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ANALYTIC CRACK KINKING TRANSFORMATIONS FOR WILLIAMS EXPANSION AND PRESSURIZED CRACK

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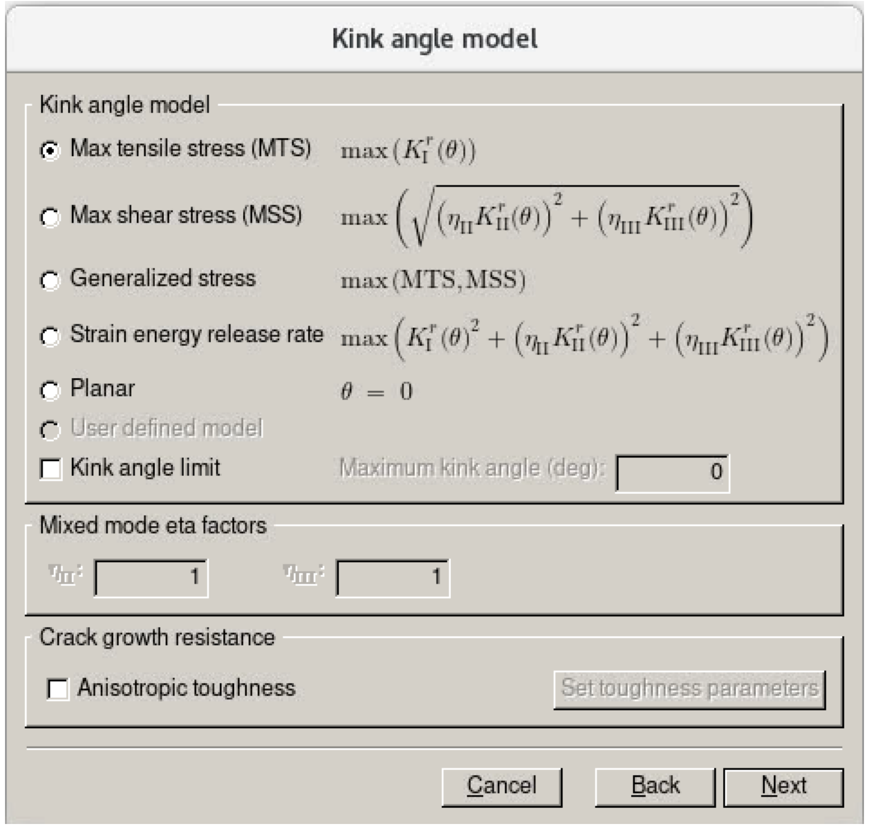
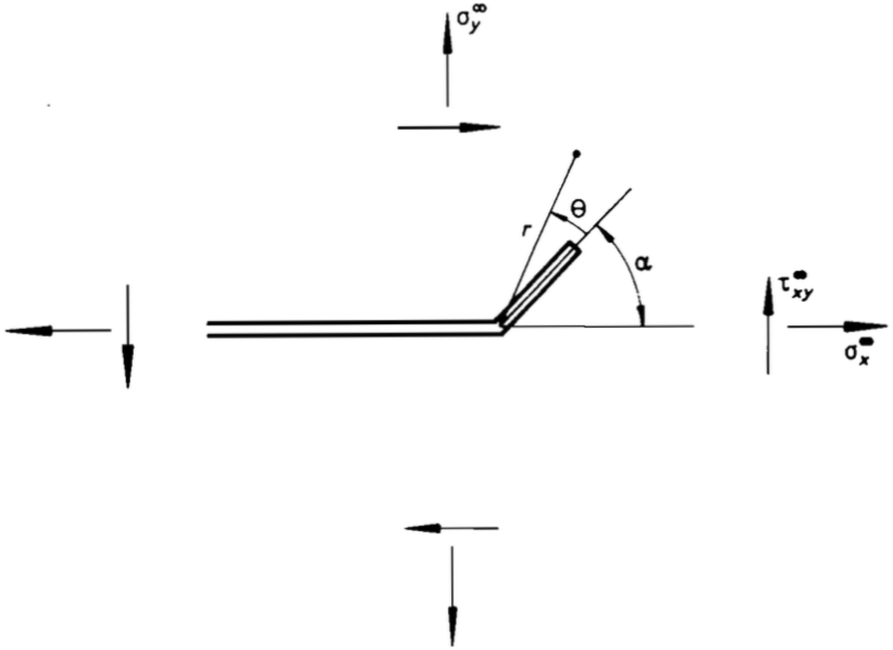
MOTIVATION

Crack kinking methods are needed for propagation path modeling



$$K_I = K_{I\alpha} k_\alpha, \quad K_{II} = K_{II\alpha} k_\alpha$$



PREVIOUS RESULTS: ELASTIC DISLOCATION METHOD

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Energy-Release Rate and Crack Kinking Under Combined Loading

Based on the maximum energy-release-rate criterion, kinking from a straight crack is in vestigated under the plane strain condition. Solutions are obtained by the method tha models a kink as a continuous distribution of edge dislocations. The energy-release ra is expressed as a quadratic form of the stress-intensity factors that exist prior to the ons of kinking, and the coefficients of this quadratic form are tabulated for various values the kink angle. The examination of the results shows that Irwin's formula for the energ release rate remains valid for any kink angle provided that the stress-intensity facto in the formula are taken equal to those existing at the tip of a vanishingly small kink.

