV. $N_F = 1$ CASE

For E_6 with one flavor, there is no continuous global symmetry. While there is a nonanomalous $\mathbb{Z}_6$ symmetry which acts on $\psi \to e^{2\pi i/6}\psi$. As long as $S \neq 0$, it is broken to $\mathbb{Z}_3$, which is identical to the center of the E_6 gauge group. Thus, there is no nontrivial global symmetry left. The F_4 gaugino condensate generates a nonperturbative superpotential for the theory. The exact form of the superpotential is uniquely fixed up to an overall scale factor to be a function of the gauge and global invariant combination of the fundamental field ψ. In terms of the gauge invariant IR field S, the superpotential takes the form

$$W_{N_F=1} = \left(\frac{\Lambda^{33}}{S^2}\right)^{1/9}. \tag{9}$$

A single D-flat direction breaks E_6 to F_4. Without loss of generality, we can restrict to the first 3×3 submatrix of the **27** of E_6. We use the $\text{SU}(3) \times \text{SU}(3)$ gauge symmetry to rotate the D-flat direction into the form

$$\langle \psi \rangle = \begin{pmatrix} v & & \\ & v & \\ & & v \end{pmatrix}. \tag{10}$$

In terms of v, the superpotential simplifies to

$$\langle W_{N_F=1} \rangle = \left(\frac{\Lambda^{33}}{6v^6}\right)^{1/9}. \tag{11}$$

The total potential including the terms from AMSB (see Fig. 1) is

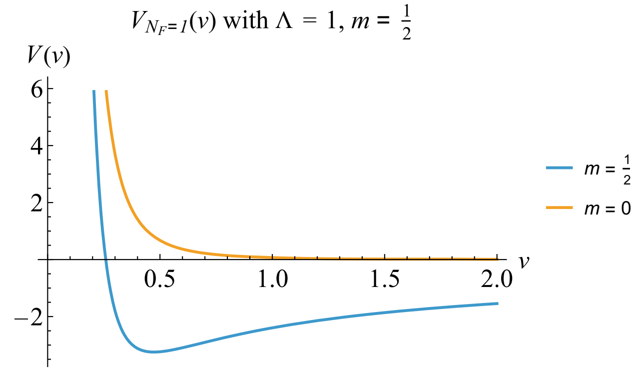

FIG. 1. A plot of the $N_F = 1$ scalar potential as a function of v with $\Lambda = 1$, and $m = 1/2$. The plot shows the potential with AMSB $m = 1/2$, and without AMSB $m = 0$.

$$\langle V_{N_F=1} \rangle = \frac{2 \times 6^{5/9}\Lambda^{22/3}}{81 v^{10/3}} - \frac{99 \times 6^{7/9} m \Lambda^{11/3}}{81 v^{2/3}}, \tag{12}$$

which can be easily minimized to find the VEV,

$$\langle v \rangle = \frac{2^{7/24}}{3^{5/6}} \left(\frac{5\Lambda^{11/3}}{11m}\right)^{3/8}. \tag{13}$$

Thus, the minimum value of the scalar potential is

$$\langle V_{N_F=1} \rangle_{\min} = -\frac{33 \times 3^{1/3}}{5 \times 2^{5/12}} \left(\frac{11 m^5 \Lambda^{11}}{4}\right)^{1/4}. \tag{14}$$

There is no global symmetry to check for the 't Hooft anomaly, so the mass spectrum only consists of the Higgs superfield which breaks supersymmetry. The corresponding fermion and scalar Higgs masses are

$$M_f^2 = \frac{121}{9} m^2, \qquad M_{b,\text{light}}^2 = \frac{242}{45} m^2, \qquad M_{b,\text{heavy}}^2 = \frac{968}{45} m^2, \tag{15}$$

which satisfies the supertrace rule.

V. $N_F = 2$ CASE

In this section, we discuss the features of E_6 with two fields in the fundamental representation. In addition to the gauge group E_6, there is also a global flavor symmetry $\text{SU}(2)_F$. We expect the gauge symmetry to break according to the following pattern,

$$E_6 \to \text{SO}(10) \times \text{U}(1)_1 \to \text{SO}(8) \times \text{U}(1)_1 \times \text{U}(1)_2. \tag{16}$$

The SO(8) gaugino condensate generates a nonperturbative superpotential fixed in terms of the gauge and global invariant S^4 to be of the form

TABLE II. Charges of the triplet of scalars in the embedding of SO(8) into a fundamental representation of E_6 for the case of $N_F = 2$ flavors. The gauge symmetry $U(1)_1 \times U(1)_2$ is part of the six-dimensional maximal torus of E_6.

	X^+	X^-	Y
$U(1)_1$	2	2	−4
$U(1)_2$	1	−1	0

$$W_{N_F=2} = \left(\frac{\Lambda^{30}}{S^4}\right)^{1/6}. \tag{17}$$

We construct the S^4 global invariant from the standard normalization of the $SU(2)_F$ Clebsch-Gordan coefficients. Since the only gauge invariant is the four-component S tensor, the minimum must break the global $SU(2)_F$ at least partially. In addition, the global symmetry suffers from the Witten anomaly as this theory has 27 massless fermions in the UV. Thus, we expect the ground state produced from AMSB to fully break the global symmetry.

To come up with a general D-flat parametrization, we consider the symmetry breaking pattern discussed above. In breaking $E_6 \to SO(10) \times U(1)_1$, the branching rule for **27** of E_6 is that $\mathbf{27} \to (\mathbf{16}, \mathbf{-1}) \oplus (\mathbf{10}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{-4})$, so the embedding of SO(10) into the fundamental representation of E_6 is accomplished with a spinor, vector, and scalar of SO(10). Branching rules down to SO(8) gives us $\mathbf{8}_v, \mathbf{8}_s, \mathbf{8}_c$ and three scalars. We can set the nontrivial representations of SO(8) to vanish in determining the general D-flat configuration, since the embedding of SO(8) into E_6 yields six SO(8) scalars in the two fundamental fields of E_6. The D-flat conditions from the remaining $U(1)_1 \times U(1)_2$ fix two of these scalars, leaving four fields to parametrize our four D-flat directions. Each triplet of scalars under SO(8) has the charges given in Table II.

