Two loop renormalization of scalar theories using a geometric approach

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ABSTRACT: We derive a general formula for two-loop counterterms in Effective Field Theories (EFTs) using a geometric approach. This formula allows the two-loop results of our previous paper to be applied to a wide range of theories. The two-loop results hold for loop graphs in EFTs where the interaction vertices contain operators of arbitrarily high dimension, but at most two derivatives. We also extend our previous one-loop result to include operators with an arbitrary number of derivatives, as long as there is at most one derivative acting on each field. The final result for the two-loop counterterms is written in terms of geometric quantities such as the Riemann curvature tensor of the scalar manifold and its covariant derivatives. As applications of our results, we give the two-loop counterterms and renormalization group equations for the $O(n)$ EFT to dimension six, the scalar sector of the Standard Model Effective Field Theory (SMEFT) to dimension six, and chiral perturbation theory to order p^6.

Contents

1 Introduction

Observables in quantum field theory (QFT), such as S-matrix elements, are unchanged under field redefinitions of the Lagrangian, even though Green functions do change. Thus, observable quantities are determined in terms of field-redefinition invariants constructed from the Lagrangian. The fields in a QFT take values in a manifold $\mathcal{M}$ such as the curved three-sphere S^3 for SU(2) chiral perturbation theory. Field redefinitions correspond to coordinate transformations on $\mathcal{M}$, so field-redefinition invariants are coordinate invariant quantities built out of geometrical objects such as the Riemann curvature tensor of $\mathcal{M}$. The one-loop renormalization of scalar QFTs using a geometrical approach was studied in refs. [1, 2]. The method was generalized to include gauge fields [3, 4] and fermions [5–7]. Closely related work by other groups can be found in [3, 8–16]. In this paper, we restrict our discussion to scalar QFTs.

't Hooft [17] computed the one-loop counterterm for a renormalizable scalar field theory containing terms with up to two derivatives in the Lagrangian. The one-loop counterterm formula of 't Hooft [17] requires the scalar kinetic term be canonical, i.e.

$$\mathcal{L}_{\text{KE}} = \frac{1}{2}(D_\mu \phi)^i (D^\mu \phi)^i, \tag{1.1}$$

as in a renormalizable QFT. Effective field theories (EFTs) such as the Standard Model Effective Field Theory (SMEFT) or chiral perturbation theory (χPT) contain non-trivial

kinetic terms of the form

$$\mathcal{L}_{\text{KE}} = \frac{1}{2} g_{ij}(\phi)(D_\mu \phi)^i (D^\mu \phi)^j \,, \tag{1.2}$$

where $g_{ij}(\phi)$, the metric on $\mathcal{M}$, in general is non-trivial. Refs. [1, 2] derived a generalization of 't Hooft's formula that is valid for a general kinetic energy eq. (1.2). Radiative corrections were given in terms of the Riemann curvature tensor constructed from g_{ij}.

In a recent paper [18], referred to as paper I, we derived the two-loop counterterms and anomalous dimensions for a scalar field theory with interactions up to two derivatives and with a canonically normalized kinetic energy term, which is the two-loop version of 't Hooft's formula. In this paper, we use the geometric approach of refs. [1, 2] to extend the results of paper I to an arbitrary scalar theory with interactions up to two derivatives. Our two-loop result allows for a purely algebraic computation of the two-loop counterterms, i.e. we have a universal formula for the counterterms in terms of the Riemann curvature R_{abcd}, which is determined from derivatives of the metric g_{ij}.

In this paper, we use the Riemann normal coordinate expansion to compute the fluctuation Lagrangian. The covariant formalism allows for a transformation to a local Cartan frame, where the results of paper I can be used to compute the renormalization counterterms for a general two-derivative scalar theory. The final result is coordinate invariant, so after deriving it, we can use it in any coordinate system. The universal two-loop formula is applied to a number of examples. It is used to compute the two-loop renormalization group equations (RGE) for the $O(n)$ EFT to dimension six, the two-loop RGE for the Standard Model Effective Field Theory scalar sector to dimension six, and the two-loop RGE of SU(n) chiral perturbation theory to order p^6. Our geometric approach greatly simplifies the computation — (a) The corrections are grouped into geometric objects such as the Riemann curvature tensor, which transform covariantly under field redefinitions. (b) The two-derivative cubic interaction in the fluctuation Lagrangian $(D_\mu \eta)(D_\nu \eta)\eta$ vanishes in Riemann normal coordinates, even though it is present under the usual background field method of taking $\phi \to \overline{\phi} + \eta$ and expanding in η. Thus, the use of Riemann normal coordinates removes a large number of counterterms that would otherwise have to be computed. (c) Factorizable diagrams do not contribute to the anomalous dimensions. (d) The field theory for η has an $O(n)$ gauge symmetry, noted by 't Hooft [17], which restricts the form of the counterterms, and allows one to determine them without having to compute higher-point amplitudes, even in an EFT. (e) The computations are algebraic, i.e. one writes the metric $g_{ij}(\phi)$ as a polynomial in fields, computes the Riemann curvature tensor in terms of derivatives of g_{ij}, and takes traces. No Feynman diagrams need to be computed. The basic Feynman graphs give counterterm coefficients which are universal for all EFTs and are tabulated in paper I.

This paper is organized as follows. In section 2, we discuss the Riemann normal coordinate expansion, which is used in the derivation of the geometrical version of the two-loop formula. Section 3 then derives the geometrical version of the two-loop results of paper I. Next, we consider applications to specific scalar field theories. In sections 4, 5 and 6, the algebraic two-loop formula is applied to the $O(n)$ effective field theory to dimension six, SMEFT to dimension six, and chiral perturbation theory to order p^6, respectively. Our results for the $O(n)$ EFT agree with known results [19, 20] for $n = 1, 2$. Our results for SMEFT agree with

known two-loop results for the SM when higher dimension operators are dropped [21–23], and our results for χPT agree with ref. [24] . Finally, we present conclusions in section 7.

2 Riemann normal coordinate expansion

The theory we consider is an arbitrary scalar theory with up to two derivatives coupled to external gauge fields. We follow the notation and analysis of refs. [1, 2]. The Lagrangian is

$$\mathcal{L} = \frac{1}{2} g_{ij}(\phi) (D_\mu \phi)^i (D^\mu \phi)^j - V(\phi) \tag{2.1}$$

where ϕ are real scalar fields which take values in a scalar manifold $\mathcal{M}$. Complex scalar fields are written in terms of real scalar fields. The scalar field metric is $g_{ij}(\phi)$, and transforms as a metric tensor under field redefinitions (i.e. coordinate transformations of $\mathcal{M}$),

$$g'_{ij} = \left(\frac{\partial \phi^k}{\partial \phi'^i} \right) \left(\frac{\partial \phi^l}{\partial \phi'^j} \right) g_{kl} \,. \tag{2.2}$$

Indices on $\mathcal{M}$ are lowered and raised using g_{ij} and its inverse g^{ij}.

If the manifold $\mathcal{M}$ on which the scalar fields live has a symmetry, then some of them can be local (gauge) symmetries. In this case, the potential $V(\phi)$ must also be invariant under the gauged symmetries. The symmetries of $\mathcal{M}$ are generated by Killing vectors $t_\alpha(\phi)$ with vanishing Lie derivative of the metric, $\mathscr{L}_{t_\alpha} g = 0$. They satisfy Killing's equation

$$\nabla_i t_{j\alpha} + \nabla_j t_{i\alpha} = 0 \,, \tag{2.3}$$

where the covariant derivative ∇ uses the metric compatible torsion-free Christoffel connection computed from g_{ij}. For symmetries which are gauged, we also require $\mathscr{L}_{t_\alpha} V = 0$ so that the Lagrangian is gauge invariant.

The covariant derivative is

$$(D_\mu \phi)^i = \partial_\mu \phi^i(x) + t^i_\alpha(\phi) A^\alpha_\mu(x) \tag{2.4}$$

where the gauge fields $A_\mu(x)$ do not depend on ϕ. The coupling constant and a factor of i has been absorbed into t_α. The Killing vectors satisfy the Lie algebra

$$[t_\alpha, t_\beta] = f_{\alpha\beta}{}^\gamma t_\gamma \tag{2.5}$$

where the l.h.s. is a Lie bracket. Explicit expressions for the terms in eq. (2.1) in SMEFT can be found in [1–3, 10].

Quantum corrections to eq. (2.1) are computed by writing $\phi = \overline{\phi} + \eta$, where $\overline{\phi}$ is an external (background) field and η is an internal (quantum) field which is integrated over. The one-loop correction is computed from the quadratic term in η, i.e. the second functional derivative of the action. The expansion $\phi = \overline{\phi} + \eta$ introduces non-covariant terms in the functional derivatives of the action [25, 26]. Under a change of coordinates, a vector V^i transforms as

$$V'^i = \left(\frac{\partial \phi'^i}{\partial \phi^j} \right) V^j \,. \tag{2.6}$$
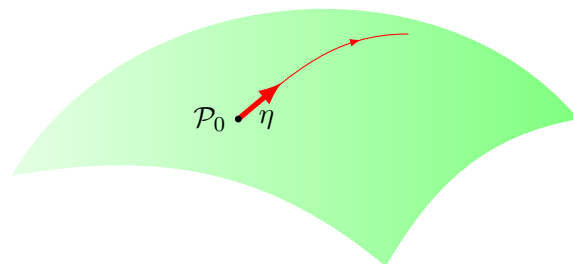
Figure 1. Riemann normal coordinates on $\mathcal{M}$. The red curve is a geodesic starting at $\mathcal{P}_0$ with tangent vector η.

Note that ϕ^i is a coordinate which does not transform as a vector. In contrast, $\eta^i = \delta\phi^i$, the variation of ϕ^i, is a vector, but its derivatives are not [1, 2, 25, 26]. This introduces non-covariant terms in the functional derivatives and one-loop corrections. However, as pointed out in the introduction, we can make field redefinitions without changing the S-matrix. One can make a change of variables to restore manifest covariance of the functional derivatives and the radiative corrections. The idea is to use Riemann normal coordinates to parameterize the quantum fluctuations, which turns functional derivatives into covariant functional derivatives [1, 2, 25, 26].