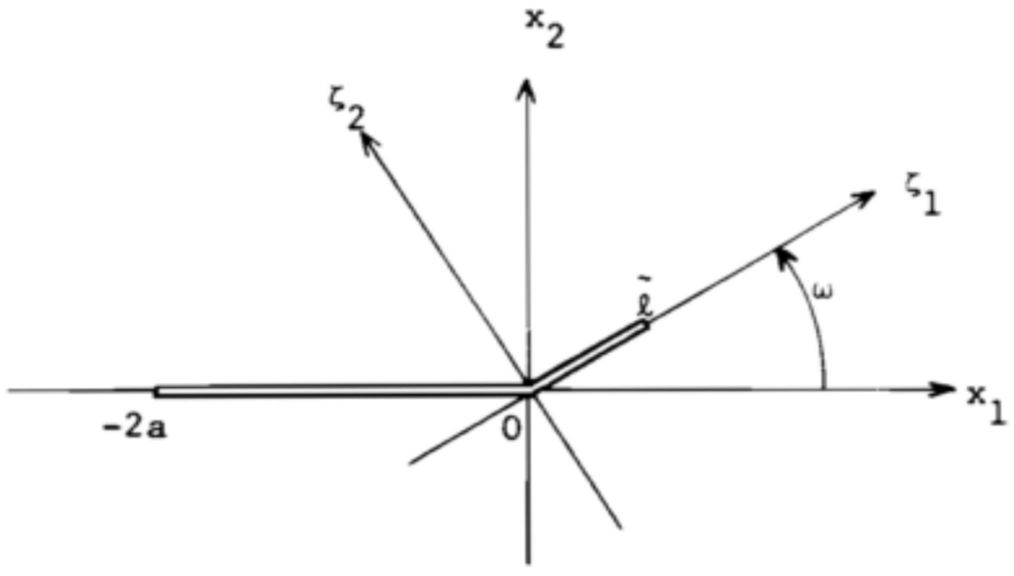


Fig. 1 Geometry and coordinate systems

$$K_{\mathrm{I}} = K_{\mathrm{I}\alpha} k_\alpha, \quad K_{\mathrm{II}} = K_{\mathrm{II}\alpha} k_\alpha$$

Table 3 Coefficients of (38); $K_{II1}(\omega) = -K_{II1}(-\omega)$, $K_{II2}(\omega) = K_{II2}(-\omega)$, $K_{I1}(\omega) = K_{I1}(-\omega)$, $K_{I2}(\omega) = -K_{I2}(-\omega)$

ω/π	K_{II1}	K_{II2}	K_{I1}	K_{I2}
0.0	0.0	1.0	1.0	0.0
-0.04	-0.06251	0.98772	0.99410	0.18770
-0.08	-0.12320	0.95131	0.97655	0.37069
-0.12	-0.18029	0.89211	0.94794	0.54441
-0.16	-0.23219	0.81224	0.90913	0.70469
-0.20	-0.27751	0.71460	0.86127	0.84784
-0.24	-0.31514	0.60262	0.80579	0.97083
-0.28	-0.34430	0.48016	0.74427	1.07134
-0.32	-0.36449	0.35137	0.67837	1.14784
-0.36	-0.37560	0.22048	0.60981	1.19960
-0.40	-0.37782	0.09165	0.54024	1.22672
-0.44	-0.37159	-0.03116	0.47126	1.22996
-0.48	-0.35768	-0.14440	0.40426	1.21082
-0.52	-0.33705	-0.24495	0.34049	1.17129
-0.56	-0.31080	-0.33023	0.28095	1.11390
-0.60	-0.28024	-0.39818	0.22632	1.04149
-0.64	-0.24697	-0.44724	0.17664	0.95746
-0.68	-0.21803	-0.47259	0.12248	0.87185
-0.72	-0.17069	-0.49145	0.10537	0.75787
-0.76	-0.13673	-0.48255	0.07269	0.65584
-0.80	-0.10410	-0.45761	0.04716	0.55040

PREVIOUS RESULTS: MELLIN TRANSFORM

Journal of Applied Mechanics

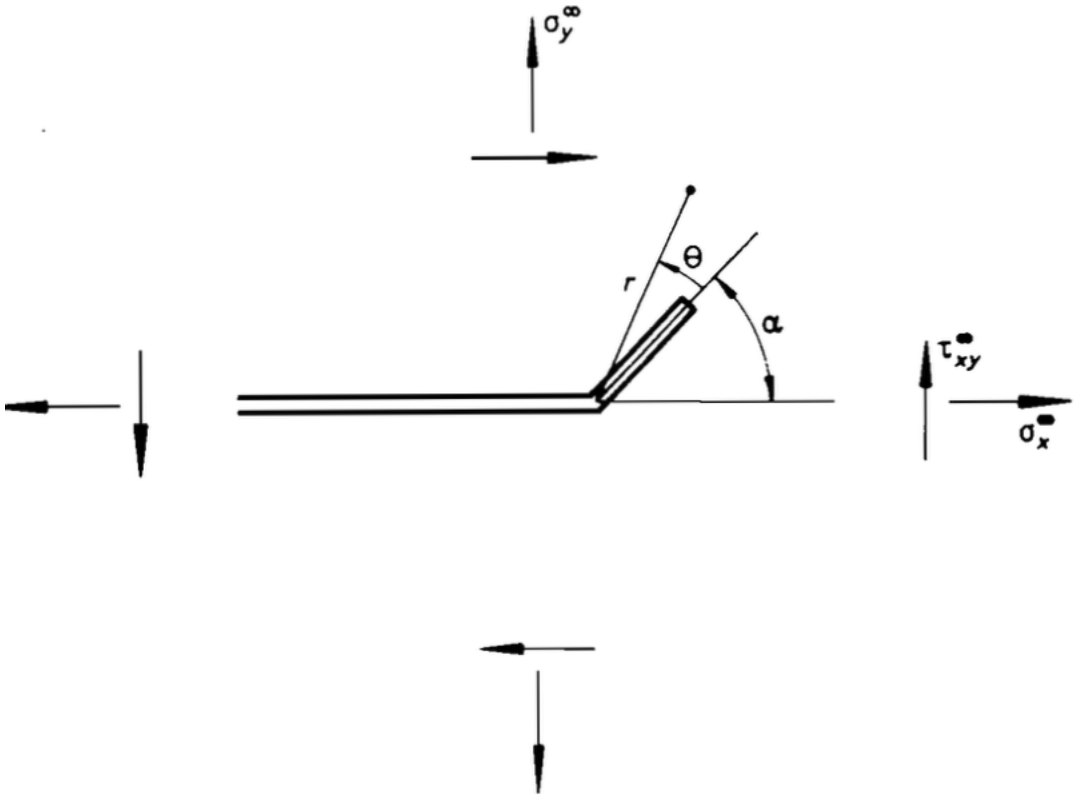
Brief Notes

A Brief Note is a short paper that presents a specific solution of technical interest in mechanics but which does not necessarily contain new general methods or results. A Brief Note should not exceed 1500 words *or equivalent* (a typical one-column figure or table is equivalent to 250 words; a one line equation to 30 words). Brief Notes will be subject to the usual review procedures prior to publication. After approval such Notes will be published as soon as possible. The Notes should be submitted to the Technical Editor of the JOURNAL OF APPLIED MECHANICS. Discussions on the Brief Notes should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N. Y. 10017, or to the Technical Editor of the JOURNAL OF APPLIED MECHANICS. Discussions on Brief Notes appearing in this issue will be accepted until two months after publication. Readers who need more time to prepare a Discussion should request an extension of the deadline from the Editorial Department.