The D-flat conditions for two pairs of these SO(8) singlets are then

$$\begin{aligned} 2|X^+|^2 + 2|X^-|^2 - 4|Y^2|^2 &= 0, \\ |X^+|^2 - |X^-|^2 &= 0, \end{aligned} \tag{18}$$

where $X^\pm, Y$ are doublets of the global $SU(2)_F$. Assuming $X^\pm, Y$ are real, the general solution is

$$X^+ = v\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}, \qquad X^- = v\begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix}, \qquad Y = v\begin{pmatrix}\cos\chi \\ \sin\chi\end{pmatrix}. \tag{19}$$

These scalars fit into the fundamental E_6 fields to produce the full D-flat configuration,

$$\langle\psi_1\rangle = v\begin{pmatrix}\cos\theta & & \\ & \cos\phi & \\ & & \cos\chi\end{pmatrix}, \qquad \langle\psi_2\rangle = v\begin{pmatrix}\sin\theta & & \\ & \sin\phi & \\ & & \sin\chi\end{pmatrix}. \tag{20}$$

Using the global $SU(2)_F$ flavor symmetry, we rotate the field χ to 0 without loss of generality.

In terms of this D-flat parametrization, the superpotential becomes

$$\langle W_{N_F=2}\rangle = \left(\frac{315}{7\sqrt{3} - 3\sqrt{7}}\right)^{1/6} \times \frac{\Lambda^5}{v^2(\sin(\theta-\phi)\sin(\theta)\sin(\phi))^{1/3}}. \tag{21}$$

The scalar potential thus takes the form

$$\langle V_{N_F=2}\rangle = \frac{35^{1/6}}{3\times 3^{2/3}}\left[\sqrt{2}(15(5+\sqrt{21}))^{1/6}\frac{\Lambda^{10}(2+\csc^2(\theta-\phi)+\csc^2(\theta)+\csc^2(\phi))}{v^6(\sin(\theta-\phi)\sin(\theta)\sin(\phi))^{2/3}} - \frac{90m\Lambda^5}{(7\sqrt{3}-3\sqrt{7})^{1/6}v^2}(\sin(\theta-\phi)\sin(\theta)\sin(\phi))^{-1/3}\right]. \tag{22}$$

We find a unique minimum where $\langle\theta\rangle = \pi/3$ and $\langle\phi\rangle = 2\pi/3$, and

$$\langle v\rangle = \frac{2^{7/16}(7(5+\sqrt{21}))^{1/48}}{3^{1/16}\times 5^{5/24}}\left(\frac{\Lambda^5}{m}\right)^{1/4}. \tag{23}$$

The $\langle S_{111}\rangle$ and $\langle S_{122}\rangle$ components are nonvanishing in this case, breaking the global $SU(2)_F$ fully. We also find no massless fermions in the spectrum in Table III since the global symmetry fully breaks so the 't Hooft anomaly and Witten anomaly are trivially satisfied.

Figure 2 shows the geometry of the scalar potential after introducing the AMSB term. In the angular (θ, ϕ) space,

TABLE III. Masses of the D-flat directions for $N_F = 2$ flavors. We have divided up the masses of the scalars to those heavier than the fermions and lighter than the fermions, denoted by $M^2_{b,\text{heavy}}$ and $M^2_{b,\text{light}}$. Higgs refers to the supermultiplet in the field direction of the VEV. The global $\mathrm{SU}(2)_F$ symmetry is fully broken, so that there are three Nambu-Goldstone bosons in the mass spectrum. In addition, there are no massless fermions which we attribute to the 't Hooft anomaly and Witten anomaly being trivially satisfied. The supertrace also vanishes as expected.

	Higgs	$m_{(NG)}$	$m_f = 0$	m_s
		$\mathrm{SU}(2)_F \to 1$		
$M^2_{b,\text{heavy}}$	$\frac{100}{3}m^2$	$\frac{50}{9}m^2$	$\varnothing$	$\varnothing$
M^2_f	$25m^2$	$\frac{25}{9}m^2$	$\varnothing$	$\varnothing$
$M^2_{b,\text{light}}$	$\frac{50}{3}m^2$	0	$\varnothing$	$\varnothing$
1	1	3	$\varnothing$	$\varnothing$

the scalar potential is divided into triangular subregions bounded by enhanced SO(9) gauge symmetric parametrizations. The potential is periodic by π in both θ and ϕ, and exhibits a reflection symmetry $\theta \leftrightarrow \phi$. There is then a unique minimum at $\theta = \pi/2$ and $2\pi/3$, which corresponds geometrically to the centroid of the upper left triangle in Fig. 2.

In addition to the global $\mathrm{SU}(2)_F$ symmetry which is completely broken, there is a discrete $\mathbb{Z}_{12}$ symmetry where $\psi_{1,2} \to e^{2\pi i/12}\psi_{1,2}$. With $S \neq 0$, it is broken to $\mathbb{Z}_3$, which is identical to the center of the E_6 gauge group. Therefore, there is no nontrivial global symmetry left.