Let $\mathcal{P}_0$ be a point on $\mathcal{M}$ with coordinates ϕ_0^i. This is the point around which we compute the fluctuation Lagrangian. The geodesic $\phi^i(\eta, \lambda)$ is a curve starting at $\mathcal{P}_0$ with tangent vector η_0 and parameter λ,

$$\phi^i(\eta, 0) = \phi_0^i, \qquad \frac{\mathrm{d}\phi^i(\eta, \lambda)}{\mathrm{d}\lambda} = \eta^i(\lambda), \qquad \eta(\lambda = 0) = \eta_0\,, \qquad (2.7)$$

which satisfies the geodesic equation $\nabla_\eta \eta = 0$, or in coordinates,

$$\frac{\mathrm{d}^2\phi^i}{\mathrm{d}\lambda^2} + \Gamma_{jk}^i(\phi)\frac{\mathrm{d}\phi^j}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi^k}{\mathrm{d}\lambda} = 0\,. \qquad (2.8)$$

The geodesic equation eq. (2.8) is homogeneous in λ, so that $\phi^i(\eta, \lambda) = \phi^i(s\eta^i, \lambda/s)$ under rescaling by any constant s. A point $\mathcal{P}$ with coordinates $\phi^i(\mathcal{P})$ has Riemann normal coordinates $\eta^i(\mathcal{P})$, where $\phi^i(\mathcal{P}) = \phi^i(\eta(\mathcal{P}), 1)$, i.e. the starting velocity at $\mathcal{P}_0$ such that the geodesic reaches $\mathcal{P}$ at $\lambda = 1$ is the Riemann normal coordinate $\eta^i(\mathcal{P})$. The advantage of Riemann normal coordinates is that $\eta(\mathcal{P})$ transforms as a vector at the point $\mathcal{P}_0$ under coordinate transformations.[1] The coordinate ϕ^i in Riemann normal coordinates is given by the series expansion

$$\phi^i = \phi_0^i + \lambda\eta^i - \frac{1}{2}\lambda^2\,\Gamma_{jk}^i(\phi_0)\,\eta^j\eta^k - \frac{1}{3!}\lambda^3\,\Gamma_{jkl}^i(\phi_0)\,\eta^j\eta^k\eta^l - \frac{1}{4!}\lambda^4\,\Gamma_{jklr}^i(\phi_0)\,\eta^j\eta^k\eta^l\eta^r + \dots \qquad (2.9)$$

where the series coefficients are evaluated by taking successive derivatives of the geodesic equation eq. (2.8), and at each step replacing second derivatives of ϕ^i by first derivatives

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[1]Note that it is a vector at $\mathcal{P}_0$, not $\mathcal{P}$.
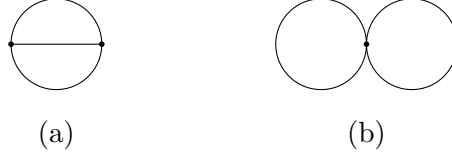
Figure 2. Skeleton graphs for two-loop corrections to the action. There can be an arbitrary number of $\eta\eta$ vertex insertions on the lines.

using eq. (2.8). The higher order terms involve generalized Christoffel symbols [27], which are functions of the standard Christoffel symbols and their derivatives. The background field $\phi_0(x)$ is a function of the spacetime point x, so the above procedure gives the quantum field $\eta(x)$ by using eq. (2.9) point-by-point in x to parameterize the fluctuation $\phi(x) - \phi_0(x)$.

The variation of the action is computed by differentiating $S[\phi(\eta, \lambda)]$ w.r.t. λ, and evaluating the derivatives at $\lambda = 0$. Expanding the action in the geodesic fluctuation η to quadratic order in η yields

$$S[\phi + \eta] = S[\phi] + \frac{\delta S}{\delta \phi^i} \eta^i + \frac{1}{2} \left(\frac{\delta^2 S}{\delta \phi^i \delta \phi^j} - \Gamma^k_{ij} \frac{\delta S}{\delta \phi^k} \right) \eta^i \eta^j + \mathcal{O}(\eta^3) , \qquad (2.10)$$

where the terms transform as tensors at $\mathcal{P}_0$. The second functional derivative of S has been replaced by the covariant second functional derivative when using Riemann normal coordinates.

The two-loop contributions to the functional integral are shown in figure 2. The graphs involve η^2, η^3 and η^4 interactions, so we need the expansion of the action to fourth order in η. This can be done efficiently by using a coordinate-free notation, and differentiating the action four times w.r.t. the geodesic parameter λ. We start by computing the derivatives of $D_\mu \phi$, which are needed in the evaluation of derivatives of the action. Its first derivative is

$$\nabla_\lambda (D_\mu \phi) = \nabla_\lambda (\partial_\mu \phi + t_\alpha A^\alpha_\mu) = \nabla_\lambda (\partial_\mu \phi) + (\nabla_\lambda t_\alpha) A^\alpha_\mu \qquad (2.11)$$

since the gauge field is independent of the scalar field. The connection is torsion free, $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0$, and $[\partial_\lambda, \partial_\mu] = 0$, since partial derivatives commute, so

$$\nabla_\lambda (\partial_\mu \phi) = \nabla_\mu (\partial_\lambda \phi) = \nabla_\mu \eta , \qquad (2.12)$$

and

$$\nabla_\lambda (D_\mu \phi) = \nabla_\mu \eta + (\nabla_\lambda t_\alpha) A^\alpha_\mu = \nabla_\mu \eta + (\eta^j \nabla_j t_\alpha) A^\alpha_\mu \equiv \mathscr{D}_\mu \eta , \qquad (2.13)$$

where the covariant derivative $\mathscr{D}_\mu \eta$ was defined in [1, 2]. The covariant derivative $\mathscr{D}$ acts on tensors, and can be used to differentiate tensors with an arbitrary number of indices. In coordinates,

$$\begin{aligned} (\mathscr{D}_\mu \eta)^i &= \left(\partial_\mu \eta^i + \Gamma^i_{kj} \partial_\mu \phi^k \eta^j \right) + A^\beta_\mu \left(t^i_{\beta,j} + \Gamma^i_{jk} t^k_\beta \right) \eta^j = \nabla_\mu \eta^i + A^\beta_\mu \eta^j \nabla_j t^i_\beta \\ &= \left(\partial_\mu \eta^i + A^\beta_\mu t^i_{\beta,j} \eta^j \right) + \Gamma^i_{kj} \left(\partial_\mu \phi^k + t^k_\beta A^\beta_\mu \right) \eta^j = (D_\mu \eta)^i + \Gamma^i_{kj} (D_\mu \phi)^k \eta^j . \end{aligned} \qquad (2.14)$$

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The first form shows that $\mathscr{D}_\mu\eta$ is coordinate invariant, and the second form shows that $\mathscr{D}_\mu\eta$ is gauge invariant. The gauge generators acting on vectors such as η^i are $t^i_{\beta,j}$, as shown in [1].

The geodesic derivative of $\mathscr{D}_\mu\eta$ is

$$\nabla_\lambda \mathscr{D}_\mu\eta = \nabla_\lambda \left[\nabla_\mu\eta + (\nabla_\lambda t_\alpha)A^\alpha_\mu \right] = \nabla_\lambda\nabla_\mu\eta + (\nabla_\lambda\nabla_\lambda t_\alpha)A^\alpha_\mu. \tag{2.15}$$

The first term is

$$\nabla_\lambda\nabla_\mu\eta = [\nabla_\lambda, \nabla_\mu]\eta + \nabla_\mu\nabla_\lambda\eta = R(\eta, \partial_\mu\phi)\eta \tag{2.16}$$

since $\nabla_\lambda\eta = \nabla_\eta\eta = 0$ by the geodesic equation, and using the definition of the Riemann curvature, $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$. Eq. (2.16) is the geodesic deviation equation,

$$\nabla_\lambda\nabla_\lambda(\partial_\mu\phi) = R(\eta, \partial_\mu\phi)\eta. \tag{2.17}$$

We also have

$$\nabla_\lambda\nabla_\lambda t_\alpha = R(\eta, t_\alpha)\eta \tag{2.18}$$

which follows from Kostant's equation for a Killing vector

$$\nabla_j\nabla_k t^i_\alpha = R^i{}_{kjl} t^l_\alpha. \tag{2.19}$$

Combining eq. (2.16) and eq. (2.18) gives the gauge invariant form of the geodesic deviation equation,

$$\nabla_\lambda(\mathscr{D}_\mu\eta) = R(\eta, D_\mu\phi)\eta, \tag{2.20}$$

for the derivative of $\mathscr{D}_\mu\eta$. Eq. (2.13) and eq. (2.20) are sufficient to compute all higher λ derivatives of $D_\mu\phi$. For example, the next derivative is

$$\begin{aligned}
\nabla_\lambda\nabla_\lambda(\mathscr{D}_\mu\eta) &= \nabla_\lambda\left[R(\eta, D_\mu\phi)\eta\right] = (\nabla_\lambda R)(\eta, D_\mu\phi)\eta + R(\eta, \nabla_\lambda D_\mu\phi)\eta \\
&= (\eta^i\nabla_i R)(\eta, D_\mu\phi)\eta + R(\eta, \mathscr{D}_\mu\eta)\eta
\end{aligned} \tag{2.21}$$

since $\nabla_\lambda\eta = 0$, and is cubic in η.

We can now compute the variation of the Lagrangian eq. (2.1) to fourth order by repeated application of eq. (2.13), eq. (2.20), and metric compatibility of the connection, $\nabla g = 0$. The variations are

$$\begin{aligned}
\delta\mathcal{L} &= g(\mathscr{D}_\mu\eta, D^\mu\phi) - \nabla_\lambda V, \\
\delta^2\mathcal{L} &= \frac{1}{2}g(R(\eta, D_\mu\phi)\eta, D^\mu\phi) + \frac{1}{2}g(\mathscr{D}_\mu\eta, \mathscr{D}^\mu\eta) - \frac{1}{2}\nabla_\lambda\nabla_\lambda V, \\
\delta^3\mathcal{L} &= \frac{1}{6}g((\nabla_\lambda R)(\eta, D_\mu\phi)\eta, D^\mu\phi) + \frac{2}{3}g(R(\eta, D_\mu\phi)\eta, \mathscr{D}^\mu\eta) - \frac{1}{6}\nabla_\lambda\nabla_\lambda\nabla_\lambda V, \\
\delta^4\mathcal{L} &= \frac{1}{24}g((\nabla_\lambda\nabla_\lambda R)(\eta, D_\mu\phi)\eta, D^\mu\phi) + \frac{1}{4}g((\nabla_\lambda R)(\eta, D_\mu\phi)\eta, \mathscr{D}^\mu\eta) + \frac{1}{6}g(R(\eta, \mathscr{D}_\mu\eta)\eta, \mathscr{D}^\mu\eta) \\
&\quad + \frac{1}{6}g(R(\eta, D_\mu\phi)\eta, R(\eta, D^\mu\phi)\eta) - \frac{1}{24}(\nabla_\lambda\nabla_\lambda\nabla_\lambda\nabla_\lambda V),
\end{aligned} \tag{2.22}$$

where we have included the $1/n!$ prefactor in the nth variation, and used the symmetry of the Riemann tensor $g(R(A, B)D, C) = g(R(C, D)B, A)$.[2] The second order term eq. (2.22) was given in [2, (50)][1, (3.42)], and used to compute the one-loop corrections. Here we need the third and fourth order terms to compute the two-loop corrections. The Lagrangian expansion in terms of quantum fields defined with respect to geodesic coordinates found in eq. (2.22) has previously been derived in the context of the non-linear sigma models in two space-time dimensions [28–30]. In the next section, we combine eq. (2.22) with the result in paper I to obtain the general two-loop counterterms.