Accurate Data for Stress Intensity Factors at Infinitesimal Kinks

S. Melin[1]

$$\bar{g}(\theta,\rho) = \int_0^\infty r^\rho g(\theta,r)\,dr$$



SUMMARY OF COMPLEX VARIABLE ELASTICITY METHODS

- Observe that stresses derived from Airy stress function naturally satisfy equilibrium

$$\sigma_{11,1} + \sigma_{12,2} = 0, \quad \sigma_{12,1} + \sigma_{22,2} = 0$$

$$\sigma_{11} = \phi_{,22}, \quad \sigma_{22} = \phi_{,11}, \quad \sigma_{12} = -\phi_{,12}$$

- Combine this with isotropic elasticity and strain compatibility to show that Airy is bi-harmonic

$$\nabla^2 \nabla^2 \phi = 0$$

- We can use complex analysis to express a general solution to the bi-harmonic equation in terms of two analytic functions, from which we can derive displacement and stress

$$2\mu(u_x + iu_y) = \kappa\Omega(z) - z\overline{\Omega'(z)} - \overline{\Phi(z)}$$

$$\sigma_{xx} + \sigma_{yy} = 2\left[\Omega'(z) + \overline{\Omega'(z)}\right]$$

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = -2\left[z\overline{\Omega''(z)} + \overline{\Phi'(z)}\right]$$

COMPLEX STRESS FUNCTIONS FOR WILLIAMS EXPANSION

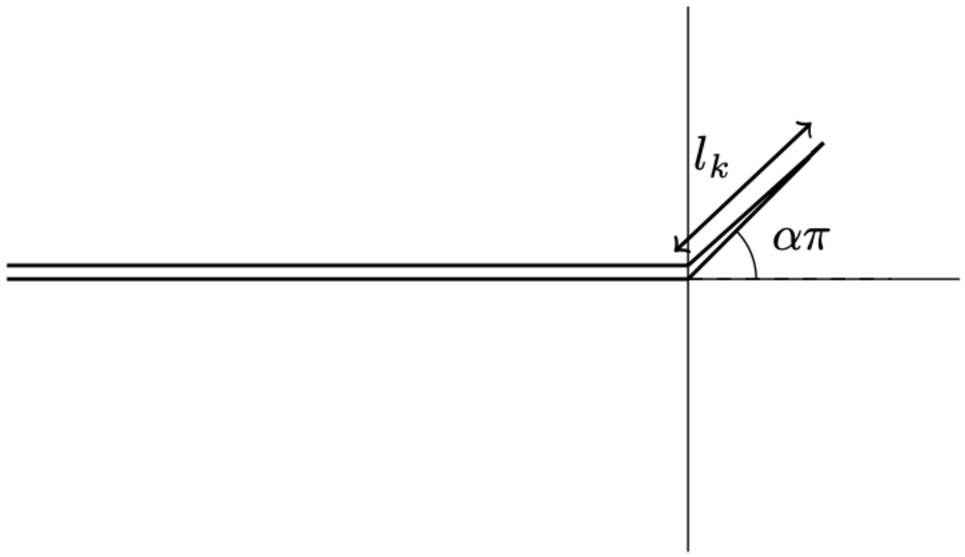
$$\sigma_{\theta\theta} + i\sigma_{r\theta} = \phi'(z) + \overline{\phi'(z)} + z\phi''(z) + \frac{z}{\bar{z}}\psi'(z)$$

$$\phi'(z) = Cz^{\lambda}$$
$$\psi'(z) = Dz^{\lambda}$$

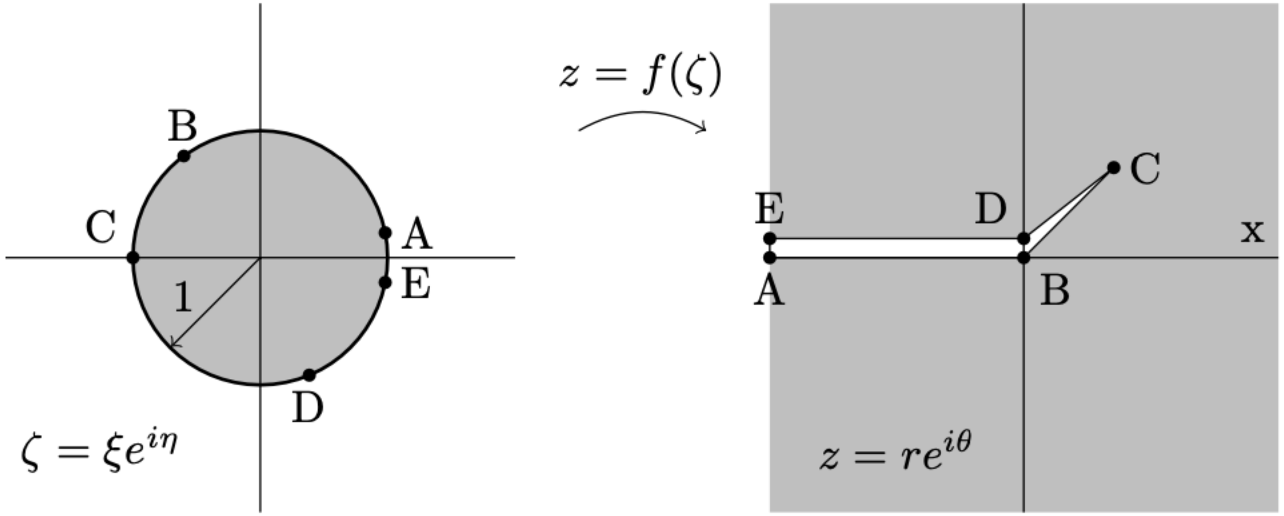
$$\sigma_{\theta\theta} + i\sigma_{r\theta} = \left[C(1+\lambda) + \overline{C}e^{-2i\lambda\theta} + De^{2i\theta}\right] z^{\lambda}$$
$$= B\left(\alpha, \lambda;\ \theta\right) z^{\lambda}$$

$$D = -C(1+\lambda) - \overline{C}e^{-2i\lambda\pi}$$

$$t\left(z;\ \alpha, \lambda, p\right) = -B\left(\alpha, \lambda;\ \alpha\pi\right) z^{\lambda} - p\,,$$
$$= -B\left(\alpha, \lambda\right) z^{\lambda} - p\,.$$

