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Open World Dempster-Shafer Theory/The Transferable Belief Model with Intervals

A Practitioner's Guide to DST and TBM

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1 Introduction

Dempster-Shafer theory (DST) [1] is a mathematical framework that allows for uncertainty or ignorance to be quantified and included when making predictions from evidence. This is in contrast to Bayesian theory, which does not allow for any quantification of ignorance. The framework is described in great detail in [7]. DST is particularly useful for problems where the inclusion of additional evidence (for example, data from another sensor) could lead to a different conclusion. Thus, it is a useful data fusion method, especially in applications not suited to maximum likelihood or maximum a posteriori estimations due to limited samples or incomplete prior knowledge.

The DST framework allows for a degree of belief to be assigned to collections of indistinguishable events given the evidence by expanding the concept of a probability distribution function into mass functions. This generalization enables computations of imprecise probabilities.

Traditionally DST operates in a closed world, meaning that all possible outcomes are known *a priori*. In reality, this is rarely the case, prompting the need for an open world formalism, where the outcomes are not necessary to be known *a priori*. This document serves two purposes. The first sections are meant to be an introduction to traditional DST machinery, with all relevant definitions introduced. We then introduce an open-world DST framework. We also discuss the concept of continuous variables, and how they fit into the DST framework. Relevant examples are given.

2 Closed World DST

Here we introduce definitions for the traditional closed world DST, and provide examples using weather phenomenology.

2.1 Focal Elements

Definition 1. *A frame of discernment, $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$, is a set of N mutually exclusive hypotheses or propositions.*

Definition 2. *A hypothesis (proposition, classification, or category), ω_i is an element of the Frame of discernment Ω.*

A general representation of a hypothesis is

$$\omega_i \in \Omega.$$

Example 1. *METAR weather codes are short abbreviations used to indicate types of weather events*[1]*: for example, rain (RA), snow (SN), or hail/graupel (GR). See Appendix A for an expanded list. A possible frame of discernment that considers, rain, snow, and hail as the only possible types of weather would be denoted by Ω = {RA, SN, GR}.*

Definition 3. *The power set of a set, Ω, is the set of all subsets of Ω. It is denoted by 2^Ω:*

$$2^\Omega = \{\emptyset, \{\omega_1\}, \{\omega_2\}, ..., \{\omega_1, \omega_2\}, ..., \Omega\}.$$

The power set of a set of N elements will have 2^N elements.

Example 2. *For the frame of discernment, Ω in Example 1, the power set has 8 elements,*

$$2^\Omega = \{\emptyset, \{RA\}, \{SN\}, \{GR\}, \{RA, SN\}, \{RA, GR\}, \{SN, GR\}, \Omega\} .$$

Definition 4. *A focal element, a, is an element of the powerset of the frame of discernment, $a \in 2^\Omega$.*

The general representation of a focal element is

$$a = \{\omega_i, \omega_j, ...\} \in 2^\Omega,$$

where a can be composed of any number and combination of distinct hypotheses in Ω. Some focal elements are of particular note. $\emptyset = \{\} \in 2^\Omega$ is called the “conflict" term, indicating that there is a fundamental conflict of beliefs (i.e. a belief that none of the propositions in the frame of discernment are true). On the other extreme, $\Omega \in 2^\Omega$ is called “vacuous", corresponding to a statement of “no content" or simply saying “something must be true". Focal elements with a single proposition, $\{\omega_i\}$, are referred to as singletons.

Example 3. *For the frame of discernment Ω from Example 1, $a = \{RA, SN\} \in 2^\Omega$ is an example of a focal element. This focal element represents the hypothesis that an observation of the weather will be either rain, or snow, but not both, and represents the* logical_or *($\vee$) of its member propositions. There are three singletons for Ω: $\{RA\}, \{SN\}$, and $\{GR\}$, as each of these subsets of Ω only have a single proposition.*

[1] https://www.weather.gov/media/wrh/mesowest/metar_decode_key.pdf

Closed World DST

Example 4. *A meteorologist awakes in a room of a tin-roofed hut to the sound of precipitation falling. Using the frame of discernment defined in Example 1, (Ω = {RA, SN, GR}), if the sound is loud, they may consider that it is either raining, or hailing* $\{RA, GR\}$*. If it is quiet, they may consider the proposition that it is snowing* $\{SN\}$.

2.2 Mass Functions

Definition 5. *A mass function, or basic probability assignment, is a function from focal elements to the real numbers*

$$m : 2^\Omega \to \mathbb{R},$$

which assigns mass to the focal elements such that,

$$m(\emptyset) = 0$$
$$0 \leq m(a) \leq 1 \textit{ for } a \in 2^\Omega$$
$$\sum_{a \in 2^\Omega} m(a) = 1 \ .$$

The Transferable Belief Model (TBM) [10] is a generalization of DST that eliminates the $m(\emptyset) = 0$ constraint in Definition 5. This generalization enables one to keep track of how much conflict appears when combining multiple data sources, but can suffer from unrepresentative mass distributions without particular care. Throughout this manuscript we work with the constraint $m(\emptyset) = 0$.

In mass functions, focal elements of indiscernible propositions are assigned a mass representing the degree of evidence in the logical_or ($\vee$) of those propositions. Note that focal elements cannot be used to construct the logical_and ($\wedge$) of singletons.

Example 5. *A meteorologist, using the frame of discernment from Example 1 that only considers rain, snow, and graupel, (i.e. Ω = {RA, SN, GR}) estimates there is at least a 50% chance of rain and an additional 20% chance of rain or snow. They concede a 30% chance the weather could do anything within the frame of discernment. This prediction could be assigned the following mass function, m, defined over the frame of discernment, 2^Ω, as*

$$m = \{\{RA\} : 0.5, \{RA, SN\} : 0.2, \{RA, SN, GR\} : 0.3\} \ .$$

Note that there is no explicit assignment of mass to the rest of the focal elements besides those listed above. It is understood that all those focal elements missing mass assignments are implicitly assigned a mass of zero, such as $m(\{SN\}) = 0$.

From the example it is apparent that mass functions are different from a standard statistical or probabilistic analysis, particularly concerning their ability to handle uncertainty. DST is a generalization of probability theory not constrained by the axiom of probability that for any countable sequence of disjoint sets $E_0, E_1, \ldots$, that $P(\bigcup_{i=0}^{\infty} E_i) = \sum_{i=0}^{\infty} P(E_i)$. For instance in Example 5,

$$m[\{RA, SN\}] = 0.2 \neq m[\{RA\}] + m[\{SN\}] = 0.5 + 0 \ .$$

Focal Element	$\{\}$	$\{RA\}$	$\{SN\}$	$\{GR\}$	$\{RA,SN\}$	$\{RA,GR\}$	$\{SN,GR\}$	$\{RA,SN,GR\}$
Belief	0.00	0.50	0.00	0.00	0.70	0.50	0.00	1.00
Pignistic	0.00	0.70	0.20	0.10	0.90	0.80	0.30	1.00
Plausibility	0.00	1.00	0.50	0.30	1.00	1.00	0.50	1.00

Table 2-1: The belief, pignistic, and plausibility of each focal element computed from m.

2.3 Belief, Plausibility, and the Pignistic

While mass functions are useful data structures to work with imprecise probabilities, it is often insightful to get bounds on those imprecise probabilities. These lower and upper bounds are called the belief and plausibility, respectively.

Definition 6. *The belief of a focal element is the lower bound of its probability:*

$$\text{belief}_m(a) = \sum_{\substack{b\in 2^\Omega \\ b\neq\emptyset \\ b\subseteq a}} m(b) \; .$$

Definition 7. *The plausibility of a focal element is the upper bound of its probability:*

$$\text{plausibility}_m(a) = \sum_{\substack{b\in 2^\Omega \\ b\cap a\neq\emptyset}} m(b) \; .$$

While the belief and plausibility bound the probability of a focal element, it is often desirable to associate a single or characteristic probability to a focal element. There are many ways to do this ([5]). The pignistic (betting) probability computes the probability under the uniform prior assumption: that all propositions are equally likely without any additional information, and can be considered the simplest characteristic probability to assign to a focal element.

Definition 8. *The pignistic of a focal element is the betting probability of that focal element under the assumption of a uniform prior:*

$$\text{pignistic}_m(a) = \sum_{\substack{b\in 2^\Omega \\ b\neq\emptyset}} m(b)\frac{|a\cap b|}{|b|} \; .$$

The pignistic function can be thought of as a uniform allocation of probability over the singletons within a focal element [11]. The pignistic probabilities are convenient as the sum of the pignistic probabilities of the singletons sum to 1. This means that the pignistic can be treated as a probability distribution and pignistic probabilities obey all axioms of probability theory.

Example 6. *Consider the mass function from Example 5,*

$$m_1 = \{\{RA\} : 0.5, \{RA, SN\} : 0.2, \{RA, SN, GR\} : 0.3\} \; .$$

The belief, pignistic, and plausibility of each focal element is computed in Table 2-1 and visualized in Figure 2-1.