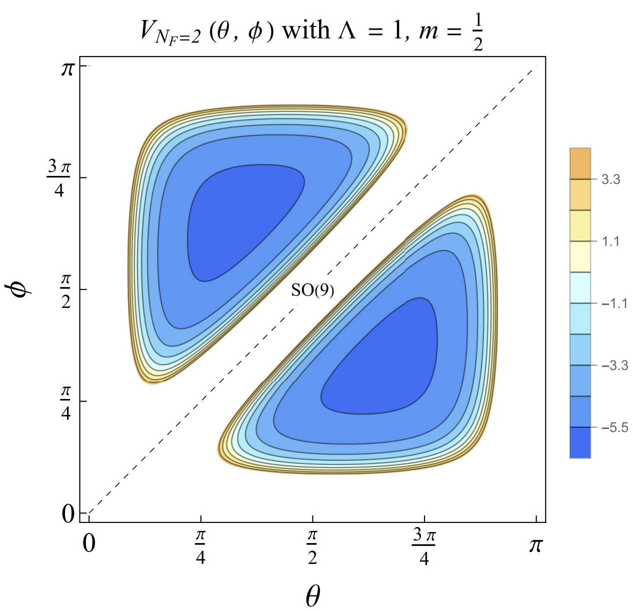

FIG. 2. A plot of the $N_F = 2$ scalar potential over the angular coordinates θ, ϕ with $\Lambda = 1$, and $m = 1/2$. The dashed line indicates the region where the theory exhibits an enhanced SO(9) symmetry.

VI. $N_F = 3$ CASE

In this section, we discuss the case with a gauged E_6 and 3 fields in the fundamental representation, producing a global $\mathrm{SU}(3)_F$ flavor symmetry. In addition to the usual S invariant used in the previous cases, there is also a new invariant T as shown in Eq. (3). The superpotential is now generated by the SU(3) gaugino condensate and in terms of S and T. There are three combinations of S and T that are gauge and global invariant and with the same $\mathrm{U}(1)_R$ charge: T^3, TS^4, and S^6. All of these terms appear in the superpotential,

$$W_{N_F=3} = \left(\frac{\Lambda^{27}}{aT^3 + bTS^4 + cS^6}\right)^{1/3}, \tag{24}$$

where $a = 1, b = 6\sqrt{3}, c = -28\sqrt{3}/5$ are coefficients determined by requiring the superpotential to blow up when the gauge symmetry doesn't fully break to SU(3) (see Appendix A for additional details). Using the Hilbert Series technique, we know that S^4 and S^6 are the only unique global invariants produced from the S invariant. T is a singlet of the global $\mathrm{SU}(3)_F$; a vacuum which only gives T a VEV would preserve the global symmetry, thereby requiring a nontrivial 't Hooft anomaly matching condition.

There are eleven total D-flat directions which break E_6 down to SU(3). These match the eleven independent parameters found in the invariant polynomials, one in the singlet T and ten in S. We describe a partial 10-dimensional parametrization of these D-flat directions in Appendix B. We could not find a closed form 11-dimensional parametrization as the D-flat constraints yield a nontrivial set of algebraic equations. However we found that a three-dimensional submanifold sufficed for finding possible ground states of the theory. Additional testing through numerical methods presented in Sec. VII justifies this simplification.

Since we are unable to find a full parametrization of the D-flat directions, we start with a single parameter that corresponds to the parametrizing the T tensor. From here, we can add two more parameters that act as spherical parameters that tune S^4 and S^6. This parametrization is given below,

$$\langle\psi_1\rangle = v\begin{pmatrix} \cos\theta & & \\ & \sin\theta\sin\phi & \\ & & \sin\theta\cos\phi \end{pmatrix},$$

$$\langle\psi_2\rangle = v\begin{pmatrix} & & -\sin\theta\sin\phi \\ \sin\theta\cos\phi & & \\ & \cos\theta & \end{pmatrix},$$

$$\langle\psi_3\rangle = v\begin{pmatrix} & \sin\theta\cos\phi & \\ & & \cos\theta \\ \sin\theta\sin\phi & & \end{pmatrix}. \tag{25}$$

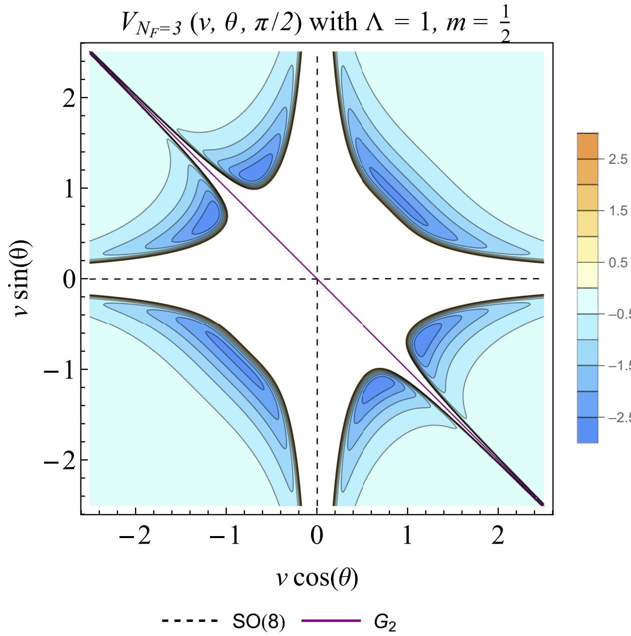

FIG. 3. A plot of the $N_F = 3$ scalar potential over the coordinates v, θ with $\phi = \pi/2$, $\Lambda = 1$, and $m = 1/2$. The dashed and solid lines indicate regions of enhanced symmetry, splitting off the potential into separate regions that each contain a minimum.