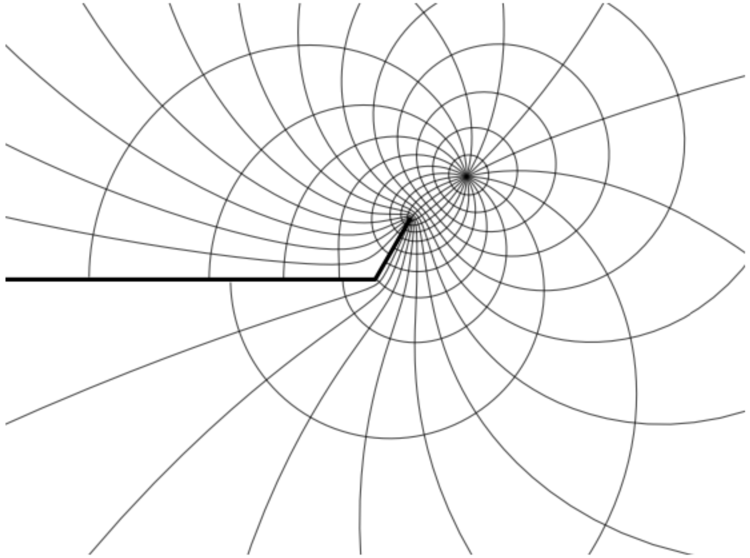


CONFORMAL MAP KINKED CRACK FROM UNIT DISK INTERIOR



$$f(\zeta) = A_1 \frac{(\zeta_B - \zeta)^{1-\alpha} (\zeta_D - \zeta)^{1+\alpha}}{(1-\zeta)^2}$$

$$f'(\zeta) = A_2 \left(\frac{\zeta_D - \zeta}{\zeta_B - \zeta} \right)^{\alpha} \frac{1+\zeta}{(1-\zeta)^3}$$



$$\zeta_B = \frac{\sqrt{l_k}(1-\alpha) + i}{\sqrt{l_k}(1-\alpha) - i}$$

$$\zeta_D = \frac{\sqrt{l_k}(1+\alpha) - i}{\sqrt{l_k}(1+\alpha) + i}$$

$$A_1 = \frac{4e^{i\pi\alpha l_k} l_k}{(\zeta_B + 1)^{1-\alpha} (\zeta_D + 1)^{1+\alpha}}$$

$$A_2 = A_1 \left[2\zeta_B\zeta_D - (1+\alpha)\zeta_B - (1-\alpha)\zeta_D \right]$$

STRESS FUNCTION FOR TRACTION BVP ON MAPPED DISK

- Traction or displacement BVPs on regions of the complex plane can be represented as a Hilbert problem
- Hilbert problems have known solutions in terms of an integral along the region boundary

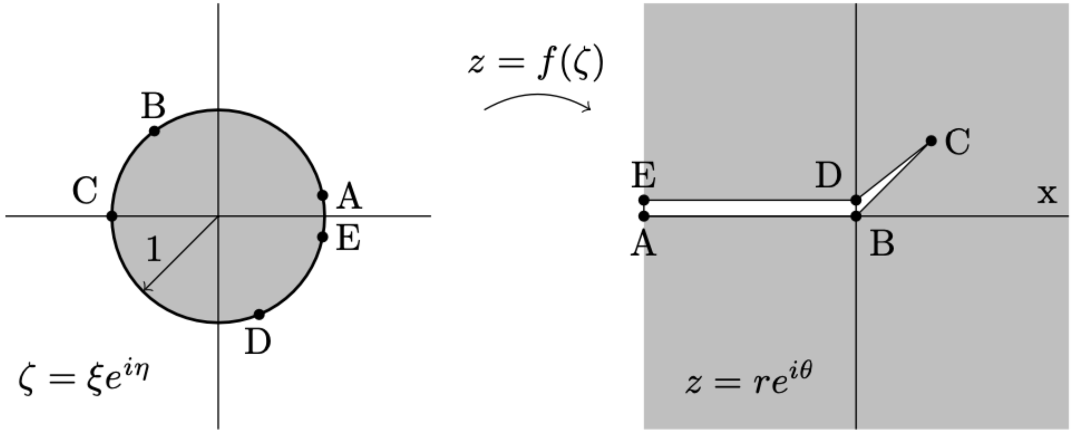
$$\sigma_{\theta\theta} + i\sigma_{r\theta} = \phi'(z) + \overline{\phi'(z)} + z\phi''(z) + \frac{z}{\bar{z}}\psi'(z)$$

$$\sigma_{\xi\xi} + i\sigma_{\xi\eta} =$$

$$g(\zeta)\left(1 + \frac{\overline{f'(1/\bar{\zeta})}}{\zeta\bar{\zeta}\overline{f'(\zeta)}}\right) + \overline{g(\zeta)} + \frac{\zeta g'(\zeta)}{\bar{\zeta}\overline{f'(\zeta)}}\left(\overline{f(\zeta)} - \overline{f(1/\bar{\zeta})}\right) + \frac{\overline{\phi'(1/\bar{\zeta})}}{\zeta\bar{\zeta}\overline{f'(\zeta)}}$$

$$g(\zeta) = \phi'(\zeta)/f'(\zeta)$$

$$\phi'(\zeta) = \frac{1}{2\pi i}\int_{\Gamma_D} \frac{t(\gamma)f'(\gamma)}{\gamma - \zeta} d\gamma - \frac{F_t}{1-\zeta} - \frac{M_t}{(1-\zeta)^2}$$