3   Derivation of the two-loop formula

We briefly summarize the results of paper I on the two-loop counterterm formula. The derivation relies on the second, third, and fourth order fluctuations terms having the form

$$
\begin{aligned}
\mathcal{L}^{(2)} &= \frac{1}{2}(D_\mu \eta)_a (D^\mu \eta)^a + \frac{1}{2}X_{ab}\eta^a \eta^b \\
\mathcal{L}^{(3)} &= A_{abc}\eta^a \eta^b \eta^c + A^\mu_{a|bc}(D_\mu \eta)^a \eta^b \eta^c + A^{\mu\nu}_{ab|c}(D_\mu \eta)^a (D_\nu \eta)^b \eta^c \\
\mathcal{L}^{(4)} &= B_{abcd}\eta^a \eta^b \eta^c \eta^d + B^\mu_{a|bcd}(D_\mu \eta)^a \eta^b \eta^c \eta^d + B^{\mu\nu}_{ab|cd}(D_\mu \eta)^a (D_\nu \eta)^b \eta^c \eta^d \,,
\end{aligned}
\tag{3.1}
$$

where

$$
(D_\mu \eta)^a \equiv \partial_\mu \eta^a + (N_\mu)_{ab}\eta^b \,,
\tag{3.2}
$$

and the coefficients $(N_\mu)_{ab}$, X_{ab}, etc. are functions of the background field, and satisfy the symmetry relations

$$
\begin{aligned}
N^\mu_{ab} + N^\mu_{ba} &= 0 \,, \\
A^\mu_{a|bc} + A^\mu_{b|ca} + A^\mu_{c|ab} &= 0 \,, \\
B^\mu_{a|bcd} + B^\mu_{b|cda} + B^\mu_{c|dab} + B^\mu_{d|abc} &= 0 \,,
\end{aligned}
\tag{3.3}
$$

i.e. the completely symmetric parts of N^μ, A^μ and $B^{\mu\nu}$ vanish. The coefficients are also symmetric under permutation of η indices, and simultaneous permutation of $D_\mu \eta$ indices. The field-strength tensor is defined by

$$
Y_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu N_\nu - \partial_\nu N_\mu + [N_\mu, N_\nu] \,.
\tag{3.4}
$$

The terms in eq. (2.22) have the structure of the terms in eq. (3.1). They involve at most two first covariant derivatives of η, $\mathscr{D}_\mu \eta$. There are no terms with second derivatives of η, $\mathscr{D}_\mu \mathscr{D}_\nu \eta$. The cubic term has either η^3 terms or $\eta\eta\mathscr{D}\eta$ terms; there is no $\eta\mathscr{D}\eta\mathscr{D}\eta$ term with two first derivatives. This feature greatly simplifies the computations required for the two-loop correction. In the extension of 't Hooft's formula to two loops, we do not need to include the $A^{\mu\nu}_{ab|c}$ interaction. If we had instead used a naive expansion $\phi \rightarrow \overline{\phi} + \eta$, there would in general be such two-derivative terms in the cubic Lagrangian, making the two-loop

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[2]In component notation $g(R(A, B)D, C) = R_{ijkl}C^i D^j A^k B^l$.

result far more complicated. The quartic term has η^4, $\eta^3 \mathscr{D}\eta$ and $\eta^2 \mathscr{D}_\mu \eta \mathscr{D}_\mu \eta$ terms. The term with two first derivatives in the original Lagrangian eq. (2.1) has their Lorentz indices contracted, so the $B^{\mu\nu}_{ab|cd}$ interaction is proportional to $\eta^{\mu\nu}$.

Loop corrections to the action eq. (2.1) can be computed using the expansion in quantum fluctuations eq. (2.22). We are free to make field redefinitions, which leave the S-matrix invariant. It is not possible to make field redefinitions (coordinate transformations) in the original Lagrangian eq. (2.1) to make the metric trivial, $g_{ij}(\phi) \to \delta_{ij}$. The obstruction to finding such a transformation is the non-vanishing of the Riemann curvature tensor $R_{ijkl}(\phi)$, which depends on second derivatives of the metric g_{ij}. The 't Hooft formula is not directly applicable for general Lagrangians such as eq. (2.1), since the formula is only valid for a canonical kinetic term proportional to δ_{ij}.

A big advantage of using Riemann normal coordinates is that the expansion is in powers of η^i which transforms like a vector, unlike ϕ^i which is a coordinate. To simplify the radiative corrections, we can go to a local orthonormal (Cartan) frame by introducing vielbeins,

$$g_{ij}(\phi) = e^a_i(\phi) e^b_j(\phi) \delta_{ab} \tag{3.5}$$

and transform all tensors to the local frame. This frame is a non-coordinate basis, i.e. we can make a local choice of tangent vectors satisfying eq. (3.5), but we cannot integrate this to a change of coordinates which makes the metric trivial if the Riemann curvature is non-vanishing. Local frame indices will be denoted by $a, b, c, \dots$. The metric tensor in the local frame is δ_{ab}. The tangent vector η^i in the local frame becomes

$$\eta^a = e^a_i(\phi) \eta^i \,, \tag{3.6}$$

which is a field redefinition on the quantum field, and does not change the S-matrix.

The variation of the action in the local frame takes the same form as eq. (2.22) with coordinate indices $i, j, k, \dots$ replaced by local frame indices $a, b, c, \dots$. The second variation of the action eq. (2.22) is

$$\delta^2 \mathcal{L} = \frac{1}{2}(\mathscr{D}_\mu \eta)_a (\mathscr{D}^\mu \eta)^a + \frac{1}{2} R_{abcd}(D_\mu \phi)^a \eta^b \eta^c (D^\mu \phi)^d - \frac{1}{2}\eta^a \eta^b \nabla_a \nabla_b V \,. \tag{3.7}$$

This second variation has the same form as eq. (3.1) with

$$X_{ab} = -R_{acbd}(D_\mu \phi)^c (D^\mu \phi)^d - \nabla_a \nabla_b V \,. \tag{3.8}$$

The covariant derivative eq. (2.14) is

$$(\mathscr{D}_\mu \eta)^a = \partial_\mu \eta^a + \omega^a_{ib} \partial_\mu \phi^i \eta^b + \nabla_b t^a_\alpha \eta^b A^\alpha_\mu \,, \tag{3.9}$$

where ω^a_{ib} is the Cartan connection. Comparing with eq. (3.2), N^μ is

$$N^\mu_{ab} = \omega_{iab} \,\partial^\mu \phi^i + \nabla_b t^\alpha_a A^\mu_\alpha \,, \tag{3.10}$$

where we can lower local frame indices using the metric δ_{ab}. The metric compatible Cartan connection ω_{iab} is antisymmetric in a, b, as is the derivative of the Killing vector $\nabla_b t^\alpha_a$, so

that N^μ_{ab} is automatically antisymmetric in a, b, as required in eq. (3.3). The commutator of two covariant derivatives eq. (3.4) gives [1]

$$[Y_{\mu\nu}]^a{}_b = [\mathscr{D}_\mu, \mathscr{D}_\nu] = R^a{}_{bcd}(D_\mu\phi)^c(D_\nu\phi)^d + \nabla_b t^a_\alpha F^\alpha_{\mu\nu} \tag{3.11}$$

using properties of the Killing vector. The one-loop counterterm formula of 't Hooft is

$$\mathcal{L}^{(1)}_{\text{c.t.}} = \frac{1}{16\pi^2\epsilon}\left[-\frac{1}{4}X_{ab}X^{ba} - \frac{1}{24}(Y_{\mu\nu})_{ab}(Y^{\mu\nu})^{ba}\right] \tag{3.12}$$

which is only valid if the kinetic term has canonical normalization δ_{ab}, and $(N_\mu)_{ab}$ is antisymmetric. Both conditions are satisfied in the local Cartan frame, so we can use eq. (3.12) with X and $Y_{\mu\nu}$ given in eq. (3.8) and eq. (3.11).

At first sight, this seems like an involved computation, having to first determine the vielbeins which satisfy eq. (3.5), transform to the local frame, etc. However, the final expression for the counterterm eq. (3.12) is coordinate invariant and written in terms of tensor quantities. It can be evaluated in any frame. In particular, it can be evaluated in the original coordinate basis — i.e. the entire transformation to the local frame is not actually needed to compute the counterterms; it is only needed to make sure that the formulæ in paper I can be applied. For example, the one-loop counterterm in the local frame eq. (3.12) becomes

$$\mathcal{L} = \frac{1}{16\pi^2\epsilon}\left[-\frac{1}{4}X_{ij}X_{kl}\,g^{ik}g^{jl} - \frac{1}{24}[Y^{\mu\nu}]^i{}_j[Y_{\mu\nu}]^j{}_i\right] \tag{3.13}$$

in the original coordinate basis, where we have to be careful to raise and lower indices with the metric tensor g_{ij} and its inverse g^{ij}. The coordinate basis expressions for X and Y are

$$X_{ij} = -R_{ikjl}(D_\mu\phi)^k(D_\mu\phi)^l - \nabla_i\nabla_j V\,,$$
$$(Y_{\mu\nu})^i{}_j = [\mathscr{D}_\mu, \mathscr{D}_\nu]^i{}_j = R^i{}_{jkl}(D_\mu\phi)^k(D_\nu\phi)^l + \nabla_j t^i_\alpha F^\alpha_{\mu\nu}\,, \tag{3.14}$$

and $D_\mu\phi$ is given in eq. (2.3). Eq. (3.14) is the result derived in refs. [1, 2]. Eq. (3.14) can be computed using the original Lagrangian, without having to transform to a local frame. The terms have the same form as eq. (3.8) and eq. (3.11) with local frame indices replaced by coordinate indices. This method was used to compute the one-loop SMEFT anomalous dimensions in the bosonic sector to dimension eight in ref. [3].