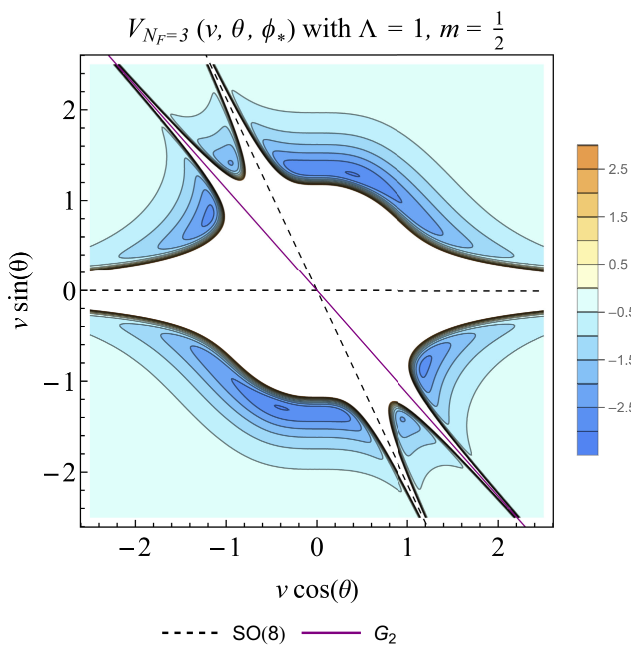

FIG. 4. A plot of the $N_F = 3$ scalar potential over the coordinates v, θ with $\phi_* \approx 0.57827$, $\Lambda = 1$, and $m = 1/2$. The dashed and solid lines indicate regions of enhanced symmetry, splitting off the potential into separate regions that each contain a minimum. This particular cross section of the moduli space contains the deepest vacuum state with a value of around $V = -3.01811$.

Unlike most theories in which SUSY is broken via AMSB, we find three vacua with different symmetry breaking patterns. Moreover, all of these cases satisfy 't Hooft anomaly matching conditions, so these minima are candidates in the non-SUSY limit.

We ascribe the multiple distinct vacua to the structure of the superpotential in Eq. (24). The singular nature of the superpotential along enhanced symmetry regions corresponding to SO(8) and G_2 (see Figs. 3 and 4) separates the scalar potential into three different regions. In each of these regions, AMSB can produce at most a single vacuum, and in our case of an E_6 gauge theory with $N_F = 3$ flavors, we have found a maximal set of three valid vacua.

A. Unbroken global SU(3) vacuum

The vacuum that leaves the global symmetry fully unbroken corresponds to a vanishing S invariant. Since the T invariant is transforms trivially under the flavor symmetry when $N_F = 3$, a VEV parametrized by T can lead to a potential vacuum that leaves the global symmetry unaffected upon integrating out the broken gauge superfields. A nontachyonic mass spectrum and matching 't Hooft anomaly further vindicates this vacuum as a valid minimum.

We can achieve a vanishing S invariant from Eq. (25) by setting $\langle\theta\rangle = \pi/4$ and $\langle\phi\rangle = \pi/2$. Along this D-flat direction, the T invariant is parametrized by the scale of the UV fields by $\langle T\rangle = v^6/2$. Since $\langle S^4\rangle = \langle S^6\rangle = 0$ here, the superpotential takes a more convenient form,

$$\langle W_{N_F=3}\rangle|_T = \frac{\Lambda^9}{a^{1/3}T} = \frac{2\Lambda^9}{a^{1/3}v^6}. \tag{26}$$

The scalar potential then has the explicit form,

$$\langle V_{N_F=3}\rangle|_T = \frac{48\Lambda^{18}}{a^{2/3}v^{14}} - \frac{36m\Lambda^9}{a^{1/3}v^6}, \tag{27}$$

which can be quickly minimized to obtain the VEV

$$\langle v\rangle|_T = \left(\frac{28\Lambda^9}{9a^{1/3}m}\right)^{1/8}. \tag{28}$$

In the limit $m \ll \Lambda$, the VEV becomes parametrically large, so that our analysis starting with the UV fields is justified over working directly with the gauge invariants. The vacuum energy in this case is

$$\begin{aligned}\langle V_{N_F=3}\rangle|_{T,\min} &= -\frac{108\sqrt{6}}{7^{7/4}a^{1/12}}(m^7\Lambda^9)^{1/4} \\ &= -8.78168\ldots \times (m^7\Lambda^9)^{1/4}. \end{aligned} \tag{29}$$

At this vacuum, the global symmetry is left fully unbroken, so the mass spectra and 't Hooft anomaly should agree with this symmetry pattern. We find ten massless fermions in the spectrum, which correspond to the S tensor due to the lack of T in the superpotential along this direction. Indeed, the UV anomaly $\mathcal{A}_{\rm UV} = 27$ matches the IR anomaly for the massless **10** of $\mathrm{SU}(3)_F$ with $\mathcal{A}_{\rm IR} = (3+3)(3+6)/2 = 27$.

B. Unbroken global $\mathbf{U(1)\times U(1)}$ vacuum

The second vacuum is less trivial and breaks the global symmetry partially from $\mathrm{SU}(3)_F$ to $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$. In this case, the global invariants S^4 and S^6 are algebraically dependent through the polynomial $P(S^4, S^6) = (S^4)^{1/4} - \alpha(S^6)^{1/6}$ for some constant α. Moreover, the T invariant is nonvanishing, so all terms in the denominator of the superpotential equation (24) contribute towards the scalar potential.