STRESS FUNCTION FOR TRACTION BVP ON MAPPED DISK

- Use analytic continuation to express stresses inside unit ζ-disk in terms of a single stress function
- Traction or displacement BVPs on regions of the complex plane can be represented as a Hilbert problem
- Hilbert problems have known solutions in terms of an integral along the region boundary

$$\sigma_{\theta\theta} + i\sigma_{r\theta} = \phi'(z) + \overline{\phi'(z)} + z\phi''(z) + \frac{z}{\bar{z}}\psi'(z)$$

$$\sigma_{\xi\xi} + i\sigma_{\xi\eta} =$$

$$g(\zeta)\left(1 + \frac{\overline{f'(1/\bar{\zeta})}}{\zeta\bar{\zeta}\overline{f'(\zeta)}}\right) + \overline{g(\zeta)} + \frac{\zeta g'(\zeta)}{\bar{\zeta}\overline{f'(\zeta)}}\left(\overline{f(\zeta)} - \overline{f(1/\bar{\zeta})}\right) + \frac{\overline{\phi'(1/\bar{\zeta})}}{\zeta\bar{\zeta}\overline{f'(\zeta)}}$$

$$g(\zeta) = \phi'(\zeta)/f'(\zeta)$$

$$\phi'(\zeta) = \frac{1}{2\pi i}\int_{\Gamma_D} \frac{t(\gamma)f'(\gamma)}{\gamma - \zeta}d\gamma - \frac{F_t}{1-\zeta} - \frac{M_t}{(1-\zeta)^2}$$

$$\phi'(\zeta) = \sum_{j=1}^{2} \beta_j(\alpha,\lambda)\int_{\Gamma_k} \frac{q_j(\gamma;\ \alpha,\lambda)(1+\gamma)}{\gamma-\zeta}\,d\gamma - \frac{F_t}{1-\zeta} - \frac{M_t}{(1-\zeta)^2}$$

$$\beta_1(\alpha,\lambda) = \frac{-B(\alpha,\lambda)\,A_1^{\lambda}A_2 l_k^{\lambda+1}}{2\pi i}$$

$$q_1(\gamma;\ \alpha,\lambda) = \frac{(\zeta_B-\gamma)^{p_B}(\zeta_D-\gamma)^{p_D}}{(1-\gamma)^{2\lambda+3}}$$

$$\beta_2(\alpha,p) = \frac{-A_2 l_k p}{2\pi i}$$

$$q_2(\gamma;\ \alpha) = q_1(\gamma;\ \alpha,\lambda=0)$$

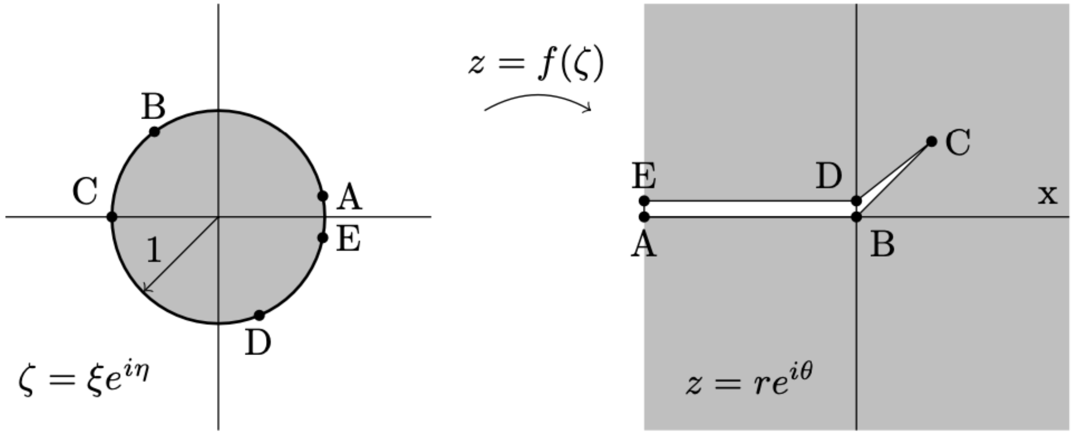
$$p_B = \lambda - \alpha\lambda - \alpha$$

$$p_D = \lambda + \alpha\lambda + \alpha$$

CALCULATING KINK-TIP STRESS INTENSITY FACTORS

- Multiply stresses by $r_k^{1/2}$, this counteracts K-field singularity
- Take the limit of the resulting quantity as $r_k \to 0^+$, approaching along the kink direction

$$\begin{aligned} k &= k_I - ik_{II} \\ &= \lim_{\substack{r \to l_k^+ \\ z = re^{i\alpha\pi}}} (\sigma_{\theta\theta} + i\sigma_{r\theta}) \left(ze^{-i\alpha\pi} - l_k\right)^{1/2} \\ &= \lim_{\substack{\xi \to -1^+ \\ \mathrm{Im}\zeta = 0}} (\sigma_{\xi\xi} + i\sigma_{\xi\eta}) \left[f(\xi)e^{-i\alpha\pi} - l_k\right]^{1/2} \end{aligned}$$



$$\bar{k} = k_I + ik_{II} = e^{-i\alpha\pi}\sqrt{1-\alpha^2}\sum_{j=1}^{2}\int_{\Gamma_k} Q_j(\gamma)\, d\gamma\,.$$

$$Q_j(\gamma) = \left[1 - \frac{5}{8}(1+\gamma)\right]\beta_j q_j(\gamma) + \frac{1+\overline{\gamma}}{8}\overline{\beta_j q_j(\gamma)}$$

NUMERICAL INTEGRATION OF INTEGRAL ALONG Γ_k

- Noting power-law nature of integrand at end points, integral is approximated using Gauss-Jacobi quadrature

$$\int_{-1}^{1} w(x) f(x)\, dx \approx \sum_{l=1}^{n} w_l f(x_l)\,, \qquad w(x) = (1-x)^{\alpha}(1+x)^{\beta}$$