For the two-loop correction, we need the cubic plus quartic terms in eq. (2.22), from which we can read off the expansion coefficients in eq. (3.1)

$$A_{abc} = -\frac{1}{6}\nabla_{(a}\nabla_b\nabla_{c)}V - \frac{1}{18}\left(\nabla_a R_{bdce} + \nabla_b R_{cdae} + \nabla_c R_{adbe}\right)(D_\mu\phi)^d(D^\mu\phi)^e$$

$$A^\mu_{a|bc} = \frac{1}{3}\left(R_{abcd} + R_{acbd}\right)(D^\mu\phi)^d$$

$$A^{\mu\nu}_{ab|c} = 0$$

$$B_{abcd} = -\frac{1}{24}\nabla_a\nabla_b\nabla_c\nabla_d V - \frac{1}{24}\nabla_a\nabla_d R_{becf}(D_\mu\phi)^e(D^\mu\phi)^f$$
$$\qquad + \frac{1}{6}R_{eabf}R_{ecdg}(D_\mu\phi)^f(D^\mu\phi)^g \quad \text{sym}(abcd)$$

$$B^\mu_{a|bcd} = \frac{1}{4}\left(\nabla_d R_{abce}\right)(D^\mu\phi)^e \quad \text{sym}(bcd)$$

$$B^{\mu\nu}_{ab|cd} = -\frac{1}{12}\eta^{\mu\nu}\left(R_{acbd} + R_{adbc}\right) \tag{3.15}$$

The notation sym(*abcd*) and sym(*bcd*) means the expressions have to be completely symmetrized in *abcd* and in *bcd*, i.e. we sum over the $n!$ permutations and divide by $n!$. A remarkable fact is that the coefficients A^μ and B^μ automatically satisfy the symmetry relations eq. (3.8) using the Bianchi identities $R_{abcd} + R_{bcad} + R_{cabd} = 0$ and $\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0$, just like the earlier result that N^μ was automatically antisymmetric in the local frame. We also find that $A^{\mu\nu}$ automatically vanishes. The above equations are tensorial, and so can be used in the original coordinate basis $i, j, k, \ldots$ being careful about raising and lowering indices using g_{ij}.

This derivation has been a lengthy one, but the final result is simple — Compute the quantities $A, A^\mu, B, B^\mu, B^{\mu\nu}$ using the expressions in eq. (3.15) in the original coordinate frame, and substitute into the counterterm expressions eq. (4.1) and eq. (4.3) of paper I, contracting upper indices with lower indices and raising and lowering indices using g_{ij}.

The counterterms in paper I depend on the tensors $X, Y, A, A^\mu, B, B^\mu, B^{\mu\nu}$ and their covariant derivatives. Multiple covariant derivatives are given by repeated applications of the covariant derivative $\mathscr{D}$ in eq. (2.14) which acts on tensors. ϕ is not a tensor, and its first derivative is $D_\mu \phi = \partial_\mu \phi + t_\alpha A_\mu^\alpha$. Since $D_\mu \phi$ is a tensor, subsequent covariant derivatives are given by $\mathscr{D}$. There are two formulæ which can be used to simplify multiple covariant derivatives,

$$\mathscr{D}_\mu (D_\nu \phi)^i - \mathscr{D}_\nu (D_\mu \phi)^i = t_\alpha^i F_{\mu\nu}^\alpha \,,$$
$$[\mathscr{D}_\mu, \mathscr{D}_\nu]^i{}_j = (Y_{\mu\nu})^i{}_j \,. \tag{3.16}$$

These can be used to reorder the indices on multiple covariant derivatives of ϕ and transform them to the form $\mathscr{D}_\mu(D^\mu \phi)$ which can be removed by a field redefinition. The tensors $X, Y, A, A^\mu, B, B^\mu, B^{\mu\nu}$ also contain the Riemann curvature tensor, the potential V, and their derivatives. Their Lie derivatives w.r.t. the Killing vectors t_α vanish. Any tensor $T(\phi)$ with arbitrary number of upper and lower indices and vanishing Lie derivative w.r.t. the Killing vectors t_α has covariant derivative

$$\mathscr{D}_\mu T = (D_\mu \phi)^i \left(\nabla_i T \right) . \tag{3.17}$$

If a tensor T satisfies $\mathscr{L}_{t_\alpha} T = 0$, then so does it covariant derivative ∇T, because $[\mathscr{L}_{t_\alpha}, \nabla] = 0$ when t_α is a Killing vector, which can be proved from the Kostant formula for a Killing vector eq. (2.19). Eq. (3.17) then gives the covariant derivatives of A, A^μ, etc. needed to compute the counterterms.

An EFT has an expansion in powers of $1/M$, where M is a high-energy scale much larger than the scales in the EFT. The potential V up to terms of order ϕ^4 is from the renormalizable dimension-four part of the Lagrangian, so the terms up to four derivatives in V power-count as dimension-four terms. Note that even terms such as $m^2 \phi^2$ are power-counted as dimension four terms rather than dimension two, because m is a low scale in the EFT. The mass term can be written in terms of the EFT power counting as $(m/M)^2 M^2 \phi^2$, i.e. as a dimension two operator with a m^2/M^2 suppressed coefficient, and so acts as dimension four. The same argument applies to other terms with dimension < 4 such as ϕ^3 interactions. The curvature R_{abcd} has terms with two derivatives of the metric, or the square of one derivative of the metric. These arise from $(\phi/M)^2$ or the square of ϕ/M terms in the metric respectively,

so $R_{abcd} \sim 1/M^2$ is a dimension-six object. Similarly, $\nabla R \sim 1/M^3$ is dimension seven, and $R^2 \sim 1/M^4$ and $\nabla\nabla R \sim 1/M^4$ are dimension eight.

4 The $O(n)$ EFT

The $O(n)$ EFT is the $O(n)$ model including higher dimension operators. Our formalism applies to Lagrangians with up to two derivatives, so the $O(n)$ Lagrangian has the form

$$\mathcal{L} = \frac{1}{2}A\,(\partial_\mu\phi \cdot \partial^\mu\phi) + B\,(\phi \cdot \partial_\mu\phi)(\phi \cdot \partial^\mu\phi) - V \tag{4.1}$$

where A, B and V are functions of $(\phi \cdot \phi)$, and ϕ is an n-component real scalar field. Loop corrections will generate higher derivative terms in the Lagrangian, which are implicitly added to eq. (4.1). While these are not included in the formulæ derived earlier, we can still consistently compute radiative corrections using eq. (4.1) — the result is the value of the counterterms in the complete theory when the higher derivative coefficients are set to zero. For example, in SMEFT we can compute the β-function for the dimension eight H^4D^4 operators $(D_\mu H^\dagger D_\nu H)(D^\nu H^\dagger D^\mu H)$ from insertions of dimension-six operators, but not the contribution of the dimension-eight operator to its own β-function.

One can make a field redefinition,

$$\phi^i = \varphi^i f(\varphi \cdot \varphi) \tag{4.2}$$

and eliminate the B term in eq. (4.1). We will work to dimension six using the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi \cdot \partial^\mu\phi) - \Lambda - \frac{1}{2}m^2(\phi \cdot \phi) - \frac{\lambda}{4}(\phi \cdot \phi)^2$$
$$+ C_E(\phi \cdot \phi)(\partial_\mu\phi \cdot \partial^\mu\phi) + C_1(\phi \cdot \phi)^3 + C_2(\phi \cdot \partial_\mu\phi)^2\,, \tag{4.3}$$

where Λ is the cosmological constant, and C_E, C_1, C_2 are coefficients of the dimension-six terms and have mass dimension -2. The field redefinition eq. (4.2) allows one to remove a linear combination of C_E and C_2; the linear combination which cannot be eliminated is $2C_E - C_2$. We work in the basis where $C_2 = 0$. The loop corrections generate C_2, and we use field redefinitions to eliminate C_2 and go back to the $C_2 = 0$ basis.

The metric and inverse metric are

$$g_{ij} = \delta_{ij}(1 + 2C_E(\phi \cdot \phi))\,, \qquad\qquad g^{ij} = \delta^{ij}(1 - 2C_E(\phi \cdot \phi))\,, \tag{4.4}$$

the Christoffel symbol is

$$\Gamma^i_{jk} = 2C_E\left[\delta^i_k\phi_j + \delta^i_j\phi_k - \delta_{jk}\phi^i\right]\,, \tag{4.5}$$

and the curvature is

$$R_{ijkl} = 4C_E\left[\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right]\,, \tag{4.6}$$

where the difference between up and down indices for Γ^i_{jk} and R_{ijkl} is dimension eight. Including counterterms, the bare Lagrangian is

$$\mathcal{L} = \frac{1}{2}Z_\phi(\partial_\mu\phi \cdot \partial^\mu\phi) - [\Lambda + \Lambda_{\text{c.t.}}] - \frac{1}{2}Z_\phi(m^2 + m^2_{\text{c.t.}})(\phi \cdot \phi) - \frac{1}{4}\mu^{2\epsilon}Z_\phi^2(\lambda + \lambda_{\text{c.t.}})(\phi \cdot \phi)^2$$
$$+ \mu^{2\epsilon}Z_\phi^2(C_E + C_{E\text{c.t.}})(\phi \cdot \phi)(\partial_\mu\phi \cdot \partial^\mu\phi) + \mu^{4\epsilon}Z_\phi^3(C_1 + C_{1\text{c.t.}})(\phi \cdot \phi)^3\,. \tag{4.7}$$

The explicit expressions for the counterterms are given in appendix A. The anomalous dimensions and 't Hooft consistency conditions can be computed from the counterterms, using the expressions given in paper I. The anomalous dimensions are (where $\dot{C} \equiv \mu \, \mathrm{d}C/\mathrm{d}\mu$)[3]

$$\dot{\Lambda} = \left\{ \frac{1}{2} n m^4 \right\}_1 + \{0\}_2 \tag{4.8}$$

$$\dot{m}^2 = \left\{ 2(n+2)\lambda m^2 - 8nm^4 C_E \right\}_1 + \left\{ -10(n+2)\lambda^2 m^2 + \frac{80}{3}(n+2)\lambda m^4 C_E \right\}_2 \tag{4.9}$$

$$\dot{\lambda} = \left\{ 2(n+8)\lambda^2 - 16(n+3)\lambda m^2 C_E - 24(n+4)m^2 C_1 \right\}_1$$
$$+ \left\{ -12(3n+14)\lambda^3 + \frac{32}{3}(22n+113)\lambda^2 m^2 C_E + 480(n+4)\lambda m^2 C_1 \right\}_2 \tag{4.10}$$

$$\dot{C}_E = \left\{ 4(n+2)\lambda C_E \right\}_1 + \left\{ -34(n+2)\lambda^2 C_E \right\}_2 \tag{4.11}$$

$$\dot{C}_1 = \left\{ 20\lambda^2 C_E + 6(n+14)\lambda C_1 \right\}_1 + \left\{ -\frac{8}{3}(23n+259)\lambda^3 C_E - 42(7n+54)\lambda^2 C_1 \right\}_2 \tag{4.12}$$

where the subscripts outside the curly braces denote the one-loop and two-loop contributions. The 't Hooft consistency conditions are satisfied. The anomalous dimensions of the $O(n)$ EFT were compute previously in refs. [19, 20] for $n = 1$ and $n = 2$. We agree with their results for $n = 1$ to two-loop order; they have results to 5 loops. They use a dimension-six basis different from ours for $n = 2$. The parameters in eq. (4.3) in terms of those of refs. [19, 20] are

$$\lambda = \frac{1}{4}g, \qquad C_E = \frac{1}{4}g C_4^{(6)}, \qquad C_2 = -\frac{1}{4}g C_4^{(6)}, \qquad C_1 = \frac{1}{3! \, 3! \, 8} g^2 C_6^{(6)}. \tag{4.13}$$

Our two-loop results for $n = 2$ agree with refs. [19, 20], which give results up to four loops.[4]

We find that the 't Hooft consistency condition is not satisfied for Z_ϕ, which implies that the field anomalous dimension is infinite,

$$\gamma_\phi = \left\{ -4(n-1)m^2 C_E \right\}_1 + \frac{1}{\epsilon} \left\{ 4(n-1)(n+2)\lambda m^2 C_E \right\}_2 + \left\{ (n+2)\lambda^2 - \frac{8}{3}(n+2)\lambda m^2 C_E \right\}_2 \tag{4.14}$$

The infinite contribution to γ_ϕ is proportional to m^2 times a dimension-six coefficient, and to $(n-1)$. The anomalous dimensions eq. (4.8)–(4.12) are finite, even though γ_ϕ enters in their computation through the Z_ϕ factors in eq. (4.7).