We can achieve a partially broken vacuum of $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ with a nonzero VEV for only $\langle S_{123}\rangle$ of the S invariant. With our parametrization in Eq. (25), we set $\langle\phi\rangle = \pi/2$ to produce this desired behavior. From our normalization convention described in Appendix A, the global invariants are parametrized by v, θ via

$$\langle S^4\rangle_{\phi=\pi/2} = -\frac{v^{12}}{18\sqrt{3}}(\cos^3\theta - \sin^3\theta)^4,$$
$$\langle S^6\rangle_{\phi=\pi/2} = -\frac{5v^{18}}{378\sqrt{3}}(\cos^3\theta - \sin^3\theta)^6,$$
$$\langle T\rangle_{\phi=\pi/2} = \frac{v^6}{48}(10 + 6\cos 4\theta + 15\sin 2\theta - 5\sin 6\theta). \qquad (30)$$

Along these D-flat directions, the superpotential and scalar potential are

$$\langle W_{N_F=3}\rangle_{\phi=\pi/2} = \frac{\Lambda^9 2^{2/3}\csc^2 2\theta}{v^6(\cos^3\theta + \sin^3\theta)^{2/3}}$$
$$\langle V_{N_F=3}\rangle_{\phi=\pi/2} = \frac{\Lambda^{18}\csc^{16}\theta\sec^6\theta}{192\times 2^{2/3}v^{14}(1+\cot^3\theta)^{10/3}} \times (55 + 12\cos 4\theta - 3\cos 8\theta + 24\sin 2\theta - 8\sin 6\theta) - \frac{9m\Lambda^9}{2^{1/3}v^6}\frac{\csc^4\theta\sec^2\theta}{(1+\cot^3\theta)^{10/3}}. \qquad (31)$$

With this form of the scalar potential, we can numerically minimize over the parameters v, θ to obtain a vacuum solution with

$$\langle v\rangle = \left(5.06183\ldots\times\frac{\Lambda^9}{m}\right)^{1/8}, \quad \langle\theta\rangle = 2.61549\ldots, \qquad (32)$$

where $\langle\theta\rangle$ does not depend on the particular value of m or Λ. The vacuum energy in this case has the form

$$\langle V_{N_F=3}\rangle_{\min} = -9.91804\ldots\times(m^7\Lambda^9)^{1/4}. \qquad (33)$$

TABLE IV. This table lists the masses of the D-flat directions as well as their representations under the global symmetries at tree level for each of the three vacua. For those with no non-Abelian global symmetries remaining, the number instead denotes the multiplicity of the fields in each category. We have divided up the masses of the scalars to those heavier than the fermions and lighter than the fermions, denoted by $M^2_{b,\text{heavy}}$ and $M^2_{b,\text{light}}$. Higgs refers to the supermultiplet in the field direction of the VEV. NG refers to Nambu-Goldstone supermultiplets for spontaneously broken global symmetries. Only the first two vacua have massless fermions; their scalar partners acquire positive mass squared from two-loop AMSB. The second vacua has one fermion and two scalars that were neither rational fractions nor proportional to common irrational numbers up to high numerical precision; these masses together fulfill the sum rule, a nontrivial test.

	Higgs	$m_{(NG)}$	$m_f = 0$	m_s
		$\mathrm{SU}(3)_F \to \mathrm{SU}(3)_F, V = -2.61081\ldots$		
$M^2_{b,\text{heavy}}$	$\frac{648}{7}m^2$	∅	0	∅
M^2_f	$81m^2$	∅	0	∅
$M^2_{b,\text{light}}$	$\frac{486}{7}m^2$	∅	0	∅
$\mathrm{SU}(3)_F$	1	∅	$\square\square\square$	∅
		$\mathrm{SU}(3)_F \to \mathrm{U}(1)_1\times\mathrm{U}(1)_2, V = -2.61549\ldots$		
$M^2_{b,\text{heavy}}$	$\frac{648}{7}m^2$	$\frac{162}{49}m^2$	0	$\approx 15.53m^2$
M^2_f	$81m^2$	$\frac{81}{49}m^2$	0	$\approx 11.22m^2$
$M^2_{b,\text{light}}$	$\frac{486}{7}m^2$	0	0	$\approx 6.91m^2$
$\mathrm{U}(1)_1\times\mathrm{U}(1)_2$	1	6	3	1
		$\mathrm{SU}(3)_F \to 1, V = -3.01811\ldots$		
$M^2_{b,\text{heavy}}$	$\frac{648}{7}m^2$	$\frac{162}{49}m^2$	∅	$\frac{486}{49}m^2$
M^2_f	$81m^2$	$\frac{81}{49}m^2$	∅	$\frac{324}{49}m^2$
$M^2_{b,\text{light}}$	$\frac{486}{7}m^2$	0	∅	$\frac{162}{49}m^2$
1	1	8	∅	2

TABLE V. Charges of the unbroken $U(1)_1 \times U(1)_2$ global symmetry for the massless UV and IR fermionic modes.

	ψ_1	ψ_2	ψ_3	S_{111}	S_{222}	S_{333}
$U(1)_1$	0	−1	1	0	−1	1
$U(1)_2$	2	−1	−1	6	−3	−3

The corresponding mass spectra is shown in the second section of Table IV where we see that the supertrace rule is satisfied and that there are no tachyons in the spectrum.

The less trivial check that this solution is a candidate vacuum is to show that the 't Hooft anomaly matches in the UV and IR. The unbroken global symmetry $U(1)_1 \times U(1)_2$ correspond to the Cartan generators of $SU(3)_F$. We have chosen our parametrization, Eq. (25), to already be diagonal under the $U(1)_1 \times U(1)_2$ subgroup, so that our massless UV and IR fields transform as charges given in Table V, from which we see that all of the 't Hooft anomalies match.

C. Fully broken global symmetry vacuum

The final vacuum solution has vanishing T and S^4 global invariants, fully breaking the global $SU(3)_F$ symmetry. Thus, only the S^6 global invariant acquires a nonzero VEV, and in this case we obtain the deepest scalar potential value at tree level for our choice of m and Λ.

We numerically minimize (see Sec. VII) the scalar potential over all three parameters v, θ, ϕ in Eq. (25) and obtain

$$\langle v\rangle = \left(5.55590\ldots \times \frac{\Lambda^9}{m}\right)^{1/8},$$
$$\langle\theta\rangle = 1.27330\ldots, \qquad \langle\phi\rangle = 2.05978\ldots, \tag{34}$$

with a corresponding vacuum energy of

$$\langle V_{N_F=3}\rangle_{\min} = -10.15167\ldots \times (m^7\Lambda^9)^{1/4}. \tag{35}$$

The analytical expression for the global invariants, superpotential, and scalar potential are lengthy and can be found in Appendix C.