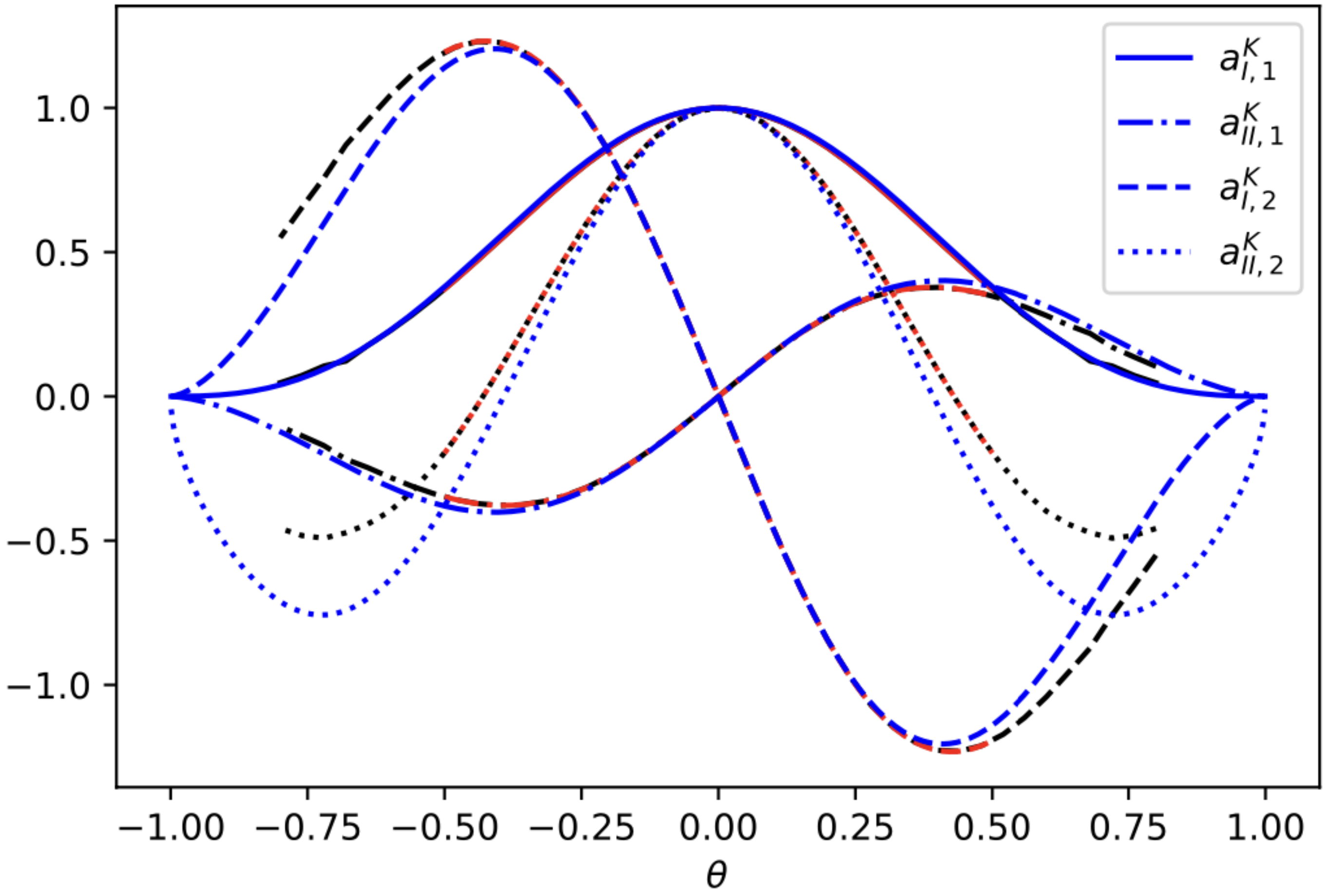
$$q_1\left(\gamma;\ \alpha, \lambda\right) = \frac{\left(\zeta_B - \gamma\right)^{p_B}\left(\zeta_D - \gamma\right)^{p_D}}{\left(1-\gamma\right)^{2\lambda+3}}$$

$$\int_{\Gamma_l} q_1(\gamma;\ \alpha, \lambda)\, d\gamma \approx \left(\frac{\zeta_D - \zeta_B}{2}\right)^{p_B+p_D+1} e^{i\pi p_D} \sum_{l=1}^{n} w_l \left(1-\gamma_l\right)^{-2\lambda-3}$$

where $\gamma_l = (\zeta_D - \zeta_B)/2 \cdot x_l + (\zeta_D + \zeta_B)/2$ and w_l and x_l are the weights and zeros of the n-order Jacobi polynomial, $J_n^{p_D, p_B}$.

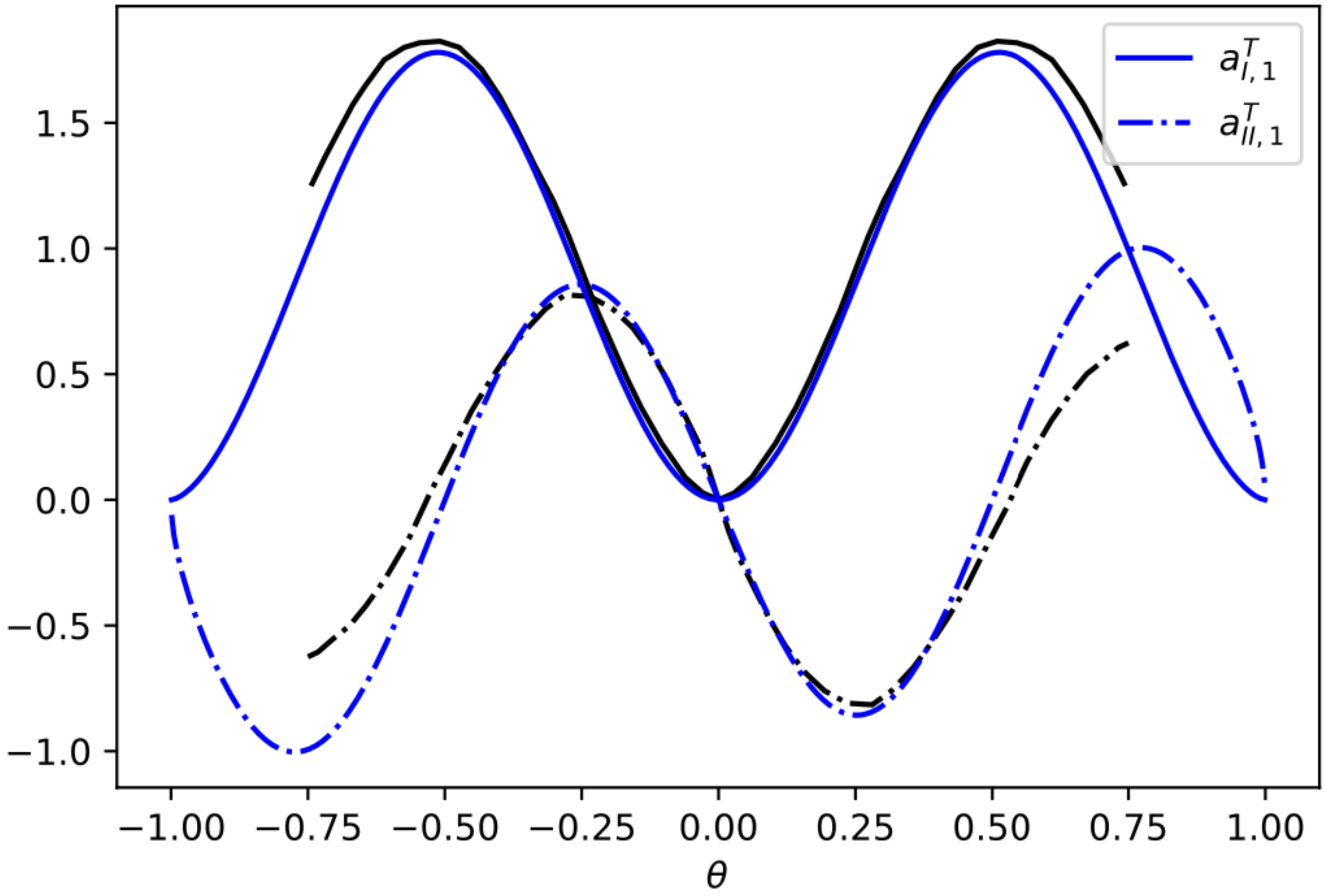
RESULTS: $\lambda = -1/2$ (K-FIELD TRANSFORMATION)