An infinite γ_ϕ arises because of the use of field redefinitions. Instead of the usual shift $\phi \to \overline{\phi} + \eta$, we use the expansion in eq. (2.9), which corresponds to a non-linear redefinition of η, $\eta \to \eta - \Gamma^i_{jk} \eta^j \eta^k / 2 + \ldots$. We can understand the origin of infinite field anomalous dimensions from the well-studied case of penguin graphs in the weak interactions. Consider the one-loop graph of figure 3 due to the insertion of a four-fermion operator $(\overline{\psi}\gamma_\mu\psi)(\overline{\psi}\gamma^\mu\psi)$, which leads to a divergence proportional to $(\overline{\psi}\gamma_\mu\psi)\partial_\nu F^{\mu\nu}$. One can make a field redefinition to

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[3]All $\{\ \}_1$ terms should be multiplied by $1/(16\pi^2)$ and all $\{\ \}_2$ terms by $1/(16\pi^2)^2$.

[4]There is a typo in refs. [19, 20], where the ϕ^6 operator is given with a normalization $1/(6! \, 8)$ instead of $1/(3! \, 3! \, 8)$. We thank Jasper Nepveu for informing us of this. With this change, our result differs in the sign of the $45g^2/2$ two-loop term in the $C_6^{(6)} - C_4^{(6)}$ entry of the anomalous dimension matrix.
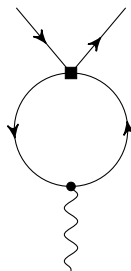
Figure 3. Penguin graph.
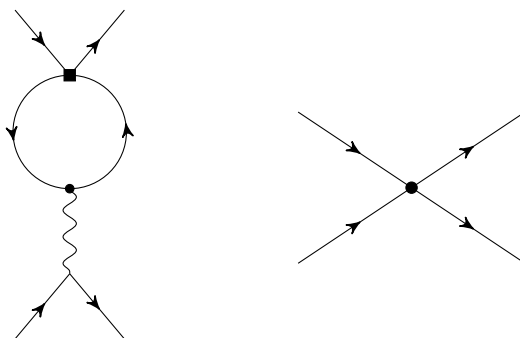
Figure 4. $\psi + \psi \to \psi + \psi$ scattering amplitude.

replace the $(\overline{\psi}\gamma_\mu\psi)\partial_\nu F^{\mu\nu}$ by a four-fermion counterterm $(\overline{\psi}\gamma_\mu\psi)(\overline{\psi}\gamma^\mu\psi)$. The $\psi + \psi \to \psi + \psi$ scattering amplitude has contributions from the penguin graph and the ψ^4 counterterm, as shown in figure 4, and the sum of the two graphs is finite, and gives a finite S-matrix element. However, the three-point correlation function $\left\langle \overline{\psi}\psi A_\mu \right\rangle$ given by the penguin graph in figure 3 is infinite, as there is no longer a $(\overline{\psi}\gamma_\mu\psi)\partial_\nu F^{\mu\nu}$ counterterm to cancel the divergence. As a result the field A_μ is not finite, and can have divergent correlation functions. The reason is that the field redefinition to remove the $(\overline{\psi}\gamma_\mu\psi)\partial_\nu F^{\mu\nu}$ operator is an infinite field redefinition. In our example, the divergent contribution to γ_ϕ is proportional to $(n-1)$, and so vanishes if there is only a single scalar field, in which case the Riemann curvature vanishes. The coupling constants m^2, λ, etc. are related to observable S-matrix elements, and so are not affected by field redefinitions.

An interesting example of an infinite field anomalous dimension was discussed in detail recently [31–33]. The authors found that the three-loop β-functions in the SM were divergent, which could be compensated by making an infinite flavor rotation of the fields.

5   SMEFT

We apply our two-loop formula to compute the two-loop scalar renormalization in SMEFT to dimension six. We include the SM Higgs sector, as well as insertions of the dimension six operators

$$C_H, \ C_{H\square}, \ C_{HD}, \ C_{HG}, \ C_{HW}, \ C_{HB}, \ C_{HWB}, \ C_{H\widetilde{G}}, \ C_{H\widetilde{W}}, \ C_{H\widetilde{B}}, \ C_{H\widetilde{W}B}, \tag{5.1}$$

and compute their anomalous dimensions from internal scalar loops. The two-loop counterterms are given in appendix B.

The anomalous dimensions due to scalar loops, with subscripts denoting the one-loop and two-loop contributions, are

$$\dot\Lambda = \left\{ \frac{1}{2} m_H^4 \right\}_1 + \{0\}_2 \,, \tag{5.2}$$

$$\dot m_H^2 = \left\{ 12\lambda - 4m_H^2 C_{H\Box} + 2m_H^2 C_{HD} \right\}_1 m_H^2 + \lambda \left\{ -60\lambda + 80 m_H^2 C_{H\Box} - 20 m_H^2 C_{HD} \right\}_2 m_H^2 \,, \tag{5.3}$$

$$\dot\lambda = \left\{ 24\lambda^2 + 12 m_H^2 C_H - 32\lambda m_H^2 C_{H\Box} + 12\lambda m_H^2 C_{HD} \right\}_1$$
$$+ \left\{ -312\lambda^3 - 240\lambda m_H^2 C_H + 1096\lambda^2 m_H^2 C_{H\Box} - 282\lambda^2 m_H^2 C_{HD} \right\}_2 \,, \tag{5.4}$$

$$\dot C_H = \left\{ 108\lambda C_H - 160\lambda^2 C_{H\Box} + 48\lambda^2 C_{HD} \right\}_1 + \left\{ -3444\lambda^2 C_H + 7968\lambda^3 C_{H\Box} - 1992\lambda^3 C_{HD} \right\}_2 \,, \tag{5.5}$$

$$\dot C_{H\Box} = \{ 24\lambda C_{H\Box} \}_1 + \left\{ -204\lambda^2 C_{H\Box} \right\}_2 \,, \tag{5.6}$$

$$\dot C_{HD} = \{ 12\lambda C_{HD} \}_1 + \left\{ -144\lambda^2 C_{HD} \right\}_2 \,, \tag{5.7}$$

$$\dot C_r = \{ 12\lambda C_r \}_1 + \left\{ -60\lambda^2 C_r \right\}_2 \,, \qquad r \in \{ HG, H\widetilde{G}, HW, H\widetilde{W}, HB, H\widetilde{B} \} \,, \tag{5.8}$$

$$\dot C_r = \{ 4\lambda C_r \}_1 + \left\{ -28\lambda^2 C_r \right\}_2 \,, \qquad r \in \{ HWB, H\widetilde{W}B \} \,. \tag{5.9}$$

The one-loop contributions agree with the known values [34–36]. The two loop contributions with the dimension-six coefficients set to zero agree with the known two loop values for the SM [21–23]. Note that there are very large coefficients in the two-loop anomalous dimensions. The 't Hooft consistency conditions for the couplings listed above are satisfied.

The Higgs field anomalous dimension is divergent at two loops, i.e. the 't Hooft consistency conditions are not satisfied,

$$\gamma_H = \left\{ 3m_H^2 C_{H\Box} \right\}_1 + \left\{ -\frac{1}{\epsilon} 18\lambda m_H^2 C_{H\Box} + 6\lambda^2 - 8\lambda m_H^2 C_{H\Box} + 2\lambda m_H^2 C_{HD} \right\}_2 \tag{5.10}$$

and the divergent term agrees with eq. (4.14) for $n = 4$ on replacing m^2 and C_E by their equivalent SMEFT couplings, $m^2 \to -m_H^2/2$, $C_E \to C_{H\Box}/2$.

6 χPT

The final example we consider is chiral perturbation theory. We study the counterterms for the QCD chiral lagrangian for n flavors, with the symmetry breaking pattern $\mathrm{SU}(n) \times \mathrm{SU}(n) \to \mathrm{SU}(n)$. The notation follows refs. [24, 37], except that we use a Minkowski signature for the spacetime metric. We restrict ourselves to the pure chiral theory without external sources. The chiral Lagrangian is written in terms of a field u,

$$u(x) = e^{\frac{i}{f}\boldsymbol{\pi}(x)} \qquad\qquad \boldsymbol{\pi}(x) = \pi^a(x) T^a \qquad\qquad \langle T^a T^b \rangle = \frac{1}{2}\delta^{ab} \tag{6.1}$$

where $f \sim 93\,\text{MeV}$ is the pion decay constant, and $\langle \cdot \rangle$ denotes a trace. Under chiral $\text{SU}(n) \times \text{SU}(n)$, $u(x)$ transforms as

$$u(x) \to R u(x) h^{-1}(x) = h(x) u(x) L^{-1} \tag{6.2}$$

where L and R are the left- and right-handed $\text{SU}(n)$ transformations, and $h(x)$ is defined implicitly through eq. (6.2). The field $\text{U}(x) = u(x)^2$ transforms as $\text{U}(x) \to R\,\text{U}(x) L^{-1}$. Note that L and R are global transformations, and do not depend on x, since we have not included external gauge fields.