We observe no massless fermions and the 't Hooft anomaly is satisfied in this vacuum, since the $SU(3)_F$ global symmetry of the theory is fully broken. The mass spectra obtained from our numerical analysis are shown in Table IV. Once again, we can see that the supertrace is satisfied, further verifying this as a candidate vacuum solution.

VII. NUMERICS

For $N_F \leq 2$ the relatively low dimensional moduli spaces allow for analytical methods. However in the case of $N_F = 3$ with the introduction of the T gauge invariant, we employ numerical techniques. We use the Python wrapper of `SymEngine`, a C++ backend symbolic manipulation library, and the *Mathematica* package GroupMath for the construction of our gauge invariants. We parse the GroupMath output using `ReGex` and create symbolic expressions in Python to differentiate and write our superpotential and thus our scalar potential.

To find an explicit expression for the T invariant, we use the definition in Eq. (4) in terms of the primitive invariant $d_{\mu\nu\lambda}$ of E_6, whose definition we use from E6Tensors [29]. We can then export this tensor as a `CSV` file into Python and continue our analysis with `SymEngine`.

We use the Broyden-Fletcher-Goldfarb-Shanno algorithm to minimize the scalar potential. To reduce computational complexity, we restrict the number of chiral superfields to the first nine components of each of our UV fields as described in Sec. II. Minimizing our scalar potential gives us numerical values for these UV fields. From here, we substitute this numerical VEV into the symbolic second derivative expression of the superpotential to obtain the fermion masses.

The complexity of our gauge invariants made obtaining scalar and pseudoscalar masses computationally challenging, so we outline a more efficient method. In this approach, we construct the superpotential and scalar potential by treating the global invariants as `SymEngine` functions of the UV fields. This enables us to treat the superpotential and scalar potential as functionals of the global invariants and differentiate implicitly twice with respect to the UV fields. Once these symbolically differentiated expressions are obtained, we evaluate the global invariants and their derivatives along a given VEV. We can reduce the amount of computations further by invoking the commutativity of derivatives. Finally, we can substitute the VEV into the second derivative of the scalar potential, where again we utilize the symmetric exchange of derivatives to decrease the computational steps.

We found this approach to be significantly more efficient compared to writing the superpotential and scalar potential explicitly in terms of the UV fields. The structure of group invariants for linear representations are polynomials, so that derivatives of these global invariants only involve the power rule. Hence, the large number of 818,100 nonvanishing terms of the T invariant does not pose an issue with runtime. The remaining derivatives only involve the global invariants, which takes minimal computation time.

With this technique, we obtain numerical mass spectra which appear in Table IV. We verify the analytical value of these masses to 512 binary bits of precision which correspond roughly to 154 decimals of precision. We perform the same numerical precision in all parts of our numerical calculations including the supertrace check. This computational approach to obtaining numerical solutions can be extended to other gauge theories as a useful tool alongside analytical methods, especially when the gauge theory and matter representations pose a high-dimensional problem that makes full analytical techniques difficult to implement.

We also include a short animation of the $N_F = 3$ potential that interpolates between Figs. 3 and 4 in an online repository [30]. We see that from these three D-flat directions in Eq. (25) there are indeed only three distinct minima for this configuration, which are split by the enhanced symmetry regions corresponding to unbroken SO(8) and G_2 gauge symmetry.

VIII. CONCLUSIONS

In this paper, we obtained exact solutions to chiral E_6 gauge theories with $N_F = 1, 2, 3$ matter fields in the **27** representation when supersymmetry is perturbed by a small supersymmetry breaking $m \ll \Lambda$ via anomaly mediation. The main results are summarized in Table I. Combined with all other results obtained using this method to date, we find quite different patterns of symmetry breaking and massless composite spectra from what could be conjectured based on the tumbling hypothesis in the nonsupersymmetric limit.

Even though there is a wealth of evidence that the exact solutions for $m \ll \Lambda$ smoothly connect to the nonsupersymmetric limit $m \gg \Lambda$ for vectorlike gauge theories [31], there could still be phase transitions between the two limits in these chiral gauge theories. Nonetheless, we would like to emphasize that naively taking the limit $m \to \infty$ does give us a self-consistent conjecture on nonsupersymmetric dynamics and it is worth exploring.

Another important application is to stay within the validity range $m \ll \Lambda$ and use it to build models of physics beyond the standard model at high energies, where supersymmetry could indeed be present with a small supersymmetry breaking. In this case, we can employ the exact solutions we have directly, to address problems such as the composite axion for the Peccei-Quinn quality problem, dark matter with composite dynamics, composite inflaton field, nonperturbative dynamics for baryogenesis, etc.

Finally, it would be useful to extend the analysis to more general types of supersymmetry breaking. We would like to give masses to both gauginos and scalars to approach the nonsupersymmetric theory and the anomaly mediation provides them in a UV-insensitive fashion. On the other hand, D-terms [both conventional U(1) as well as $U(1)_R$] are also UV-insensitive, while they do not generate gaugino masses and hence they are not sufficient for the purpose. Yet one can study the combination of them to see if there is an interesting phase diagram to be explored. In addition, some theories can incorporate a tree-level superpotential to explicitly break (a part of) the symmetries that could also provide a rich phase diagram to be studied. We hope to explore both of these directions in the near future to understand the big picture on dynamics of chiral gauge theories.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: FIXING SUPERPOTENTIAL COEFFICIENTS FOR $N_F = 3$

The superpotential generated by the SU(3) gaugino condensate is of the form,

$$W_{N_F=3} = \left(\frac{\Lambda^{27}}{aT^3 + bTS^4 + cS^6}\right)^{1/3}, \tag{A1}$$

where a, b, c are coefficients that we determine from the condition that the superpotential is poorly behaved when the gauge symmetry does not fully break to SU(3).