$$\begin{bmatrix} k_I \\ k_{II} \end{bmatrix} = \begin{bmatrix} a^K_{I,1}(\alpha) & a^K_{I,2}(\alpha) \\ a^K_{II,1}(\alpha) & a^K_{II,2}(\alpha) \end{bmatrix} \begin{bmatrix} K_I \\ K_{II} \end{bmatrix}$$



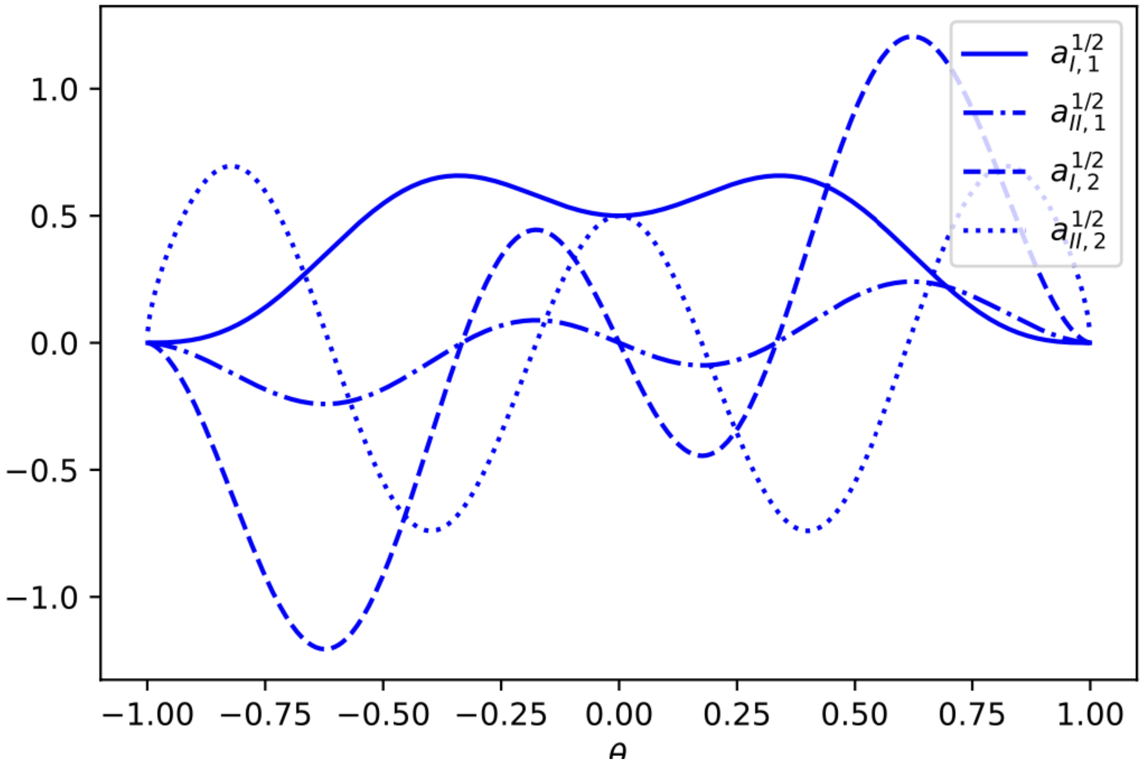
RESULTS: $\lambda = 0$ (T-FIELD TRANSFORMATION)

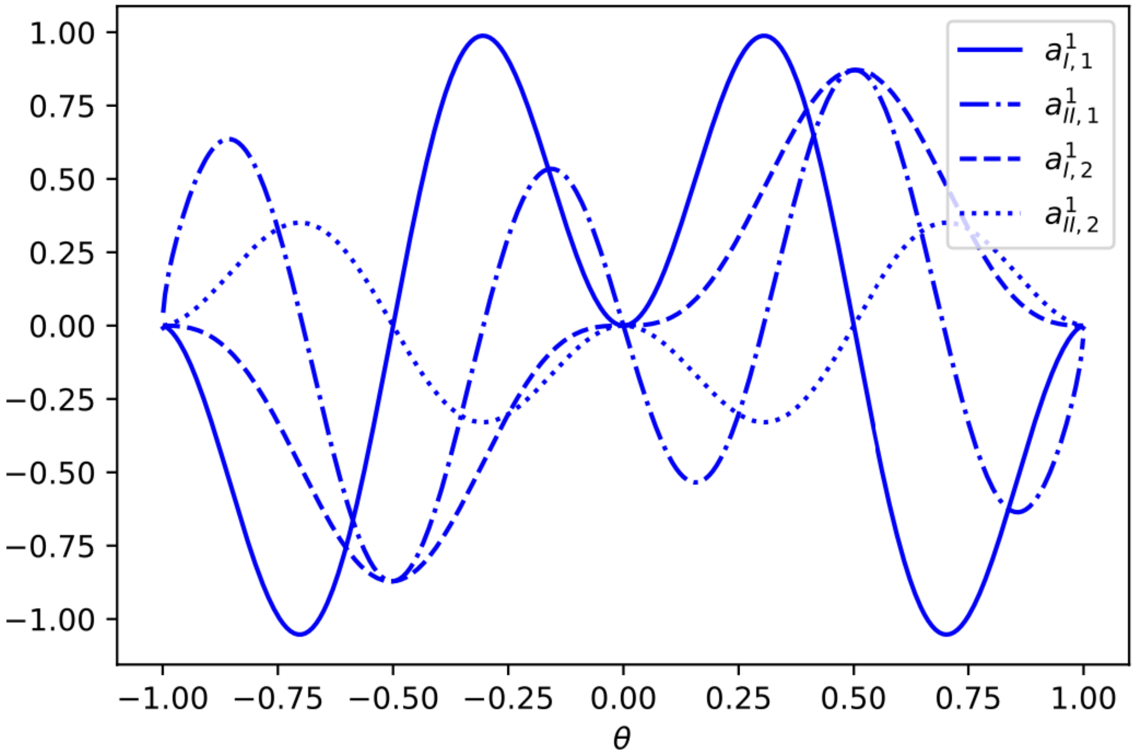
$$\begin{bmatrix} k_I \\ k_{II} \end{bmatrix} = \begin{bmatrix} a_I^T(\alpha) \\ a_{II}^T(\alpha) \end{bmatrix} T l_k^{1/2}$$