Chiral perturbation theory has a systematic expansion in powers of p^2, where p is the external momentum of the pions. The leading term is the order p^2 Lagrangian,

$$\mathcal{L}_2 = \frac{f^2}{4} \langle u_\mu u^\mu \rangle \,, \tag{6.3}$$

where

$$u_\mu = i(u^\dagger \partial_\mu u - u\, \partial_\mu u^\dagger)\,. \tag{6.4}$$

The order p^4 Lagrangian is

$$\mathcal{L}_4 = \widehat{L}_0 \langle u_\mu u_\nu u^\mu u^\nu \rangle + \widehat{L}_1 \langle u \cdot u \rangle^2 + \widehat{L}_2 \langle u_\mu u_\nu \rangle \langle u^\mu u^\nu \rangle + \widehat{L}_3 \left\langle (u \cdot u)^2 \right\rangle \tag{6.5}$$

with four independent coefficients in the absence of external sources. The coefficients in eq. (6.5) include counterterm contributions,

$$\widehat{L}_i = (c\mu)^{-2\epsilon} \left[-\frac{1}{2\epsilon} \frac{1}{16\pi^2} \widehat{\Gamma}_i + \widehat{L}_i^r(\mu) \right]\,, \tag{6.6}$$

where $\widehat{\Gamma}_i$ are the counterterms, $\widehat{L}_i^r(\mu)$ are the finite renormalized couplings, and $c^2\mu^2 = \mu^2 e^{\gamma-1}/(4\pi)$. The renormalized coefficients satisfy the renormalization group equations

$$\mu \frac{\mathrm{d}\widehat{L}_i^r}{\mathrm{d}\mu} = -\frac{1}{16\pi^2} \widehat{\Gamma}_i\,. \tag{6.7}$$

The p^6 Lagrangian is

$$\mathcal{L}_6 = \sum_{i=1}^{115} K_i Y_i \tag{6.8}$$

with coefficients K_i and operators Y_i listed in ref. [24]. The coefficients K_i include counterterm contributions, and are given in terms of renormalized couplings K_i^r by

$$K_i = \frac{(c\mu)^{-4\epsilon}}{f^2} \left[-\frac{1}{4\epsilon^2} \frac{1}{(16\pi^2)^2} \widehat{\Gamma}_i^{(2)} + \frac{1}{2\epsilon} \frac{1}{16\pi^2} \widehat{\Gamma}_i^{(1)} + \frac{1}{2\epsilon} \frac{1}{16\pi^2} \widehat{\Gamma}_i^{(L)} + K_i^r \right] \tag{6.9}$$

$\widehat{\Gamma}_i^{(1,2)}$ are the counterterms for two-loop graphs from the p^2 Lagrangian, and $\widehat{\Gamma}_i^{(L)}$ are the counterterms from one-loop graphs with an insertion of the p^4 Lagrangian. The renormalized couplings satisfy the renormalization group equations

$$\mu \frac{\mathrm{d}K_i^r}{\mathrm{d}\mu} = \frac{1}{16\pi^2} \left[2\widehat{\Gamma}_i^{(1)} + \widehat{\Gamma}_i^{(L)} \right]\,. \tag{6.10}$$

't Hooft's consistency conditions for the $1/\epsilon^2$ pole are

$$\mu \frac{\mathrm{d}\widehat{\Gamma}_i^{(L)}}{\mathrm{d}\mu} = -\frac{1}{8\pi^2}\widehat{\Gamma}_i^{(2)} \tag{6.11}$$

which become

$$2\,\widehat{\Gamma}_i^{(2)} = \sum_k \frac{\partial \widehat{\Gamma}_i^{(L)}}{\partial \widehat{L}_k^r}\,\widehat{\Gamma}_k \tag{6.12}$$

using eq. (6.7). Eqs. (6.11), (6.12) are known as Weinberg's consistency conditions [38] for χPT, and provide a check on the two-loop result.

In the presence of external sources, there are 115 terms in the p^6 Lagrangian. There are 21 operators listed in ref. [24] which are non-zero when external sources are turned off — Y_{1-6} and Y_{49-63}. However, in the absence of external sources, these operators are not all linearly independent. There are two linear relations,[5]

$$0 = Y_2 + 2Y_4 + 6Y_6 - 2Y_{50} + 2Y_{57}\,,$$
$$0 = 4Y_1 - Y_3 + 3Y_5 - 4Y_{49} - 2Y_{52} + 8Y_{54} + 2Y_{58} - 4Y_{60}\,, \tag{6.13}$$

leaving 19 independent operators. In the presence of external sources, the linear combinations in eq. (6.13) are proportional to operators involving external sources, on using field redefinitions. The choice in ref. [24] was to include the linear combinations in eq. (6.13) rather than the corresponding source-dependent operators. We have used eq. (6.13) to eliminate Y_{57} and Y_{60}.

The first relation in eq. (6.13) explains one feature of the counterterms found in ref. [24]. There is a contribution to $\widehat{\Gamma}^{(L)}$ proportional to $\widehat{L}_9^r$,

$$\widehat{\Gamma}_i^{(L)} Y_i = \widehat{L}_9^r \left(-\frac{1}{8}Y_2 - \frac{1}{4}Y_4 - \frac{3}{4}Y_6 + \frac{1}{4}Y_{50} - \frac{1}{4}Y_{57} \right)\,, \tag{6.14}$$

from table V in ref. [24]. $\widehat{L}_9$ is not a coefficient in the p^4 Lagrangian eq. (6.5), and should not contribute to the counterterm. It multiplies the operator $i\langle f_{+\mu\nu} u_\mu u_\nu \rangle$ in the p^4 Lagrangian, which vanishes when there are no external fields. The linear combination eq. (6.14) vanishes on using eq. (6.13).

Weinberg's power counting rule for χPT implies that the order p^2 scattering amplitude is given by tree graphs with vertices from the p^2 Lagrangian. The order p^4 scattering amplitude is given by tree graphs with one insertion of the p^4 Lagrangian and arbitrary p^2 vertices, or one-loop graphs with only p^2 vertices. The p^6 amplitude is given by tree graphs with one insertion of the p^6 Lagrangian or two insertions of the p^4 Lagrangian (and arbitrary p^2 vertices), one-loop graphs with one insertion of the p^4 Lagrangian (and arbitrary p^2 vertices), or two-loop graphs involving only p^2 interactions.

The fields in χPT live on a coset space G/H, which is the group manifold $\mathrm{SU}(n)$ for QCD chiral perturbation theory. The independent coordinates can be chosen to be π^a defined in eq. (6.1), and the pion covariant derivative [42, 43] is

$$D_\mu \pi = -\frac{f}{2}\,u_\mu = \partial_\mu \pi - \frac{1}{6f^2}[\pi, [\pi, \partial_\mu \pi]] + \dots \tag{6.15}$$

---

[5]The existence of two relations was noted in refs. [39–41]. The linear relations can be obtained from table 3.13 in [39] after replacing relation 26 by $2 \times (26) + (63) + 2 \times (76) + (137) = 0$.

The Riemann curvature tensor is

$$R_{abcd} = \frac{1}{f^2} f_{abg} f_{cdg} + \dots \tag{6.16}$$

where f_{abc} are the $SU(n)$ structure constants. In the notation of refs. [24, 37], the chiral covariant derivative acting on a field X that transforms as $X \to hXh^{-1}$ is

$$\nabla_\mu X = \partial_\mu X + [\Gamma_\mu, X] \qquad \Gamma_\mu = \frac{1}{2}(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger) \tag{6.17}$$

and the curvature (field-strength) is

$$[\nabla_\mu, \nabla_\nu]X = [\Gamma_{\mu\nu}, X] \qquad \Gamma_{\mu\nu} = \frac{1}{4}[u_\mu, u_\nu] - \frac{i}{2}f_{+\mu\nu} \tag{6.18}$$

where $f_{+\mu\nu} = 0$ when external sources are turned off. $\Gamma_{\mu\nu}$ is the curvature eq. (6.16) on rescaling by $(2/f)^2$ from the normalization of $D_\mu\pi$ relative to u_μ.

For a Lie group, the covariant derivative of the Riemann curvature tensor vanishes,

$$\nabla_e R_{abcd} = 0. \tag{6.19}$$

This greatly simplifies the computation of loop corrections, since many terms in the tensors $X, Y, A, A^\mu, B, B^\mu, B^{\mu\nu}$ and their covariant derivatives vanish. With external sources set to zero, the tensors take the simple form:

$$X_{ab} = -R_{acbd}(D_\mu\pi)^c(D^\mu\pi)^d,$$

$$[Y_{\mu\nu}]_{ab} = R_{abcd}(D_\mu\pi)^c(D_\nu\pi)^d,$$

$$A_{abc} = 0,$$

$$A^\mu_{a|bc} = \frac{1}{3}(R_{abcd} + R_{acbd})(D^\mu\pi)^d$$

$$A^{\mu\nu}_{ab|c} = 0$$

$$B_{abcd} = \frac{1}{6}R_{eabf}R_{ecdg}(D_\mu\pi)^f(D^\mu\pi)^g \quad \text{sym}(abcd),$$

$$B^\mu_{a|bcd} = 0$$

$$B^{\mu\nu}_{ab|cd} = -\frac{1}{12}\eta^{\mu\nu}(R_{acbd} + R_{adbc}). \tag{6.20}$$

The one-loop graphs from the p^2 Lagrangian generate order p^4 counterterms using eq. (3.12). We can expand about $\pi = 0$, and compare our counterterms with eq. (6.5) to get the one-loop counterterm coefficients

$$\widehat{\Gamma}_0 = \frac{n}{48}, \qquad \widehat{\Gamma}_1 = \frac{1}{16}, \qquad \widehat{\Gamma}_2 = \frac{1}{8}, \qquad \widehat{\Gamma}_3 = \frac{n}{24}, \tag{6.21}$$

which agrees with [24, (3.14)].

The two-loop graphs from the p^2 Lagrangian generate order p^6 counterterms $\widehat{\Gamma}^{(2)}_i$ and $\widehat{\Gamma}^{(1)}_i$ which are listed in table 1. These agree with table IV of [24] using the identities eq. (6.13) to eliminate Y_{57} and Y_{60}. The agreement provides a highly non-trivial check of our method.