We determine the a, b, c relative coefficients using two different D-flat configurations that preserve a larger gauge symmetry than SU(3). One configuration that breaks E_6 to G_2 is the following,

$$\langle\psi_1\rangle = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \qquad \langle\psi_2\rangle = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix},$$
$$\langle\psi_3\rangle = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}. \tag{A2}$$

The G_2 configuration have nonvanishing $\langle T\rangle$ and $\langle S_{123}\rangle$ components so that we obtain a nontrivial relation in setting the denominator of the superpotential equation (A1) to vanish.

A less trivial configuration that breaks E_6 to SO(8) is

$$\langle\psi_1\rangle = \begin{pmatrix} u & x & x \\ & & \\ & & \end{pmatrix}, \qquad \langle\psi_2\rangle = \begin{pmatrix} v & y & y \end{pmatrix},$$

$$\langle\psi_3\rangle = \begin{pmatrix} & & \\ & & \\ w & z & z \end{pmatrix}, \tag{A3}$$

and where the D-flat constraints require that $y^2 + u^2 = x^2 + v^2$ and $z^2 + u^2 = x^2 + w^2$.

Our choice of normalization convention for T, S^4, and S^6 is based on the GroupMath package. We generate the T tensor through its definition in Eq. (3) including a factor of $1/36$ to account for the 36 terms of products of S to match the symmetry pattern in Eq. (5). The S^4 and S^6 invariants are produced from the canonical definition of S in Eq. (2) and plugged into GroupMath to yield SU(3) invariant tensors.

With these normalizations and testing these enhanced symmetry configurations, we obtain relative coefficients

$$b = 6\sqrt{3}a, \qquad c = -\frac{28\sqrt{3}}{5}a, \tag{A4}$$

and from now on we set $a = 1$ as a convention. This exact value can be determined through an exact gaugino condensate calculation, but we find that this is not necessary for the rest of this computation.

APPENDIX B: 10D PARAMETRIZATION OF MODULI SPACE

The case of $N_F = 3$ flavors of E_6 has eleven D-flat directions. Here, we describe how to write an explicit parametrization of ten of these D-flat directions. Let us start by embedding nine vectors in $\mathbb{R}^3$ labeled $\mathbf{l}_i$, $\mathbf{m}_j$, $\mathbf{n}_k$ with $i = 1, 2, 3$ into the $(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$ block of three $\mathbf{27}$s of E_6 as

$$\langle\psi_i\rangle = \begin{pmatrix} \mathbf{l}_1 & \mathbf{m}_3 & \mathbf{n}_2 \\ \mathbf{n}_3 & \mathbf{l}_2 & \mathbf{m}_1 \\ \mathbf{m}_2 & \mathbf{n}_1 & \mathbf{l}_3 \end{pmatrix}. \tag{B1}$$

We simplify our analysis by setting $\mathbf{l}_3$, $\mathbf{m}_3$, $\mathbf{n}_3$ to vanish. The D-flat conditions then restrict the form of the remaining vectors for $i \neq j$,

$$\mathbf{l}_i \cdot \mathbf{m}_j = \mathbf{m}_i \cdot \mathbf{n}_j = \mathbf{n}_i \cdot \mathbf{l}_j = 0$$
$$\sum_{i=1}^{2} \mathbf{l}_i \cdot \mathbf{l}_i = \sum_{i=1}^{2} \mathbf{m}_i \cdot \mathbf{m}_i = \sum_{i=1}^{2} \mathbf{n}_i \cdot \mathbf{n}_i. \tag{B2}$$

These constraints can be solved for analytically using *Mathematica* to produce a 10-dimensional parametrization of the D-flat directions for E_6 with three $\mathbf{27}$s. To do this, we set the length of $|\mathbf{l}_i| = |\mathbf{m}_i| = |\mathbf{n}_i|$ and then impose orthogonality conditions. We satisfy the orthogonality constraints by letting $\mathbf{l}_i$ be general vectors, and set the complement $\mathbf{m}_j$ and $\mathbf{n}_j$ with $i \neq j$ to span the orthogonal plane to $\mathbf{l}_i$. We can then solve the remaining orthogonality condition $\mathbf{m}_i \cdot \mathbf{n}_j = 0$ explicitly in this reduced parameter space. This yields a seven-dimensional parametrization of the D-flat directions, but we can include three additional parameters corresponding to the global $\mathrm{SU}(3)_F$ symmetry.

Adding an eleventh parameter to the D-flat parametrization requires one of $\mathbf{l}_3$, $\mathbf{m}_3$, $\mathbf{n}_3$ to be nonzero. This greatly increases the number of orthogonality conditions, leading to equations that are much more difficult to separate and solve. We do not pursue a full parametrization because of this complication, and numerical checks improve our confidence that an entire parametrization of D-flatness is unnecessary for our analysis.