RESULTS: $\lambda = 1/2$ AND $\lambda = 1$

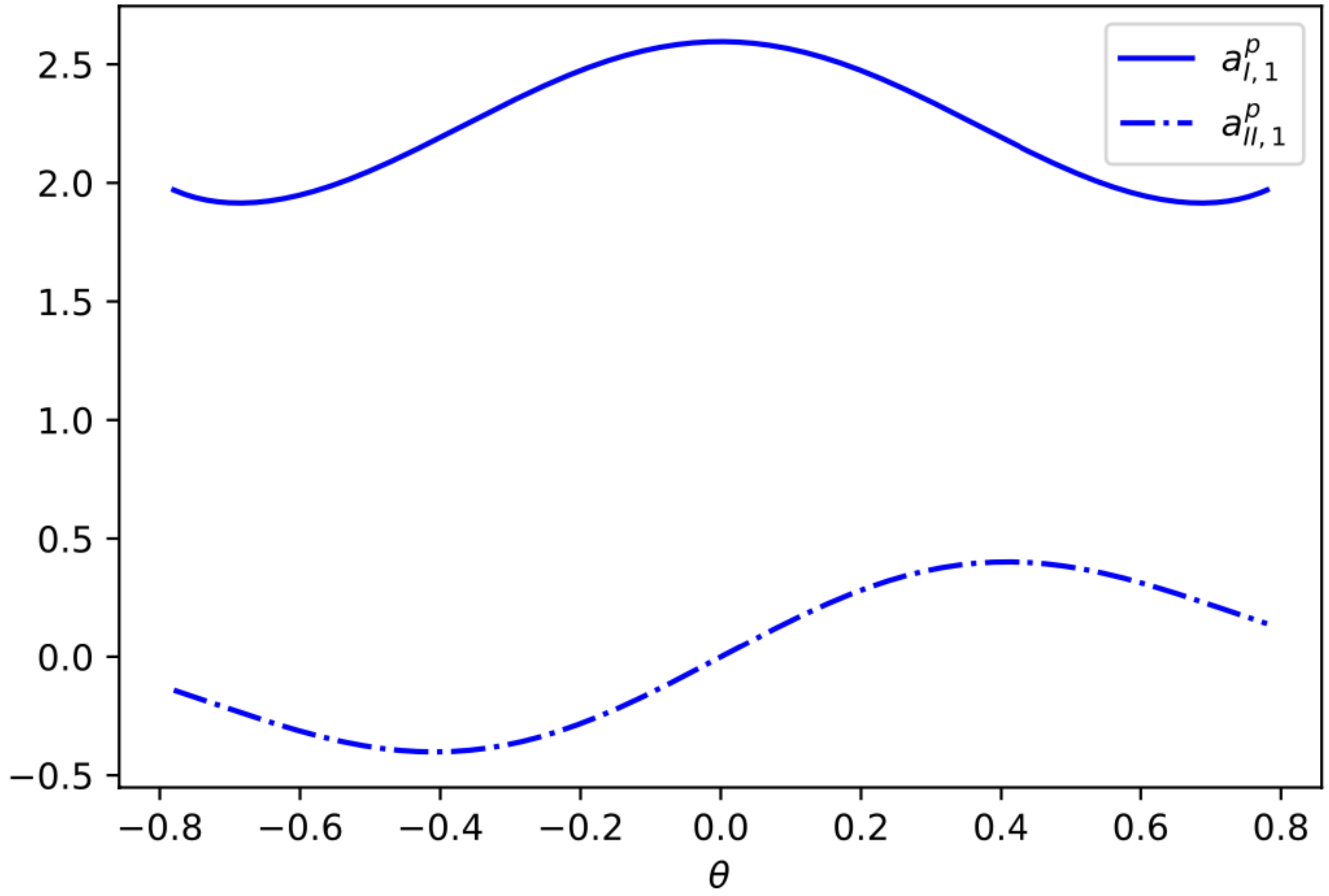
$$\begin{bmatrix} k_I \\ k_{II} \end{bmatrix} = \begin{bmatrix} a^{\lambda}_{I,1}(\alpha) & a^{\lambda}_{I,2}(\alpha) \\ a^{\lambda}_{II,1}(\alpha) & a^{\lambda}_{II,2}(\alpha) \end{bmatrix} \begin{bmatrix} \mathrm{Re}C \\ \mathrm{Im}C \end{bmatrix} l_k^{\lambda+1/2}$$





RESULTS: PRESSURIZED CRACK TRANSFORMATION

$$\begin{bmatrix} k_I \\ k_{II} \end{bmatrix} = \begin{bmatrix} a_1^p(\alpha) \\ a_2^p(\alpha) \end{bmatrix} p l_k^{1/2}$$



CONCLUSIONS

- Crack tip stress intensity transformations are presented in analytic form
- Transformation of Williams expansion terms beyond T-stress are presented for the first time
- Transformation of pressurized crack and kink is presented for the first time

QUESTIONS?

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