Closed World DST

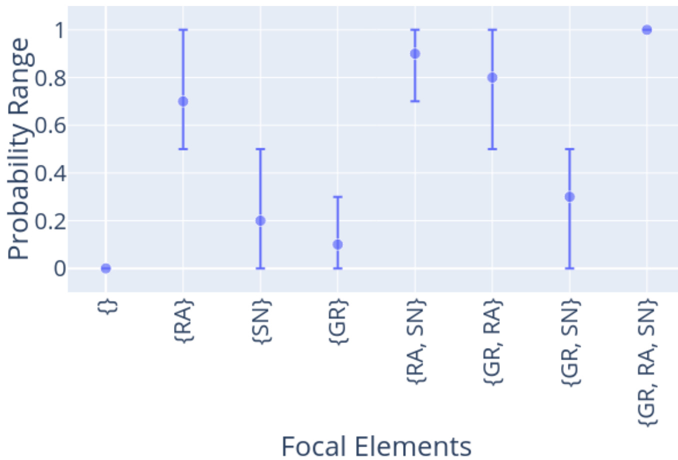


Figure 2-1: Belief (lower bounds), plausibility (upper bounds) and pignistic probability (circles) of the belief function.

Definition 9. *A mass function m can be discounted with a reliability of $0 \leq r \leq 1$ to obtain a new mass function $r \cdot m$ defined by*

$$r \cdot m = \begin{cases} m(a) \cdot r \text{ for } a \neq \Omega \\ m(a) \cdot r + (1 - r) \text{ for } a = \Omega \end{cases}$$

If the reliability is 0, this removes all information from the mass function, generating a new mass function that is 100% vacuous. A reliability of 1 indicates complete confidence in the source and generation of that mass function, and does not adjust the mass function at all. Numbers between 0 and 1 interpolate between these two extremes by moving more or less weight into Ω.

Example 7. *Consider the mass function from Example 5,*

$$m = \{\{RA\} : 0.5, \{RA, SN\} : 0.2, \{RA, SN, GR\} : 0.3\} .$$

If we know the reliability of source of the mass function to be only 0.4, then we can discount the mass function to obtain a new mass function

$$0.4 \cdot m_1 = \{\{RA\} : 0.2, \{RA, SN\} : 0.08, \{RA, SN, GR\} : 0.72\}$$

Discounting causes the feasible probability ranges to grow (seeFigure 2-2).

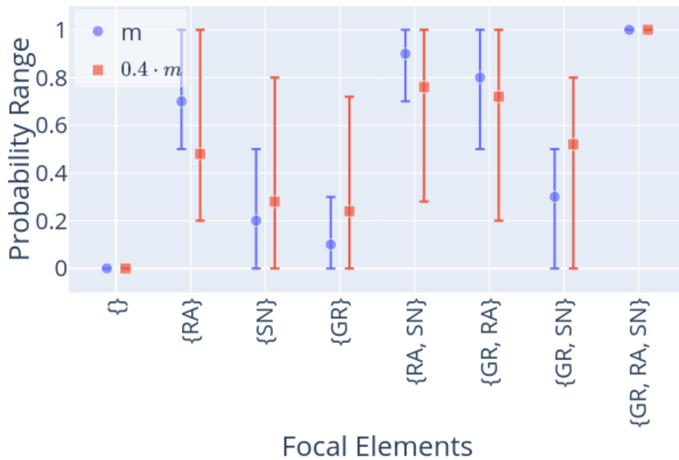


Figure 2-2: Effect of discounting a mass function on the belief, plausibility, and pignistic. Here, the mass function has been discounted by 0.4.

2.4 Rules of Combination

The mass functions introduced so far encode the uncertainty and ignorance of a single source. DST also provides a framework for combining multiple mass functions, each of which have their own uncertainties and ignorance, to generate a new mass function.

Definition 10. *Let m_1 and m_2 be mass functions on the frame of discernment Ω. The* conjunctive join *of m_1 and m_2, denoted $m_1 \bowtie m_2$, is defined by*

$$(m_1 \bowtie m_2)(a) = \sum_{b \cap c = a} m_1(b) m_2(c)\ .$$

One characteristic of the conjunctive join is that it does not necessarily ensure that $m(\emptyset) = 0$. For example, when $m_1(a) > 0$, $m_2(b) > 0$, and the focal elements are disjoint, $a \cap b = \emptyset$, the conjunctive join $m_1 \bowtie m_2$ will have non-zero mass assigned to the empty set, i.e. $(m_1 \bowtie m_2)(\emptyset) > 0$. While the result of this conjunctive join is a valid mass function in the TBM generalization of DST, this is not a valid mass function in the DST framework.

Dumpster's rule of combination adds additional normalization to the conjunctive join to ensure that $m(\emptyset) = 0$, so the resulting object is a valid mass function by DST definitions. This normalization is done by redistributing the mass assigned to the emptyset among all other focal elements.

Definition 11. *Let m_1 and m_2 be mass functions on the frame of discernment Ω. Then* Dempster's rule of combination *combines m_1 and m_2, denoted $m_1 \oplus m_2$, into a new mass function defined by*

$$(m_1 \oplus m_2)(a) = \begin{cases} 0 & \text{for } a = \emptyset \\ \frac{1}{1-K} \sum_{b \cap c = a} m_1(b) m_2(c) & \text{for } a \neq \emptyset \end{cases}$$

where $K = \sum_{b \cap c = \emptyset} m_1(b) m_2(c)$.

Example 8. *Consider two meteorologists competing predictions of the weather. Meteorologist 1 believes*

$$m_1 = \{\{RA\} : 0.5, \{RA, SN\} : 0.3, \{GR, RA, SN\} : 0.2\}$$

represents a realistic prediction of the weather. Meteorologist 2 believes

$$m_2 = \{\{GR, RA\} : 0.5, \{RA, SN\} : 0.3, \{SN\} : 0.2\}$$

is an accurate prediction. Then

$$m_1 \bowtie m_2 = \{\{RA\} : 0.55, \{\} : 0.1, \{RA, SN\} : 0.15, \{SN\} : 0.1, \{GR, RA\} : 0.1\}$$

and

$$m_1 \oplus m_2 = \{\{RA\} : 0.61, \{RA, SN\} : 0.17, \{SN\} : 0.11, \{GR, RA\} : 0.11\} .$$

These mass functions, the conjunctive join, and Dempster's rule of combination can be seen in Figure 2-3.

DST, while a powerful generalization of probability theory, can like other probability theories, lead to seemingly nonsensical results when misapplied or interpreted inconsistently. For example, Zadeh's famous example [13] (rehashed in Example 9)depicts a generalization of a phenomena seen when applying a Naive Bayes model and can lead to unintuitive results in DST as well.

Example 9. *Consider two meteorologists, Alice and Bob, predicting the weather within the frame of discernment of rain, snow, and hail, with $\Omega = \{RA, SN, GR\}$. Alice thinks its probably going to rain, with a slight chance it will hail. The corresponding mass function is:*

$$m_A = \{\{RA\} : 0.9, \{RA, SN\} : 0.1\} .$$

Bob thinks that it is likely going to snow, with a small chance of hail and generates

$$m_B = \{\{SN\} : 0.9, \{SN, GR\} : 0.1\} .$$

The combination of these two independent predictions with Dempster's rule of combination (see Figure 2-4) results in

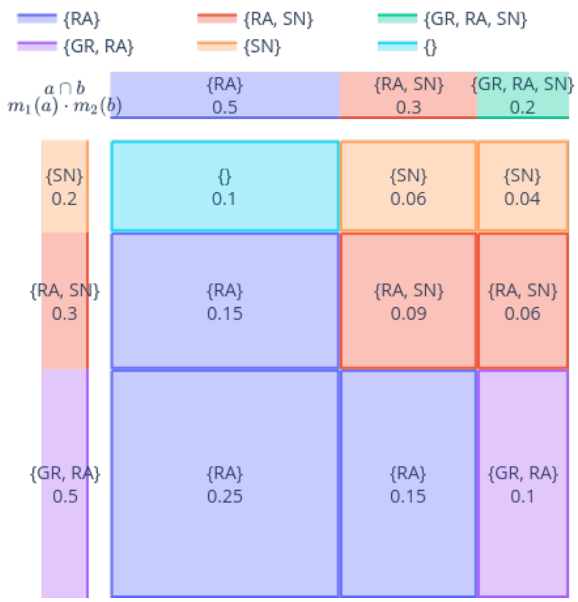
$$m_A \oplus m_B = \{\{GR\} : 1\} .$$

This results in 100% confidence that it is going to hail even though neither meteorologist believed hail to be likely.

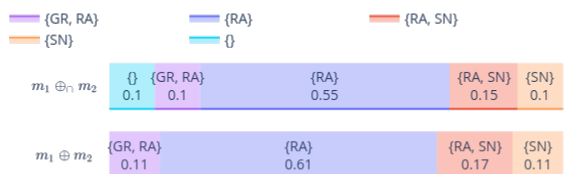
There have been rebuttals to Zadeh's example insinuating that this is a misapplication of DST [4]. First, the collection of propositions in the frame of discernment, $\Omega = \{RA, SN, GR\}$, are not actually mutually exclusive. For example, it does not consider the real world possibility that it could be simultaneously raining and snowing. The unintuitive results can be attributed to that and resolved by remedying the frame of discernment to be mutually exclusive. The standard application of METAR codes allow for concatenation of weather categories to represent the logical_and of the weather types. The collection $\{RA, SN, GR\}$ can be made mutually exclusive by considering every possible combination of no precipitation, single category precipitations, and mixed precipitations with

$$\Omega = \{\emptyset, RA, SN, GR, RASN, RAGR, SNGR, RASNGR\} .$$

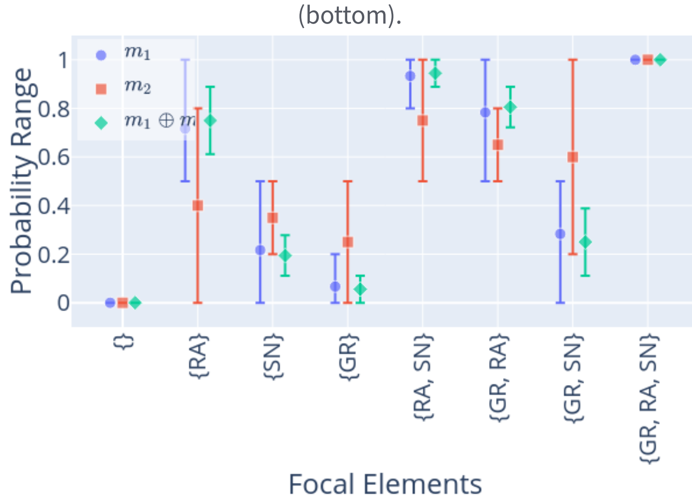
With this frame of discernment $\emptyset$ indicates no precipitation, RA indicates rain only, $RASN$ that there will be mixed rain and snow, and $RASNGR$ that there will be a rain, snow, and graupel slurry. These propositions are mutually exclusive as exactly one of these scenarios must be true.