Y	$\widehat{\Gamma}_i^{(2)}$	$16\pi^2\widehat{\Gamma}_i^{(1)}$	$\widehat{\Gamma}_i^{(L)}$
1	$\frac{5}{48}+\frac{1}{64}n^2$	$-\frac{67}{576}-\frac{7}{1728}n^2$	$\frac{1}{12}n\widehat{L}_0+3\widehat{L}_1+\frac{1}{6}\widehat{L}_2+\frac{17}{24}n\widehat{L}_3$
2	$\frac{1}{576}n$	$-\frac{31}{6912}n$	$\frac{5}{24}\widehat{L}_0^r-\frac{1}{24}n\widehat{L}_2^r+\frac{5}{48}\widehat{L}_3^r$
3	$-\frac{5}{48}+\frac{1}{2304}n^2$	$\frac{61}{2304}+\frac{11}{27648}n^2$	$\frac{1}{48}n\widehat{L}_0-\frac{7}{6}\widehat{L}_1-\frac{13}{12}\widehat{L}_2+\frac{1}{96}n\widehat{L}_3$
4	$-\frac{11}{72}n$	$-\frac{49}{3456}n$	$-\frac{5}{4}\widehat{L}_0^r-2n\widehat{L}_1^r-\frac{3}{4}n\widehat{L}_2^r-\frac{35}{24}\widehat{L}_3^r$
5	$-\frac{1}{768}n^2$	$-\frac{23}{256}-\frac{49}{27648}n^2$	$-\frac{11}{48}n\widehat{L}_0+\frac{11}{6}\widehat{L}_1-\frac{11}{12}\widehat{L}_2+\frac{5}{96}n\widehat{L}_3$
6	$-\frac{13}{32}n$	$-\frac{5}{192}n$	$-\frac{3}{4}\widehat{L}_0^r-6n\widehat{L}_1^r-\frac{9}{4}n\widehat{L}_2^r-\frac{27}{8}\widehat{L}_3^r$
49	$-\frac{5}{576}n^2$	$\frac{5}{48}+\frac{1}{2304}n^2$	$\frac{1}{4}n\widehat{L}_0-\frac{4}{3}\widehat{L}_1+\frac{2}{3}\widehat{L}_2-\frac{13}{24}n\widehat{L}_3$
50	$\frac{1}{32}n$	$\frac{5}{128}n$	$-\frac{1}{4}\widehat{L}_0^r+\frac{1}{4}n\widehat{L}_2^r+\frac{7}{8}\widehat{L}_3^r$
51	$\frac{1}{64}$	$\frac{5}{256}$	$\frac{1}{4}\widehat{L}_2^r$
52	$-\frac{11}{384}n^2$	$\frac{17}{128}+\frac{77}{13824}n^2$	$-\frac{17}{24}n\widehat{L}_0-\frac{1}{3}\widehat{L}_1+\frac{1}{6}\widehat{L}_2-\frac{49}{48}n\widehat{L}_3$
53	$-\frac{1}{64}n$	$-\frac{5}{256}n$	$\widehat{L}_0^r-\frac{5}{4}\widehat{L}_3^r$
54	$\frac{1}{48}n^2$	$-\frac{17}{64}-\frac{13}{3456}n^2$	$-\frac{1}{3}n\widehat{L}_0+\frac{2}{3}\widehat{L}_1-\frac{1}{3}\widehat{L}_2+\frac{7}{6}n\widehat{L}_3$
55	$-\frac{1}{24}n$	$-\frac{1}{72}n$	$-\frac{2}{3}\widehat{L}_0^r-\frac{2}{3}n\widehat{L}_2^r+\frac{1}{3}\widehat{L}_3^r$
56	$-\frac{1}{32}$	$\frac{3}{128}$	$-\frac{1}{2}\widehat{L}_2^r$
58	$\frac{1}{1152}n^2$	$\frac{11}{384}-\frac{13}{13824}n^2$	$\frac{1}{24}n\widehat{L}_0+\widehat{L}_1-\frac{1}{2}\widehat{L}_2+\frac{1}{48}n\widehat{L}_3$
59	$-\frac{1}{192}n$	$\frac{65}{2304}n$	$\widehat{L}_0^r-\frac{3}{4}\widehat{L}_3^r$
61	$\frac{7}{192}n$	$-\frac{23}{2304}n$	$\widehat{L}_0^r+\frac{5}{4}\widehat{L}_3^r$
62	$-\frac{1}{12}n$	$-\frac{5}{288}n$	$-\frac{13}{3}\widehat{L}_0^r-\frac{1}{3}n\widehat{L}_2^r-\frac{5}{6}\widehat{L}_3^r$
63	$-\frac{1}{8}$	$-\frac{1}{32}$	$-2\widehat{L}_2^r$

Table 1. The counterterms for the p^6 Lagrangian. The first column lists the operator Y_i. Y_{57} and Y_{60} were eliminated using eq. (6.13). The second and third columns list the $1/\epsilon^2$ and $1/\epsilon$ counterterm coefficients $\widehat{\Gamma}_i^{(2)}$ and $\widehat{\Gamma}_i^{(1)}$ from two-loop graphs from the p^2 Lagrangian. The last column lists the $1/\epsilon$ one-loop counterterm from an insertion of the p^4 Lagrangian. The normalization is given in eq. (6.9). The counterterms agree with ref. [24] after using eq. (6.13).

We also need the one-loop corrections with an insertion of the p^4 Lagrangian eq. (6.5). The p^4 Lagrangian has four derivatives, but at most one derivative acting on a single field, and our method can still be used. The one-loop graphs are computed by expanding the Lagrangian to second order in the quantum field η, i.e. by taking the second derivative of the Lagrangian in the Riemann normal coordinate expansion of section 2. Using the basic equations eq. (2.13) and eq. (2.20) for the derivatives on fields, the second variation of the p^4 Lagrangian is

$$C_{ab}\eta^a\eta^b = 2\widehat{L}_0\langle R(\eta,u_\mu)\eta\,u_\nu\,u^\mu\,u^\nu\rangle + 2\widehat{L}_1\langle R(\eta,u_\mu)\eta\,u^\mu\rangle\langle u_\nu\,u^\nu\rangle + 2\widehat{L}_2\langle R(\eta,u_\mu)\eta\,u_\nu\rangle\langle u^\mu\,u^\nu\rangle$$
$$+ \widehat{L}_3\big[\langle R(\eta,u_\mu)\eta\,u^\mu\,u_\nu\,u^\nu\rangle + \langle R(\eta,u_\mu)\eta\,u_\nu\,u^\nu\,u^\mu\rangle\big] \tag{6.22}$$

$$C_{ab}^\mu = 0 \tag{6.23}$$

$$C_{ab}^{\mu\nu}(\nabla_\mu\eta)^a(\nabla_\nu\eta)^b = \widehat{L}_0\left[4\langle\nabla_\mu\eta\,\nabla_\nu\eta\,u^\mu\,u^\nu\rangle + 2\langle\nabla_\mu\eta\,u_\nu\,\nabla^\mu\eta\,u^\nu\rangle\right]$$
$$+ \widehat{L}_1\left[2\langle\nabla_\mu\eta\,\nabla^\mu\eta\rangle\langle u_\nu\,u^\nu\rangle + 4\langle\nabla_\mu\eta\,u^\mu\rangle\langle\nabla_\nu\eta\,u^\nu\rangle\right]$$
$$+ \widehat{L}_2\left[2\langle\nabla_\mu\eta\,\nabla_\nu\eta\rangle\langle u^\mu\,u^\nu\rangle + 2\langle\nabla_\mu\eta\,u_\nu\rangle\langle\nabla^\mu\eta\,u^\nu\rangle + 2\langle\nabla_\mu\eta\,u_\nu\rangle\langle\nabla^\nu\eta\,u^\mu\rangle\right]$$
$$+ \widehat{L}_3\left[2\langle\nabla_\mu\eta\,\nabla^\mu\eta\,u_\nu\,u^\nu\rangle + 2\langle\nabla_\mu\eta\,\nabla_\nu\eta\,u^\nu\,u^\mu\rangle + \langle\nabla_\mu\eta\,u^\mu\,\nabla_\nu\eta\,u^\nu\rangle + \langle\nabla_\mu\eta\,u_\nu\,\nabla^\nu\eta\,u^\mu\rangle\right]$$
$$\tag{6.24}$$

on comparing with the general second variation eq. (3.10) in paper I. The one-loop counterterm in eq. (3.11) of paper I gives the results in the $\widehat{\Gamma}_i^{(L)}$ column of table 1, and agrees with [24] using eq. (6.13).[6]

In the general case, one-loop corrections from higher derivative terms with at most a single derivative on each field can be computed the same way. First, the fluctuation Lagrangian is computed to quadratic order in η to determine the coefficients C_{ab}, C_{ab}^μ and $C_{ab}^{\mu\nu}$, and then, the one-loop counterterms are computed from eq. (3.11) of paper I. This was also used to compute the one-loop RGE in SMEFT from dimension-eight H^4D^4 operators, which agrees with the previous calculation of ref. [44] using diagrammatic methods. The H^4D^4 contribution to the SMEFT RGEs is given in appendix C.

Two-loop corrections are computed from η^3 and η^4 fluctuation terms. Higher derivative interactions can generate $(D\eta)^2\eta$, $(D\eta)^3$, $(D\eta)^3\eta$ and $(D\eta)^4$ interactions. The two-loop counterterms can then be computed by the same method as given in paper I for the A-type and B-type two-loop counterterms. However, we have not explicitly computed the counterterm coefficients with these additional interactions, so we cannot determine the two-loop corrections from higher derivative operators without additional work.

7 Conclusions

The geometric method has been used to compute the two-loop renormalization of generic scalar QFTs, applying the counterterm formula developed in paper I. The counterterms can be derived purely algebraically, in terms of covariant derivatives of the scalar potential

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[6]In an earlier version of this paper, we found a discrepancy with the result of [24]. The source of this discrepancy was that in eq. (3.11) of paper I, the terms $\frac{1}{12}C_{ab}^{\mu\nu}Y_{bc}^{\mu\alpha}Y_{ca}^{\nu\alpha} - \frac{1}{4}C_{ab}^{\mu\nu}Y_{bc}^{\nu\alpha}Y_{ca}^{\mu\alpha}$ were instead written as $\frac{1}{12}C_{ab}^{\mu\nu}Y_{bc}^{\nu\alpha}Y_{ca}^{\mu\alpha} - \frac{1}{4}C_{ab}^{\mu\nu}Y_{bc}^{\mu\alpha}Y_{ca}^{\nu\alpha}$. We thank J. Bijnens, G. Colangelo, and G. Ecker for help resolving this discrepancy.

and Riemann curvature tensor, which are computed by taking derivatives of polynomial coefficients in the Lagrangian. We have applied the method to the $O(n)$ EFT, the Higgs sector of the SMEFT, and chiral perturbation theory. The agreement with previous results computed by other methods provides a highly non-trivial check on our calculation.

The results in this paper are for scalar loops. The method is generalizable to gauge and fermion loops, as has already been done for one-loop renormalization [3–7].