APPENDIX C: GLOBAL INVARIANTS FOR $N_F = 3$ FULLY BROKEN GLOBAL SYMMETRY CASE

In the E_6 gauge theory with $N_F = 3$ flavors, we can use the VEV in Eq. (25) to find the resultant global invariants, as well as the superpotential and scalar potential, explicitly. These expressions are analytical, but lengthy, so we have provided their form in this appendix. For brevity, we write $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$. With these simplifications, the analytical expression for the global invariants are

$$\langle T\rangle = \frac{2}{3}v^6(5s_{2\theta}^3 s_\phi^3 + s_\theta^6(5s_{2\phi}^3 + 4(s_\phi^6 + c_\phi^6)) - 40s_\theta^3 c_\theta^3 c_\phi^3 + 4c_\theta^6),$$

$$\langle S^4\rangle = \frac{v^{12}}{36\sqrt{3}}\left(512s_\theta^9 c_\theta^3(s_\phi^9 - c_\phi^9 + 57s_\phi^3 c_\phi^6 - 57s_\phi^6 c_\phi^3) + 57s_{2\theta}^6 s_{2\phi}^3 - \frac{1}{2}s_\theta^{12}(4s_\phi^3 - 3c_\phi - c_{3\phi})^4 - 96s_\theta^6 c_\theta^6(3c_{4\phi}+5)\right.$$
$$\left. - 512s_\theta^3 c_\theta^9(c_\phi^3 - s_\phi^3) - 128c_\theta^{12}\right),$$

$$\langle S^6\rangle = \frac{1280v^{18}}{189\sqrt{3}}\left(\frac{15}{16}s_\theta^6 c_\theta^{12}(17s_{6\phi} - 51s_{2\phi} - 6c_{4\phi} - 10) + 6s_\theta^3 c_\theta^{15}(s_\phi^3 - c_\phi^3) + 20s_\theta^9 c_\theta^9(s_\phi^9 - c_\phi^9 - 78s_\phi^3 c_\phi^6 + 78s_\phi^6 c_\phi^3)\right.$$
$$+ \frac{s_\theta^{18}}{4096}(4s_\phi^3 - 3c_\phi - c_{3\phi})^6 - \frac{3}{256}s_\theta^{15}c_\theta^3(-4s_\phi^3 + 3c_\phi + c_{3\phi})^3(66s_{2\phi} - 22s_{6\phi} + 3c_{4\phi} + 5)$$
$$\left. - \frac{3}{512}s_\theta^{12}c_\theta^6(9360s_{2\phi} - 520s_{6\phi} - 1560s_{10\phi} - 6330 + 12465c_{4\phi} - 4326c_{8\phi} + 751c_{12\phi}) - c_\theta^{18}\right). \quad \text{(C1)}$$

The superpotential takes a rather simple form,

$$\langle W\rangle = \frac{2^{5/3}\Lambda^9}{v^6 s_\theta^2[(c_\phi^3 + s_\phi^3)^2(c_\theta^3 - c_\phi^3 s_\theta^3)^2(c_\theta^3 + s_\theta^3 s_\phi^3)^2]^{1/3}}. \quad \text{(C2)}$$

In terms of the superpotential, the scalar potential becomes

$$\langle V\rangle = -\frac{v^{16}\langle W\rangle^4 s_\theta^4}{48\Lambda^{27}}[c_\theta^{12}(27mv^2c_\phi^6 s_\theta^2 + 54mv^2c_\phi^3 s_\theta^2 s_\phi^3 - 4\langle W\rangle c_\phi^4 + 27mv^2 s_\theta^2 s_\phi^6 - 4\langle W\rangle s_\phi^4)$$
$$+ 2c_\theta^9 s_\theta^3(-27mv^2c_\phi^9 s_\theta^2 - 27mv^2c_\phi^6 s_\theta^2 s_\phi^3 + 27mv^2c_\phi^3 s_\theta^2 s_\phi^6 + 8\langle W\rangle c_\phi^7 + 27mv^2 s_\theta^2 s_\phi^9 - 8\langle W\rangle s_\phi^7)$$
$$+ c_\theta^6 s_\theta^6(8\langle W\rangle c_\phi^4 s_\phi^6 - 16\langle W\rangle s_\phi^{10} + 2c_\phi^6(4\langle W\rangle s_\phi^4 - 81mv^2 s_\theta^2 s_\phi^6) + 2c_\phi^3(8\langle W\rangle s_\phi^7 - 27mv^2 s_\theta^2 s_\phi^9)$$
$$+ 16\langle W\rangle c_\phi^7 s_\phi^3 + 27mv^2c_\phi^{12}s_\theta^2 - 16\langle W\rangle c_\phi^{10} + 27mv^2 s_\theta^2 s_\phi^{12} - 54mv^2c_\phi^9 s_\theta^2 s_\phi^3)$$
$$- 2c_\theta^3 c_\phi^3 s_\theta^9 s_\phi^3(-27mv^2c_\phi^9 s_\theta^2 - 27mv^2c_\phi^6 s_\theta^2 s_\phi^3 + c_\phi^3(27mv^2 s_\theta^2 s_\phi^6 - 8\langle W\rangle s_\phi^4) + 8\langle W\rangle c_\phi^4 s_\phi^3$$
$$+ 16\langle W\rangle c_\phi^7 + 27mv^2 s_\theta^2 s_\phi^9 - 16\langle W\rangle s_\phi^7)$$
$$+ c_\phi^4 s_\theta^{12} s_\phi^4(c_\phi^8(27mv^2 s_\theta^2 s_\phi^2 - 4\langle W\rangle) + c_\phi^5(54mv^2 s_\theta^2 s_\phi^5 - 16\langle W\rangle s_\phi^3) - 16\langle W\rangle c_\phi^6 s_\phi^2$$
$$+ c_\phi^2(27mv^2 s_\theta^2 s_\phi^8 - 16\langle W\rangle s_\phi^6) - 16\langle W\rangle c_\phi^3 s_\phi^5 - 4\langle W\rangle s_\phi^8)$$
$$- 16\langle W\rangle c_\theta^{10} s_\theta^2(c_\phi^3 + s_\phi^3)^2 + 16\langle W\rangle c_\theta^7 s_\theta^5(c_\phi^3 - s_\phi^3)(c_\phi^3 + s_\phi^3)^2 - 4\langle W\rangle c_\theta^4 s_\theta^8(c_\phi^6 - s_\phi^6)^2]. \quad \text{(C3)}$$

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