(a) Geometric construction of the conjunctive join $m_1 \bowtie m_2$. Along the top is the mass function m_1; along the left is the mass function m_2. The center is the geometric construction of the conjunctive join.



(b) The result of $m_1 \bowtie m_2$ obtained from summing masses for like focal elements in Figure 2-3a (top). $m_1 \oplus m_2$ showing the redistribution of the mass assigned to $\emptyset$ in the conjunctive join (bottom).



(c) Belief, plausibility, and pignistic values for m_1, m_2, and $m_1 \oplus m_2$.

Figure 2-3: Depictions of the conjunctive join and Dempster's rule of combination for the two mass functions.

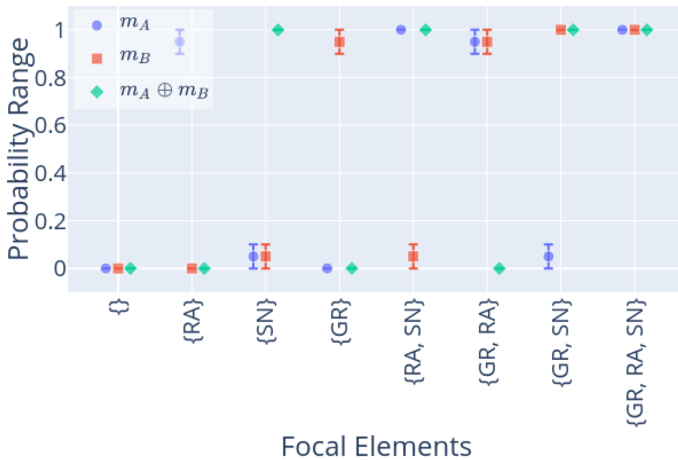


Figure 2-4: Depiction of the mass functions m_A, m_B, and $m_A \oplus m_B$.

Second, at least one of the meteorologists could be incorrect [4]. The reliability of a source of information should be taken into account through a discounting procedure. If each mass function is discounted according to the reliability of the meterologist producing the mass function, the problem is diminished, though not fully resolved.

Additionally, the apparent discrepancy with intuition can also be accommodated by using an alternate combination rule. Alternatives to Dempster's rule of combination include Yager's rule of combination [12], Dubois and Prade's Disjunctive Join [2], and Proportional Conflict Redistribution rules [9] to name a few. Some rules apply only to TBM, other to either DST or TBM. Each different rule of combination provides a distinct way to combine different pieces of evidence, and consequently the different pieces of evidence can be interpreted slightly differently.

Lastly, the limitation of closed-world assumption can also account for the unintuitive results in Example 9. In Section 3 we develop the machinery necessary to remove the closed world assumption and solve the problem in an open world.

3 Open-World DST

The closed world assumption of traditional DST is often too limiting as we know that not all possible propositions are known or accounted for. Considering the unintuitive results of Example 9, what happens if it is neither raining, snowing, or hailing, but instead sleeting? The closed world DST framework in Section 2 is not able to adequately handle this situation. This prompts the need for an open world DST, first developed in [8] and expanded upon in this work.

3.1 Open-World Focal Elements

Definition 12. *An ppen-world focal element is an element in the Cartesian product of the powerset of* Ω *and the Boolean space,* $u \in 2^{\Omega} \times \mathbb{B}$ *.*

The general representation of a focal element in the open world is

$$u = (a, \text{bool}) \in 2^{\Omega} \times \mathbb{B}$$

where a is a typical focal element, and bool is a Boolean flag, called the world Boolean. This has a value of either $True$ or $False$ and indicates whether the focal element is "closed" or "open." We use $True$ to indicate a closed world, representing the direct propositions, and $False$ to represent an open world, representing the complement of the propositions. In an open world, conflict remains encoded with $(\emptyset, True)$, meaning "nothing is true"[2]. The vacuous statement is $(\emptyset, False)$, means that "absolutely anything", which is the complement of "nothing", could be true.

It is evident that (closed-world) focal elements 2^{Ω} are embedded in open-world focal elements $2^{\Omega} \times \mathbb{B}$, as they are the same with the addition of a Boolean. This embedding prompts us to define the two morphisms (or function definitions), the "open" and "closed" functions,

$$\text{open} : 2^{\Omega} \to 2^{\Omega} \times \mathbb{B}$$
$$\text{open}(a) = (a, False)$$

and

$$\text{closed} : 2^{\Omega} \to 2^{\Omega} \times \mathbb{B}$$
$$\text{closed}(a) = (a, True)\ .$$

These morphisms map from focal elements to open-world focal elements that have either a false or true world Boolean indicating that they are either open or closed. As this notation can get cumbersome, we adopt the following shorthand notation to select which morphism to apply when casting a focal element to an open world-focal element,

$$\neg a \coloneqq \text{open}(a) = (a, False)$$
$$a \coloneqq \text{closed}(a) = (a, True)\ .$$

In this shorthand $\neg a$, or "not a", is an explicit indication of an open-world Boolean, so the resulting open focal element should be thought of as "not in a", or in the "complement of a". The lack of a "$\neg$" is an implicit indication of a closed-world Boolean flag. The implicit conversion from a focal element should always be clear from context.

[2] One advantage of DST over TBM is that in DST one does not have to attribute meaning to conflict statements such as "nothing is true", because mass is never assigned to conflict terms.

Open-World DST

Example 10. *Let*

$$a \in 2^{\Omega}$$
$$b \in 2^{\Omega} \times \mathbb{B}$$
$$c \in 2^{\Omega} \times \mathbb{B}$$

with

$$a = \{RA, SN\}$$
$$b = \{RA, SN\}$$
$$c = \neg\{RA, SN\} \ .$$

The focal element "a" represents the hypothesis that it will rain or snow. The focal element "b" has an implicit casting to a closed element and is notionally identical to "a". The focal element "c" is explicitly cast as an open focal element with the "¬" symbol, and represents all possible propositions except rain and snow.

3.2 Open-World Mass Functions

Open-world focal elements are the domain of open-world mass functions.

Definition 13. *An open-world mass function is a function from open-world focal elements to the real numbers*

$$m : 2^{\Omega} \times \mathbb{B} \to \mathbb{R}$$

which assigns mass to the focal elements such that,

$$m((\emptyset, True)) = 0$$
$$0 \leq m(x) \leq 1 \ \textit{for} \ x \in 2^{\Omega} \times \mathbb{B}$$
$$\sum_{x \in 2^{\Omega} \times \mathbb{B}} m(x) = 1 \ .$$

Example 11. *A meteorologist, using the frame of discernment $\Omega = \{RA, SN, GR\}$ estimates there is at least a 50% chance of rain and an additional 20% chance of rain or snow. They estimate a 30% chance the weather could do anything. This prediction could be assigned the following mass function,*

$$m[(\{RA\}, True)] = 0.5$$
$$m[(\{RA, SN\}, True)] = 0.2,$$
$$m[(\emptyset, False)] = 0.3 \ .$$

3.3 Open-World Belief, Plausibility, and Pignistic

With open-world mass functions, the intersection identities:

$$\begin{aligned} (a, True) \cap (b, True) &\coloneqq (a \cap b, True) \\ (a, True) \cap (b, False) &\coloneqq (a - b, True) \\ (a, False) \cap (b, True) &\coloneqq (b - a, True) \\ (a, False) \cap (b, False) &\coloneqq (a \cup b, False) \end{aligned} \tag{1}$$

and the subset identities:

$$\begin{aligned}(a, True) &\subseteq (b, True) \coloneqq a \subseteq b \\ (a, True) &\subseteq (b, False) \coloneqq a \cap b == \emptyset \\ (a, False) &\subseteq (b, True) \coloneqq False \\ (a, False) &\subseteq (b, False) \coloneqq b \subseteq a \ ,\end{aligned} \tag{2}$$

allow for extensions of the belief, plausibility, and pignistic functions to an open world. While these identities allow an immediate extension of belief and plausibility DST to an open world, the pignistic function requires some additional attention in an open-world framework.

In open world DST, the "size" of an open focal element is infinite. A consequence of this is that the ratio of sizes of open focal elements is $\frac{\infty}{\infty}$, an ill posed quantity, which means there isn't a singularly correct way to compute a pignistic value.