Acknowledgments

A Two-loop Counterterms for the $O(n)$ EFT

Using our one-loop and two-loop formulæ gives the counterterms for the $O(n)$ EFT listed below. The subscripts $1, 2$ denote the one-loop and two-loop contributions, respectively (see footnote 3).

$$
\begin{aligned}
Z_\phi = 1 + \frac{1}{\epsilon} & \left\{ 4(n-1)m^2 C_E \right\}_1 + \frac{1}{\epsilon^2} \left\{ 4(n-1)(n+2)\lambda m^2 C_E \right\}_2 \\
&+ \frac{1}{\epsilon} \left\{ -\frac{1}{2}(n+2)\lambda^2 + \frac{4}{3}(n+2)\lambda m^2 C_E \right\}_2
\end{aligned}
\tag{A.1}
$$

$$
\Lambda_{\text{c.t.}} = \frac{1}{\epsilon} \left\{ \frac{nm^4}{4} \right\}_1 + \frac{1}{\epsilon^2} \left\{ \frac{1}{4}n(n+2)\lambda m^4 - n^2 m^6 C_E \right\}_2
\tag{A.2}
$$

$$
\begin{aligned}
m_{\text{c.t.}}^2 = \frac{1}{\epsilon} & \left\{ (n+2)\lambda m^2 - 4nm^4 C_E \right\}_1 + \frac{1}{\epsilon^2} \{ (n+2)(n+5)\lambda^2 m^2 - 2(n+2)(7n+6)\lambda m^4 C_E \\
&- 6(n+2)(n+4)m^4 C_1 \}_2 + \frac{1}{\epsilon} \left\{ -\frac{5}{2}(n+2)\lambda^2 m^2 + \frac{20}{3}(n+2)\lambda m^4 C_E \right\}_2
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
\lambda_{\text{c.t.}} = \frac{1}{\epsilon} & \left\{ (n+8)\lambda^2 - 8(n+3)\lambda m^2 C_E - 12(n+4)m^2 C_1 \right\}_1 + \frac{1}{\epsilon^2} \{ (n+8)^2 \lambda^3 \\
&- 12(2n^2 + 21n + 50)\lambda^2 m^2 C_E - 36(n+4)(n+10)\lambda m^2 C_1 \}_2 \\
&+ \frac{1}{\epsilon} \left\{ -3(3n+14)\lambda^3 + \frac{8}{3}(22n+113)\lambda^2 m^2 C_E + 120(n+4)\lambda m^2 C_1 \right\}_2
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
C_{1,\text{c.t.}} = \frac{1}{\epsilon} & \left\{ 10\lambda^2 C_E + 3(n+14)\lambda C_1 \right\}_1 + \frac{1}{\epsilon^2} \{ 5(7n+62)\lambda^3 C_E + 3(n+14)(2n+25)\lambda^2 C_1 \}_2 \\
&+ \frac{1}{\epsilon} \left\{ -\frac{2}{3}(23n+259)\lambda^3 C_E - \frac{21}{2}(7n+54)\lambda^2 C_1 \right\}_2
\end{aligned}
\tag{A.5}
$$

$$
C_{E,\text{c.t.}} = \frac{1}{\epsilon} \left\{ 2(n+2)\lambda C_E \right\}_1 + \frac{1}{\epsilon^2} \left\{ 3(n+2)(n+4)\lambda^2 C_E \right\}_2 + \frac{1}{\epsilon} \left\{ -\frac{17}{2}(n+2)\lambda^2 C_E \right\}_2
\tag{A.6}
$$

B Two-loop counterterms for SMEFT

The one- and two-loop counterterms for the scalar sector of SMEFT are

$$Z_H = 1 + \frac{1}{\epsilon}\left\{-3C_{H\square}m_H^2\right\}_1 + \frac{1}{\epsilon}\left\{-3\lambda^2 + 4C_{H\square}\lambda m_H^2 - C_{HD}\lambda m_H^2\right\}_2 + \frac{1}{\epsilon^2}\left\{-18C_{H\square}\lambda m_H^2\right\}_2$$

(B.1)

$$\Lambda_{\text{c.t.}} = \frac{1}{\epsilon}\left\{\frac{1}{4}m_H^4\right\}_1 + \frac{1}{\epsilon^2}\left\{\frac{3}{2}\lambda m_H^4 - \frac{1}{2}C_{H\square}m_H^6 + \frac{1}{4}C_{HD}m_H^6\right\}_2$$

(B.2)

$$(m_H^2)_{\text{c.t.}} = \frac{1}{\epsilon}\left\{6\lambda m_H^2 - 2C_{H\square}m_H^4 + C_{HD}m_H^4\right\}_1$$
$$+ \frac{1}{\epsilon^2}\left\{54\lambda^2 m_H^2 + 18C_H m_H^4 - 78C_{H\square}\lambda m_H^4 + 30C_{HD}\lambda m_H^4\right\}_2$$
$$+ \frac{1}{\epsilon}\left\{-15\lambda^2 m_H^2 + 20C_{H\square}\lambda m_H^4 - 5C_{HD}\lambda m_H^4\right\}_2$$

(B.3)

$$\lambda_{\text{c.t.}} = \frac{1}{\epsilon}\left\{12\lambda^2 + 6C_H m_H^2 - 16C_{H\square}\lambda m_H^2 + 6C_{HD}\lambda m_H^2\right\}_1$$
$$+ \frac{1}{\epsilon^2}\left\{144\lambda^3 + 252C_H\lambda m_H^2 - 672C_{H\square}\lambda^2 m_H^2 + 216C_{HD}\lambda^2 m_H^2\right\}_2$$
$$+ \frac{1}{\epsilon}\left\{-78\lambda^3 - 60C_H\lambda m_H^2 + 274C_{H\square}\lambda^2 m_H^2 - \frac{141}{2}C_{HD}\lambda^2 m_H^2\right\}_2$$

(B.4)

$$(C_H)_{\text{c.t.}} = \frac{1}{\epsilon}\left\{54C_H\lambda - 80C_{H\square}\lambda^2 + 24C_{HD}\lambda^2\right\}_1$$
$$+ \frac{1}{\epsilon^2}\left\{1782C_H\lambda^2 - 3600C_{H\square}\lambda^3 + 1008C_{HD}\lambda^3\right\}_2$$
$$+ \frac{1}{\epsilon}\left\{-861C_H\lambda^2 + 1992C_{H\square}\lambda^3 - 498C_{HD}\lambda^3\right\}_2$$

(B.5)

$$(C_{H\square})_{\text{c.t.}} = \frac{1}{\epsilon}\left\{12C_{H\square}\lambda\right\}_1 + \frac{1}{\epsilon^2}\left\{144C_{H\square}\lambda^2\right\}_2 + \frac{1}{\epsilon}\left\{-51C_{H\square}\lambda^2\right\}_2$$

(B.6)

$$(C_{HD})_{\text{c.t.}} = \frac{1}{\epsilon}\left\{6C_{HD}\lambda\right\}_1 + \frac{1}{\epsilon^2}\left\{54C_{HD}\lambda^2\right\}_2 + \frac{1}{\epsilon}\left\{-36C_{HD}\lambda^2\right\}_2$$

(B.7)

$$(C_r)_{\text{c.t.}} = \frac{1}{\epsilon}\left\{6C_{HG}\lambda\right\}_1 + \frac{1}{\epsilon^2}\left\{54C_{HG}\lambda^2\right\}_2 + \frac{1}{\epsilon}\left\{-15C_{HG}\lambda^2\right\}_2$$

(B.8)

for $r = HG, H\widetilde{G}, HW, H\widetilde{W}, HB, H\widetilde{B},$ and

$$(C_r)_{\text{c.t.}} = \frac{1}{\epsilon}\left\{2C_{HWB}\lambda\right\}_1 + \frac{1}{\epsilon^2}\left\{14C_{HWB}\lambda^2\right\}_2 + \frac{1}{\epsilon}\left\{-7C_{HWB}\lambda^2\right\}_2$$

(B.9)

for $r = HWB, H\widetilde{W}B.$

C SMEFT H^4D^4 insertions

The contributions of H^4D^4 insertions to the SMEFT renormalization group equations up to dimension eight (with $1/(16\pi^2)$ absorbed into the definition of $\mathrm{d}/\mathrm{d}t$) are:

$$\dot{m}_H^2 = -\frac{3}{2}m_H^6\, {}^8C_{H^4D^4}^{(1)} - \frac{3}{4}m_H^6\, {}^8C_{H^4D^4}^{(2)} - \frac{9}{4}m_H^6\, {}^8C_{H^4D^4}^{(3)},$$

(C.1)

$$\dot{\lambda} = -\frac{19}{3}\lambda m_H^4\, {}^8C_{H^4D^4}^{(1)} - \frac{19}{6}\lambda m_H^4\, {}^8C_{H^4D^4}^{(2)} - \frac{23}{2}\lambda m_H^4\, {}^8C_{H^4D^4}^{(3)},$$

(C.2)

$$\dot{C}_H = -18\lambda^2 m_H^2 \, {}^8C_{H^4D^4}^{(1)} - 22\lambda^2 m_H^2 \, {}^8C_{H^4D^4}^{(2)} - 32\lambda^2 m_H^2 \, {}^8C_{H^4D^4}^{(3)} \,, \tag{C.3}$$

$$\dot{C}_{H\square} = -2\lambda m_H^2 \, {}^8C_{H^4D^4}^{(1)} + 2\lambda m_H^2 \, {}^8C_{H^4D^4}^{(2)} - 6\lambda m_H^2 \, {}^8C_{H^4D^4}^{(3)} \,, \tag{C.4}$$

$$\dot{C}_{HD} = -4\lambda m_H^2 \, {}^8C_{H^4D^4}^{(1)} + 4\lambda m_H^2 \, {}^8C_{H^4D^4}^{(2)} \,, \tag{C.5}$$

$${}^8\dot{C}_{H^8} = \frac{56}{3}\lambda^3 \, {}^8C_{H^4D^4}^{(1)} + \frac{184}{3}\lambda^3 \, {}^8C_{H^4D^4}^{(2)} + 16\lambda^3 \, {}^8C_{H^4D^4}^{(3)} \,, \tag{C.6}$$

$${}^8\dot{C}_{H^6D^2}^{(1)} = \frac{64}{3}\lambda^2 \, {}^8C_{H^4D^4}^{(1)} - \frac{124}{3}\lambda^2 \, {}^8C_{H^4D^4}^{(2)} + 44\lambda^2 \, {}^8C_{H^4D^4}^{(3)} \,, \tag{C.7}$$

$${}^8\dot{C}_{H^6D^2}^{(2)} = 20\lambda^2 \, {}^8C_{H^4D^4}^{(1)} - 20\lambda^2 \, {}^8C_{H^4D^4}^{(2)} \,, \tag{C.8}$$

$${}^8\dot{C}_{H^4D^4}^{(1)} = 8\lambda \, {}^8C_{H^4D^4}^{(1)} + \frac{8}{3}\lambda \, {}^8C_{H^4D^4}^{(2)} + \frac{8}{3}\lambda \, {}^8C_{H^4D^4}^{(3)} \,, \tag{C.9}$$

$${}^8\dot{C}_{H^4D^4}^{(2)} = \frac{8}{3}\lambda \, {}^8C_{H^4D^4}^{(1)} + 8\lambda \, {}^8C_{H^4D^4}^{(2)} + \frac{8}{3}\lambda \, {}^8C_{H^4D^4}^{(3)} \,, \tag{C.10}$$

$${}^8\dot{C}_{H^4D^4}^{(3)} = 16\lambda \, {}^8C_{H^4D^4}^{(1)} + \frac{32}{3}\lambda \, {}^8C_{H^4D^4}^{(2)} + \frac{80}{3}\lambda \, {}^8C_{H^4D^4}^{(3)} \,. \tag{C.11}$$

These agree with ref. [44].

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