One way to compute a pignstic value, is to close a mass function for a set of propositions Ω by computing the combination $m \oplus \{\Omega : 1\}$.

Example 12. *A meteorologist, using the frame of discernment $\Omega = \{RA, SN, GR\}$, creates the following open-world mass function*

$$m = \{\{RA\} : 0.4, \neg\{SN\} : 0.6\} \ .$$

We can close this mass function on Ω with $m \oplus \{\Omega : 1\}$ according to

$$m \oplus \{\{RA, SN, GR\} : 1\} = \{\{RA\} : 0.4, \{RA, GR\} : 0.6\}.$$

If we consider another frame of discernment, $\Omega' = \{RA, SN, GR, PL\}$, we can close m on Ω' to get

$$m \oplus \{\{RA, SN, GR, PL\} : 1\} = \{\{RA\} : 0.4, \{RA, GR, PL\} : 0.6\}.$$

After closeing a mass function, the computation of pignistic values is identical to that of classical DST. Note that a mass function can be closed with any frame of discernment (not just the frame of discernment it was created in), and different Ω assumptions provide different pignistic values, highlighting that if Ω is likely incomplete, the pignistic may not be a stable quantity.

Example 13. *Consider the vacuous mass function*

$$m[\neg\emptyset] = 1 \ .$$

If we take $\Omega = \{RA\}$, then the pignistic probability of rain is 1.0, as rain is the only possible outcome. If we take $\Omega = \{RA, SN\}$, then the pignistic probability of rain is 0.5, and the pignistic probability of snow is 0.5. This demonstrates that the pignistic probability for a given singleton, is dependent on the number of singletons in consideration.

Alternatively, one can compute pignistic values in the open world, but are limited to the pignistic values to closed focal elements only.

Definition 14. *The pignistic of a closed focal element in an open world is the betting probability of that focal element and can be computed with*

$$\text{pignistic}_m((a, True)) = \sum_{\substack{v \in 2^\Omega \times \mathbb{B} \\ v \neq \emptyset}} m(v) \frac{|(a, True) \cap v|}{|v|} \ .$$

Focal Element	$\{\}$	$\{RA\}$	$\{SN\}$	$\{RA, SN\}$	$\neg\{RA, SN\}$	$\neg\{SN\}$	$\neg\{RA\}$	$\neg\{\}$
Belief	0.00	0.40	0.00	0.60	0.00	0.70	0.00	1.00
Pignistic	0.00	0.85	0.15	1.00	-	-	-	-
Plausibility	0.00	1.00	0.30	1.00	0.40	1.00	0.60	1.00

Table 3-1: The belief, pignistic, and plausibility of each focal element computed from m.

This pignistic function allows for a computation of pignistic values of closed focal elements. Direct computation of pignistic values in an open world and computation of pignistic values after closing the world can both be situationally useful. For the remainder of this manuscript we directly compute the pignistic values for the closed focal elements in an open world, and report the pignistic values of open focal elements as indefinite.

Example 14. *Consider the open-world mass function*

$$m = \{\{RA\} : 0.4, \{RA, SN\} : 0.2, \neg\{SN\} : 0.3, \neg\{\} : 0.1\} .$$

The belief, pignistic, and plausibility of each focal element is computed in Table 3-1 and visualized in Figure 3-1. Note that this open world mass function has both closed and open focal elements in it. The assignment of 0.2 mass to $\{RA, SN\}$ indicates the strength of the evidence supporting only rain or snow. In contrast the mass of 0.3 assigned to $\neg\{SN\}$ is the belief that it will do anything other than snow, including unknown possibilities.

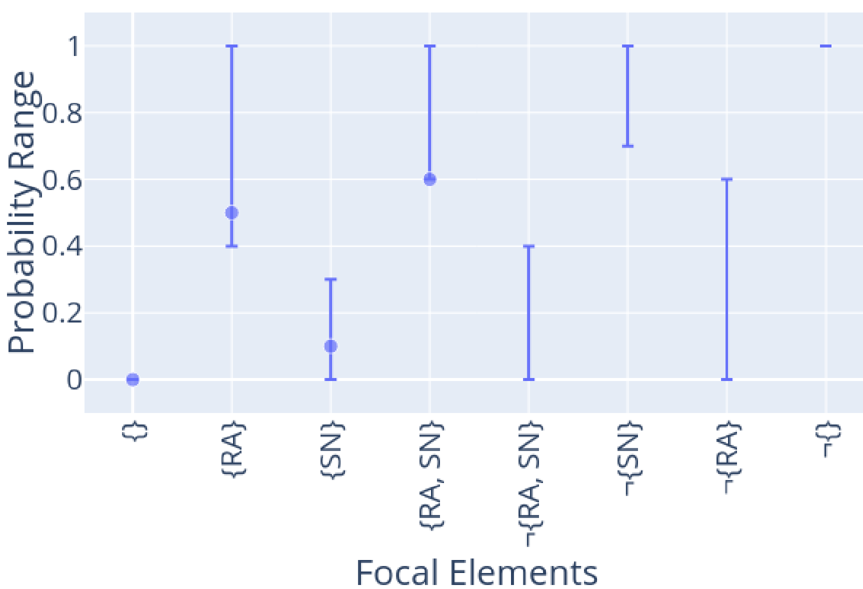


Figure 3-1: Belief (lower bounds), plausibility (upper bounds) and pignistic probability (circles) of belief function.

The ability to assign mass to open focal elements allows one to specify what is unlikely, instead of likely. With Dempster's rule of combination this quality allows for the inference of an unknown category which we can see with a revisiting of Zadeh's example in open world.

Example 15. *Revisiting Zadeh's example of predicting precipitation within the frame of discernment $\Omega = \{RA, SN, GR\}$ from Example 9, let Alice instead assert that it certainly won't hail (GR), and*

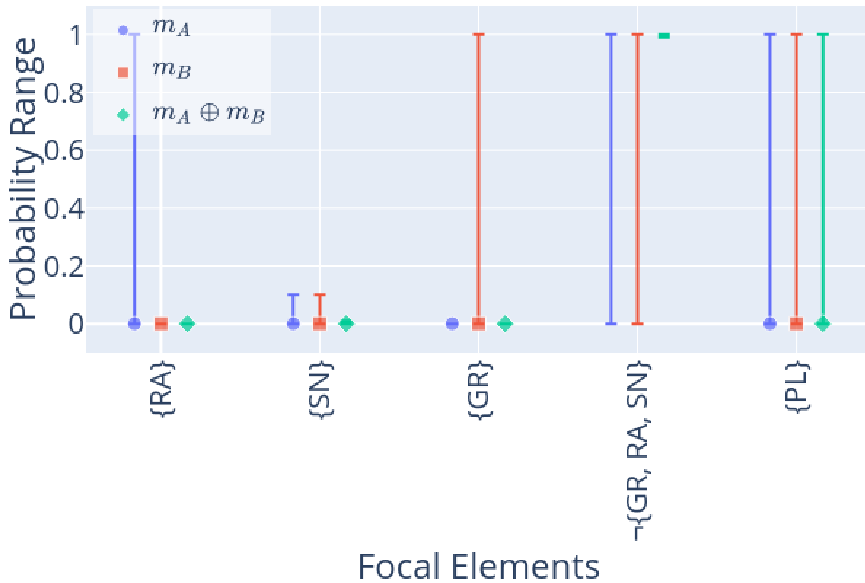


Figure 3-2: Modification of Zadeh's example showing how an open world solves the counterintuitive results.

probably won't snow with

$$m_A = \{\neg\{GR, SN\} : 0.9, \neg\{GR\} : 0.1\} .$$

Let Bob assert that it definitely won't rain, and likely won't hail with

$$m_B = \{\neg\{RA, SN\} : 0.9, \neg\{RA\} : 0.1\} .$$

Then the combination of these two mass functions instead reveals that there is a small chance that it will snow and a large chance that it is an unconsidered proposition

$$m_A \oplus m_B = \{\neg\{GR, RA, SN\} : 0.99, \neg\{GR, RA\} : 0.01\} .$$

Figure 3-2 depicts the belief, pignistic, and plausibility for some of the focal elements. Namely the singletons of Ω*, the complement of* Ω*, and the additional singleton of sleet* (PL) *which was not an element of* Ω*. In this case, 99% of the mass in* $\neg\{GR, RA, SN\}$ *indicates that it is likely an unconsidered precipitation type. The remaining mass signifies that it isn't rain or hail, providing only a 1% plausibility for snow. The open world framework allows for additional querying of propositions not in the frame of discernment. For instance the focal element containing only sleet,* $\{PL\}$*, has a belief of 0, and plausibility 1, indicating that there is no evidence for or against sleet, as it was outside the original frame of discernment.*

4 Continuous Variables

While DST fundamentally deals with categorical variables, some of the ideas of DST have also been applied to continuous variables in probability boxes [3], or probabilistic bounding analysis [6]. Probability boxes are suitable for continuous variables, focusing on the propagation of uncertainty by bounding the continuous variables within specified intervals.

4.1 Intervals

Definition 15. *An interval is a pair of real values* $[l, u] \in \mathbb{R}^2$.

An interval can be used to describe a possible region of feasibility for a continuous variable. Typically there are additional constraints such as, $l \leq u$ or $0 \leq l$ depending on the application. In some instances, it may be suitable to interpret non-physical intervals as conflict, denoted by $[\emptyset]$.

Example 16. *In the application of intervals to an amount of precipitation, an interval* $[l, u]$ *has the additional constraints* $0 \leq l \leq u$ *to ensure physicality. The interval* $[3, 5]$ *can clearly be interpreted to specify 3 to 5 inches of precipitation, while the meaning of* $[5, 3]$ *is not clear. In this context, it is suitable to interpret* $[5, 3] = [\emptyset]$ *as a way of specifying conflict as it is non-physical. Similarly intervals involving negative numbers, such as* $[-4, -2] = [\emptyset]$ *should be interpreted as conflict as there can not be a negative unit of precipitation.*

4.2 Continuous Mass Functions

Intervals can be used to generate continuous mass functions.

Definition 16. *A continuous mass function is a function from intervals to the real numbers*

$$m : \mathbb{R}^2 \to \mathbb{R}$$

which assigns mass to intervals such that,

$$\begin{aligned} m([\emptyset]) &= 0 \\ 0 \leq m(x) &\leq 1 \text{ for } x \in \mathbb{R}^2 \\ \sum_{x \in \mathbb{R}^2} m(x) &= 1 \; . \end{aligned}$$

Continuous mass functions allow for independent allocations of mass to different intervals based on the evidence.

Example 17. *A meteorologist may be 100% certain that it will precipitate between 0 and 3 inches, resulting in the continuous mass function*

$$m = \{[0, 3] : 1.0\} \; .$$

Example 18. *A meteorologist may believe that 40% of the evidence suggests it will precipitate between 2 and 3 inches, and 60% of the evidence suggests it will precipitate between 1 and 4 inches, resulting in the continuous mass function*

$$m = \{[2, 3] : 0.4, [1, 4] : 0.6\} \; .$$

Continuous Variables

4.3 Cumulative Belief, Plausibility, and Pignistic

Several of the tools from DST are applicable to continuous mass functions as well. Discounting and Dempster's rule of combination both have straightforward adaptations in the continuious case, but belief, pignistic, and plausibility need to be altered for use with continuous variables.

Definition 17. *The cumulative belief is a function on $x \in \mathbb{R}$ and is defined*

$$\text{cumulative\_belief}_m(x) = \sum_{\substack{[l,u]\in\mathbb{R}^2 \\ u \leq x}} m([l,u]) \ .$$

Definition 18. *The cumulative plausibility is a function on $x \in \mathbb{R}$ and is defined*

$$\text{cumulative\_plausibility}_m(x) = \sum_{\substack{[l,u]\in\mathbb{R}^2 \\ l \leq x}} m([l,u]) \ .$$

Like the definition of belief introduced earlier, the cumulative belief provides the lower bound for cumulative distribution functions supported by the cumulative mass function. Likewise, the cumulative plausibility provides the upper bound. This means that any function satisfying the properties of a cumulative distribution that is bounded by the cumulative belief and plausibility is a cumulative distribution function supported by the given mass function. The cumulative pignistic is one such function that is bounded by the cumulative belief and plausibility curves.

Definition 19. *The cumulative pignistic is a function on $x \in \mathbb{R}$ and is defined*

$$\text{cumulative\_pignistic}_m(x) = \sum_{\substack{[l,u]\in\mathbb{R}^2 \\ l \leq x}} m([l,u]) \frac{|[l,u] \cap [-\infty, x]|}{|[l,u]|} \ .$$

An additional complication arises with unbounded intervals, especially with the cumulative pignistic. A cumulative pignistic distribution can only be defined for a mass function with finite focal elements since ratios of sizes of infinite intervals are ill-posed. While a mass function can be made finite inside an interval $[l,u]$ with the combination $m \oplus \{(\{[l,u] : 1\}$. In practice this is often not a good idea, as any vacuous component, $[-\infty, \infty]$, dilutes the meaningful information. Depending on application, it can be better to remove all components that are unbounded, and renormalize.

These tools can be used to better understand cumulative mass functions.

Example 19. *The cumulative belief, pignistic, and plausibility for the mass function from Example 17,*

$$m = \{[0,3] : 1.0\} \ ,$$

can be seen in Figure 4-1.

The cumulative belief and plausibility are piece-wise constant, so it does not make sense to differentiate them, despite being cumulative functions. The difference of $plausibility(x) - belief(x)$ corresponds to the uncertainty of the probability that the true value is less than or equal to x.

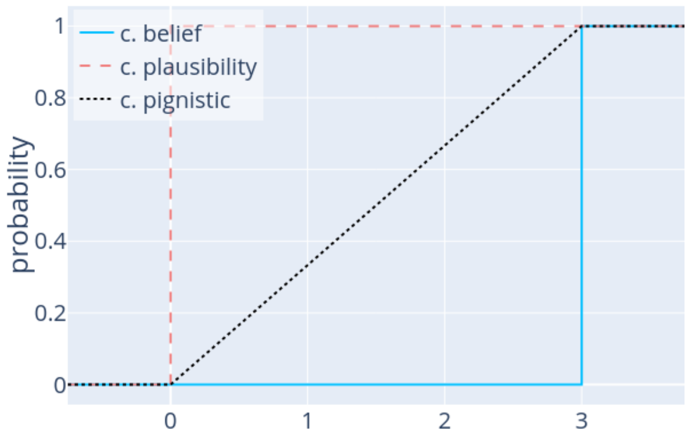


Figure 4-1: Cumulative belief, pignistic, and plausibility curves for the mass function from Example 19.

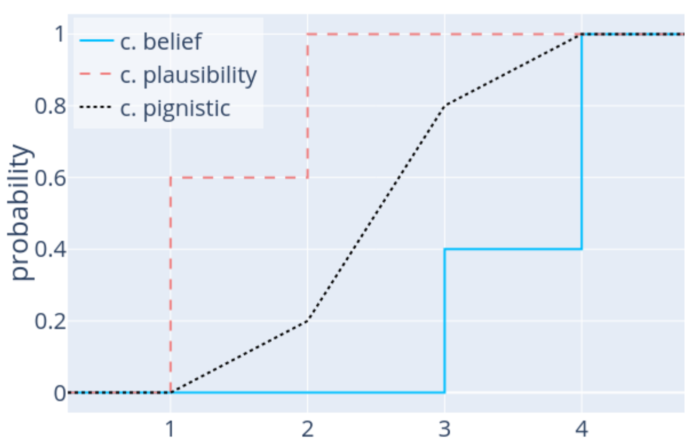


Figure 4-2: Cumulative belief, pignistic, and plausibility curves for the mass function from Example 20.

Continuous Variables

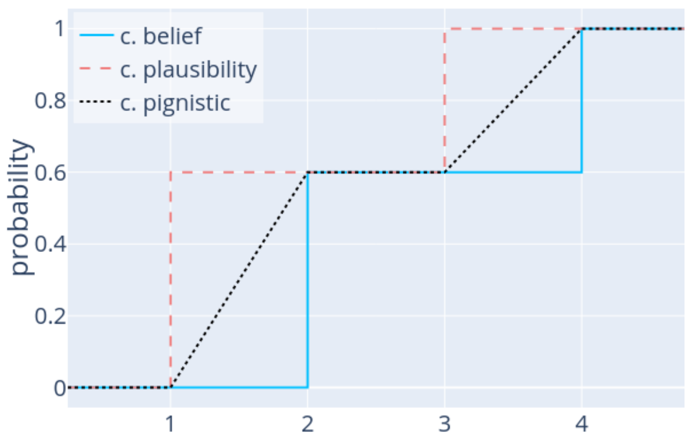


Figure 4-3: Cumulative belief, pignistic, and plausibility curves for the mass function from Example 21.

Example 20. *The cumulative belief, pignistic, and plausibility for the mass function from Example 18,*

$$m = \{[2,3] : 0.4, [1,4] : 0.6\} ,$$

can be seen in Figure 4-2.

Example 21. *The cumulative belief, pignistic, and plausibility for the mass function*

$$m = \{[1,2] : 0.6, [3,4] : 0.4\} ,$$

can be seen in Figure 4-3.

5 Categories with Continuous Variables

5.1 Category-Interval Focal Elements

Sometimes continuous variables need to be conditioned on a categorical variable. For instance, specifying the amount of precipitation may be dependent on the type of precipitation. To handle this conditional relation additional machinery is needed.

Definition 20. *A category-interval, x, is a category (or classification, or hypothesis) associated with an interval, given by the Cartesian product $x \in \Omega \times \mathbb{R}^2$.*

A general representation of a category-interval is

$$x_i = (\omega_i, [l_i, u_i]) \in \Omega \times \mathbb{R}^2$$

Example 22. *A meteorologist that wants to specify that it will snow between 5 and 6 inches, would use the category-interval given by (SN, [5,6]).*

Definition 21. *A category-interval focal element, α, is an element of the set of category-intervals*

$$2^{\Omega \times \mathbb{R}^2} = \{(\omega_i, [l_i, u_i]) : \omega_i \in a, a \in 2^{\Omega}\} \ ,$$

where $l_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are lower and upper interval bounds selected for each category ω_i.[3]

A general representation of a category-interval focal element is

$$\alpha = \{x_i, x_j, ...\} \in 2^{\Omega \times \mathbb{R}^2} \quad (3)$$

Note that the definition of category-interval focal element does not allow for multiple intervals to be associated with one category. For instance,

$$\{(\omega_i, [l_i, u_i]), (\omega_i, [l_j, u_j])\}$$

is not a valid category-interval focal element since two distinct intervals, $[l_i, u_i], [l_j, u_j]$ are associated with the same category ω_i.

Example 23. *Consider the frame of discernment, Ω = {RA, SN, GR}. The hypothesis that it will either rain between 1 and 2 inches, snow between 3 and 4 inches, but that it will not hail, would be captured by the category-interval focal element $x = \{(\mathrm{RA}, [1, 2]), (\mathrm{SN}, [3, 4])\}$. The hypothesis that it will either hail between (GR) 1 and 2 inches, rain between 1 and 2 inches, but not snow, would be captured by $y = \{(\mathrm{GR}, [1, 2]), (\mathrm{RA}, [1, 2])\}$.*

Similarly to the discussion in Section 4 about continuous variables, the state of $l_i > u_i$ for category ω_i can imply that ω_i is not a feasible category, and that category should be considered in a state of conflict. This can be interpreted as eliminating the possibility of that category from the focal element. This interpretation results in multiple equivalent representations of some focal elements, including the conflict focal element.

[3]We are being a bit informal with this (and future) definitions. A more formal mathematical approach would define category-interval focal elements as a graph of a function from Ω to $\mathbb{R}^2$, which is a subset of $2^{\Omega \times \mathbb{R}^2}$. While more technically correct, this formalism is too cumbersome for practical understanding.

Categories with Continuous Variables

Example 24. *Consider the frame of discernment, Ω = {RA, SN}. The hypothesis that it will either rain between 1 and 2 inches, but that it definitely will not snow can be represented an infinite number of ways. Some arbitrary forms of this are:*

$$\begin{aligned}\alpha &= \{(\mathrm{RA}, [1,2]), (\mathrm{SN}, [1,0])\} \\ &= \{(\mathrm{RA}, [1,2]), (\mathrm{SN}, [8,5])\} \\ &= \{(\mathrm{RA}, [1,2]), (\mathrm{SN}, [\emptyset])\} \\ &= \{(\mathrm{RA}, [1,2])\}\ .\end{aligned}$$

Similarly, there are an infinite number of ways to represent conflict. Some arbitrary forms of this are:

$$\begin{aligned}\emptyset &= \{(\mathrm{RA}, [2,1]), (\mathrm{SN}, [1,0])\} \\ &= \{(\mathrm{RA}, [\emptyset]), (\mathrm{SN}, [8,5])\} \\ &= \{(\mathrm{RA}, [\emptyset]), (\mathrm{SN}, [\emptyset])\} \\ &= \{(\mathrm{RA}, [\emptyset])\} \\ &= \{\}\end{aligned}$$

5.2 Category-Interval Mass Functions

Definition 22. *A category-interval mass function is a function from category-interval focal elements to the real numbers*

$$\mathbf{m} : 2^{\Omega \times \mathbb{R}^2} \to \mathbb{R}$$

which assigns mass to the category with interval focal element such that

$$\begin{aligned}\mathbf{m}(\emptyset) &= 0 \\ 0 \leq \mathbf{m}(\alpha) &\leq 1 \textit{ for } \alpha \in 2^{\Omega \times \mathbb{R}^2} \\ \sum_{\alpha \in 2^{\Omega \times \mathbb{R}^2}} \mathbf{m}(\alpha) &= 1\ .\end{aligned}$$

Category-interval mass functions provide significant flexibility to specify either categories of precipitation without precipitation quantities, or precipitation quantities without indicating likelihoods of any categories, or anything in between.

Example 25. *A meteorologist working in the frame of discernment $\Omega = \{RA, SN, GR\}$ predicts with 100% certainty that it will either rain between 1 and 2 inches, or snow between 3 and 4 inches, and that it will not hail is captured with the mass function,*

$$m(\ \{(RA, [1,2]), (SN, [3,4)]\}\ ) = 1\ .$$

This mass function specifies likelihoods of types of precipitations, by indicating it will either rain or snow, but not hail. Additionally it specifies the conditional amounts that it will rain and snow without positing that either one is more likely than the other.

5.3 Marginalized Belief, Plausibility, and Pignistic

For ease of interpretation of category-interval mass functions some additional tools are crucial. First, we need the ability to compute the likelihood of each category independent of the continuous value associated with it. This is accomplished with marginalized beliefs, plausibility, and pignistic functions. While these functions can be defined for general focal elements, here we define them on on propositions for simplicity.

The marginalized belief in a category is the sum of all masses associated with focal elements that have some positive intervals in that category, and are conflict in all other categories.

Definition 23. *The marginalized belief of category $\omega \in \Omega$ is defined*

$$\text{belief}_m(\omega) = \sum_{\substack{\alpha \in 2^{\Omega \times \mathbb{R}^2} \\ \alpha \subseteq \{(\omega, [-\infty, \infty])\} \\ \alpha \cap \{(\omega, [-\infty, \infty])\} \neq \emptyset}} m(\alpha) \ .$$

The marginalized plausibility of a category is the sum of all masses assigned to focal elements that have some positive interval in that category.

Definition 24. *The marginalized plausibility of category $\omega \in \Omega$ is defined*

$$\text{plausibility}_m(\omega) = \sum_{\substack{\alpha \in 2^{\Omega \times \mathbb{R}^2} \\ \alpha \cap \{(\omega, [-\infty, \infty])\} \neq \emptyset}} m(\alpha) \ .$$

The marginalized pignistic function distributes masses proportional to the number of categories with a positive interval in each focal element.

Definition 25. *The marginalized pignistic of category $\omega \in \Omega$ is defined*

$$\text{pignistic}_m(\omega) = \sum_{\alpha \in 2^{\Omega \times \mathbb{R}^2}} m(x) \frac{|\alpha \cap \{(\omega, [-\infty, \infty])\}|}{|\alpha|} \ ,$$

where $|\alpha| = \sum_{\substack{\omega \in \Omega \\ \alpha \cap \{(\omega, [-\infty, \infty])\} \neq \emptyset}} 1$, is the number of categories with a positive interval.

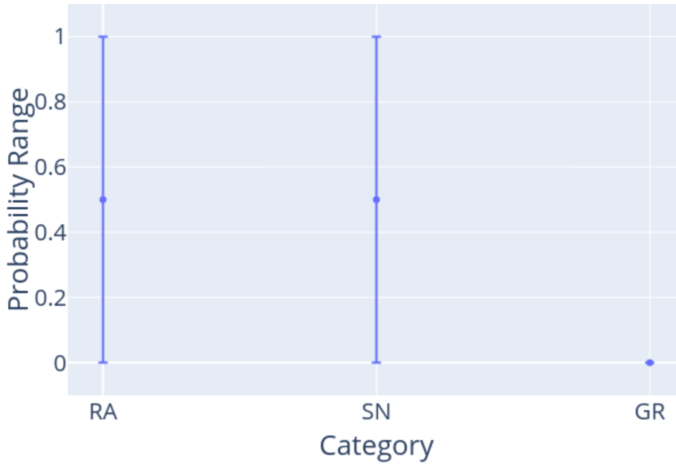
While the marginalized functions allow us to address the question of what category is most likely, the question of quantity is answered by the cumulative functions from Section 4. While one can generalize the cumulative function definitions to deal with category composites, here we consider only cumulative distributions for one category at a time. To compute a cumulative distribution for category $\omega \in \Omega$, we apply the cumulative distribution functions in Section 4 to $m \oplus \{\{(\omega, [-\infty, \infty])\} : 1.0\}$.

Example 26. *From the category-interval mass function on $\Omega = \{RA, SN, GR\}$ from Example 25,*

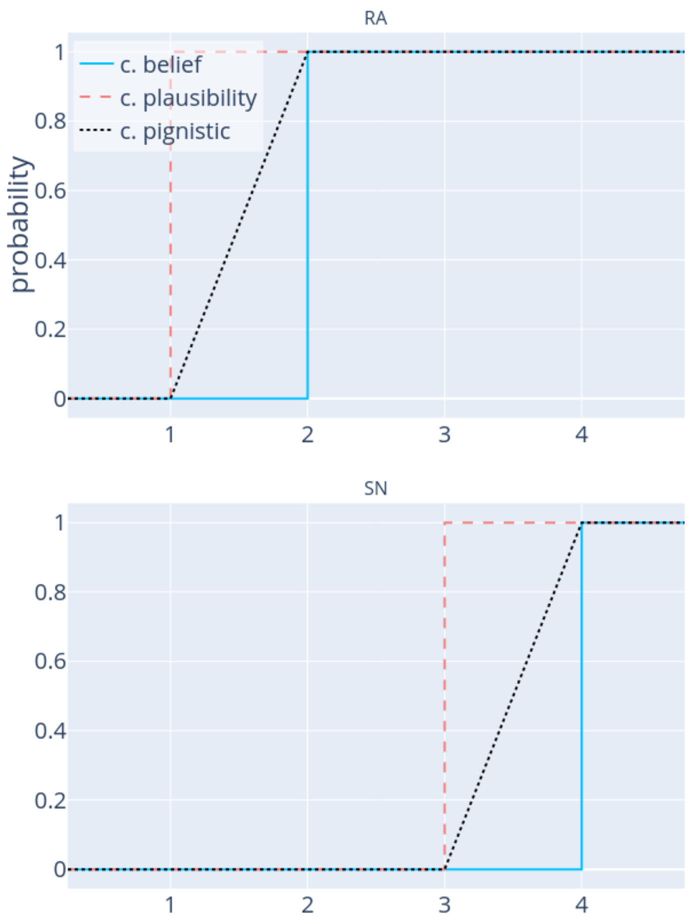
$$m(\ \{(RA, [1, 2]), (SN, [3, 4)]\}\ ) = 1 \ ,$$

we see the marginal and cumulative plots in Figure 5-1. Note that there is no cumulative plot for hail (GR) because the mass function precludes the possibility of hail.

Categories with Continuous Variables



(a)



(b)

Figure 5-1: Panel a: The marginalized belief, pignistic, and plausibility for Example 26. Panel b: The cumulative belief, pignistic, and plausibility curves for rain and snow from the same example.

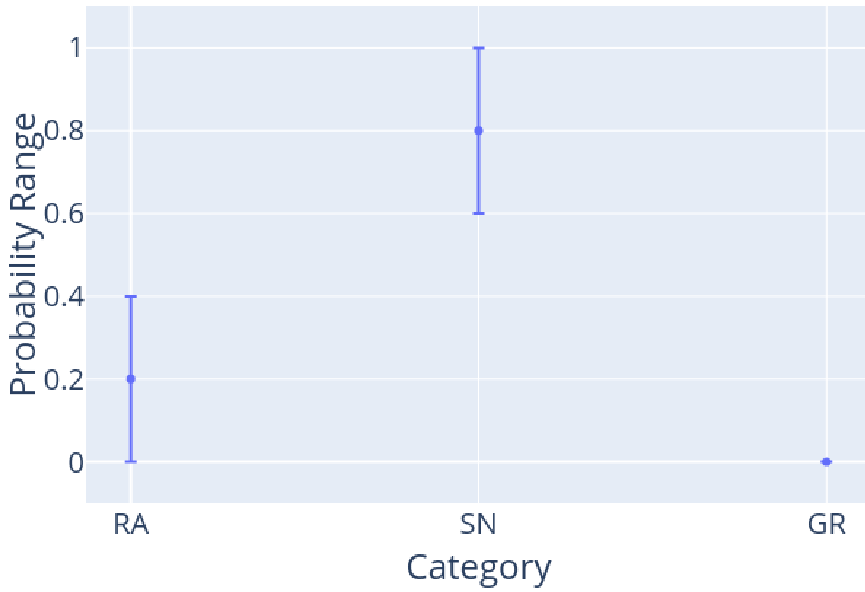


Figure 5-2: The marginal belief, pignistic, and plausibility for Example 27.

In practice, directly constructing category-interval mass functions that specify both category likelihoods and interval predictions can be arduous. It is often simpler to generate category-interval mass functions that only specify the category, or only specify likely intervals given the categories.

Example 27. *Using data from a temperature sensor, one could generate a category-interval mass function that does not indicate the amount of precipitation, but only specifies the likelihood of the category of precipitation. For instance, when it is cold, a reasonable category-interval mass function on* $\Omega = \{RA, SN, GR\}$ *may be,*

$$m_{temp}[\{(RA, [0, \infty]), (SN, [0, \infty])\}] = .4$$
$$m_{temp}[(SN, [0, \infty])\}] = .6 \ .$$

This mass function indicates that snow is most plausible, but rain is still possible. It will not hail. It does not suggest any amount of precipitation. The marginal belief, pignistic, and plausibility can be seen in Figure 5-2. With unbounded intervals the cumulative distributions cannot be plotted.

Example 28. *From a moisture sensor, one could generate a category-interval mass function that does not indicate the likelihood of the type of precipitation, but can specify the amount of each type of precipitation, if that precipitation were to happen. If there is significant moisture in the area, then a reasonable category-interval mass function on* $\Omega = \{RA, SN, GR\}$ *may be,*

$$m_{moist}[\{(RA, [.5, 1]), (SN, [5, 10]), (GR, [1, 2])\}] = 1.0 \ .$$

This mass function does not suggest that any type of precipitation is any more likely than the others, only specifying the amount of each precipitation amount if it were to happen. The marginal and cumulative belief, pignistic, and plausibility can be seen in Figure 5-3.

To combine category-interval mass functions, one first needs to generalize Dempster's rule of combination, by only defining the intersection of category-interval focal elements. While different

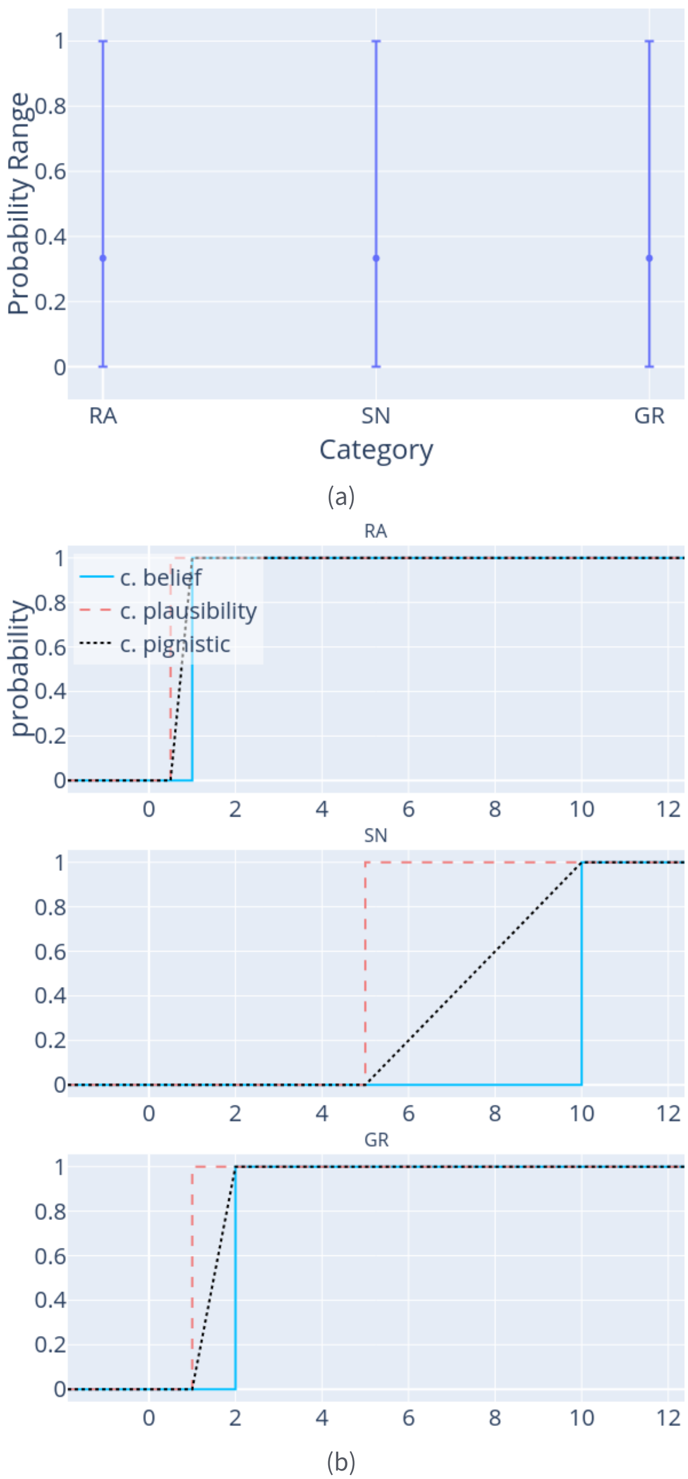


Figure 5-3: Panel a: The marginalized belief, pignistic, and plausibility for Example 28. Panel b: The cumulative belief, pignistic, and plausibility curves for rain, snow, and hail from the same example.

intersection rules can be used for different applications, a natural intersection rule for our precipitation application is

$$\alpha \cap \beta \coloneqq \{(\omega_i, [\max(l_{\alpha,i}, l_{\beta,i}), \min(u_{\alpha,i}, u_{\beta,i})]) : (\omega_i, [l_{\alpha,i}, u_{\alpha,i}]) \in \alpha, (\omega_i, [l_{\beta,i}, u_{\beta,i}]) \in \beta\} \;, \qquad (4)$$

for $\alpha \in 2^{\Omega \times \mathbb{R}^2}$ and $\beta \in 2^{\Omega \times \mathbb{R}^2}$.

Example 29. *Given* $\alpha = \{(RA, [0, 4]), (SN, [3, 4])\}$ *and* $\beta = \{(RA, [1, 6]), (SN, [0, 2]), (GR, [2, 4])\}$ *and using the intersection rule (Equation 4),*

$$\alpha \cap \beta = \{(RA, [1, 4])\}.$$

The rain intervals in x *and* y *have a nonempty intersection, so the intersected rain interval is*

$$(RA, [\max(0, 1), \min(4, 6)]) = (RA, [1, 4]) \;.$$

In contrast, the snow intervals in x *and* y *do not overlap with the intersected snow interval*

$$(SN, [\max(3, 0), \min(4, 2)]) = (SN, [3, 2]) = (SN, [\emptyset])$$

. This represents conflict in snow, and snow can be therefore be omitted from the intersected focal element. Similarly to snow, a nonempty hail interval is only contained in y*, so the intersection of hail between* x *and* y*, results in conflict and is omitted from the intersected focal element.*

Dempster's rule of combination allows for the combination of mass functions, enabling data fusion between sensors.

Example 30. *Applying the modified Dempster's rule of combination to the mass functions in Example 27 and 28 enables information level data fusion between a temperature and moisture sensor. Let*

$$\begin{aligned} m_{temp}[\{(RA, [0, \infty]), (SN, [0, \infty])\}] &= .4 \\ m_{temp}[(SN, [0, \infty])\}] &= .6 \end{aligned}$$

and

$$m_{moist}[\{(RA, [0.5, 1]), (SN, [5, 10]), (GR, [1, 2])\}] = 1.0 \;,$$

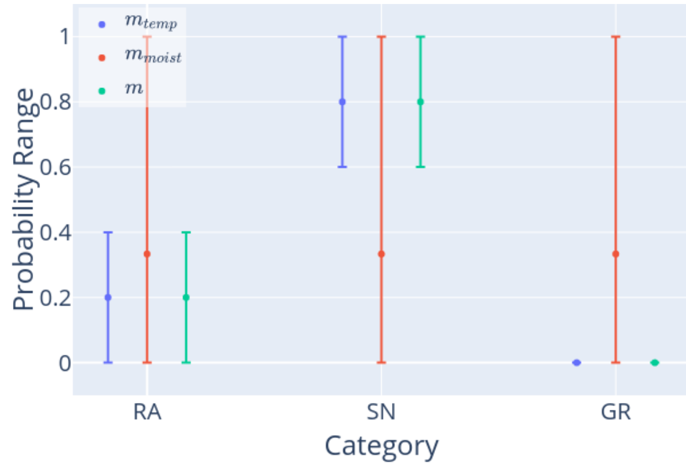
then $m = m_{temp} \oplus m_{moist}$ *is computed*

$$\begin{aligned} m[\{(RA, [0.5, 1]), (SN, [5, 10])\}] &= 0.4 \\ m[\{(SN, [5, 10])\}] &= 0.6 \;. \end{aligned}$$

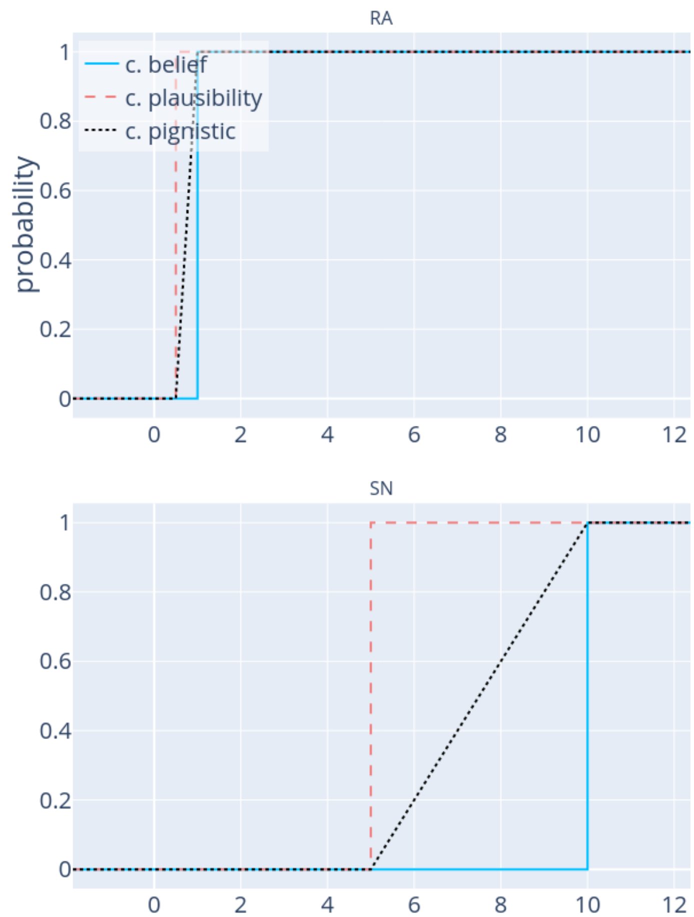
The marginal and cumulative belief, pignistic, and plausibility can be seen in Figure 5-4. We see that in this simple example, the marginal values are precisely those of m_{temp} *while the cumulative values are identical to those of* m_{moist}*.*

Additionally, Dempster's rule of combination allows for a category to emerge as the most likely, despite no individual mass function indicating that any category more likely over any other. This occurs when intervals for the same category overlap between two mass functions, as the confidence in that category is bolstered. Conversely, when intervals for a given category do not overlap, the confidence in that category is diminished. This can be used to identify a category, even when no individual source has confidence in one category over any other.

Categories with Continuous Variables



(a)



(b)

Figure 5-4: Panel a: The marginalized belief, pignistic, and plausibility for Example 30. Panel b: The cumulative belief, pignistic, and plausibility curves for rain and snow from the same example.

Example 31. *Given the category-interval mass functions,*

$$m_1[\{(RA,[1,2]),(SN,[3,4])\}] = 1$$
$$m_2[\{(RA,[0,1]),(SN,[2,4])\}] = 1$$

the combined mass function $0.5 \cdot m_1 \oplus 0.5 \cdot m_2$ *can be seen in relation to* m_1 *and* m_2 *in Figure 5-5.*

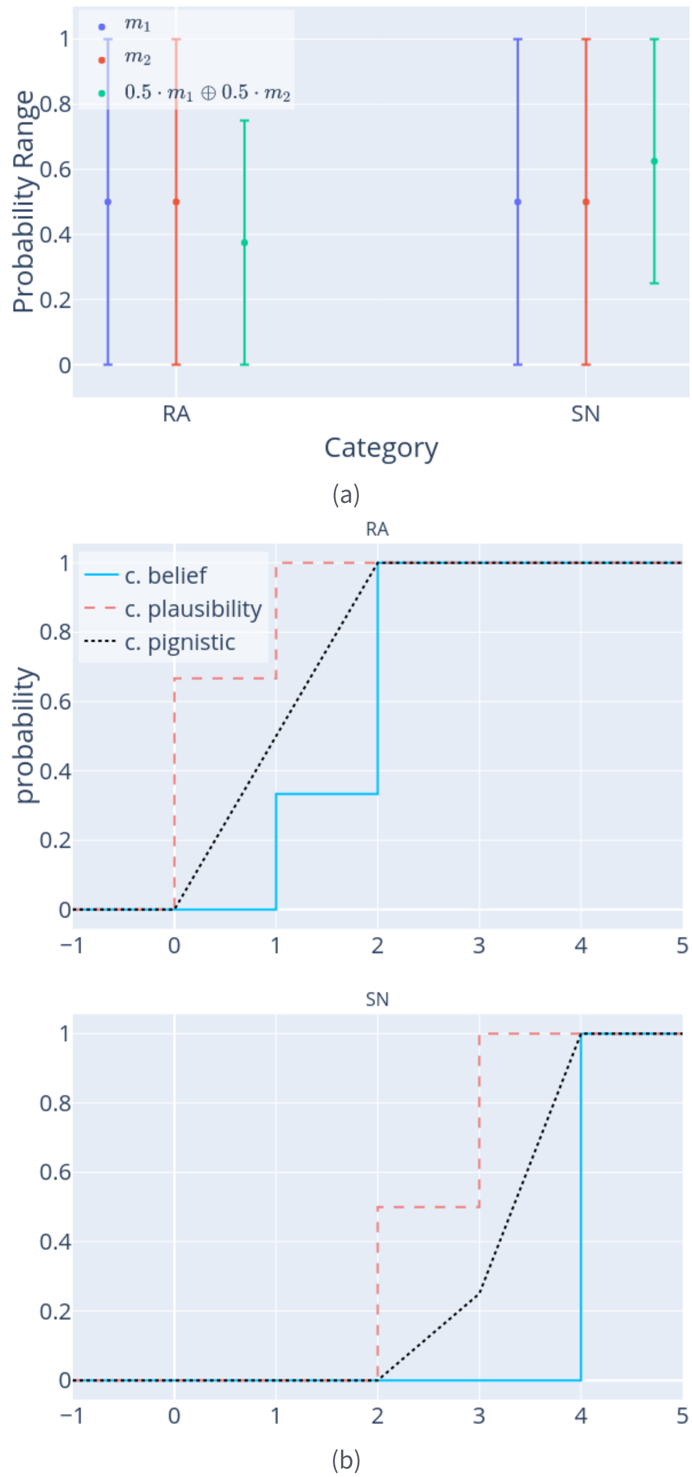


Figure 5-5: Panel a: The marginalized belief, pignistic, and plausibility for Example 31. Panel b: The cumulative belief, pignistic, and plausibility curves for rain and snow from the same example.

6 Open-World Categories with Continuous Variables

6.1 Open-World Category-Interval Focal Elements

Here, we extend the concept of category-intervals to the open world to allow for the inference of a missing category.

Definition 26. *An open-world category-interval focal element is an element in the Cartesian product of the category-interval focal elements and the Boolean space,* $x \in 2^{\Omega\times\mathbb{R}^2} \times \mathbb{B}$.

The general representation of an open-world category-interval focal element is

$$x = (\alpha, \text{bool}) \in 2^{\Omega\times\mathbb{R}^2} \times \mathbb{B}$$

Similarly to Section 3, theworld Boolean acts as a flag indicating whether the focal element is open or closed. The intersection identities are immediately analogous to those in (Equation 1) with:

$$\begin{aligned}(\alpha, True) \cap (\beta, True) &\coloneqq (\alpha \cap \beta, True)\\ (\alpha, True) \cap (\beta, False) &\coloneqq (\alpha - \beta, True)\\ (\alpha, False) \cap (\beta, True) &\coloneqq (\beta - \alpha, True)\\ (\alpha, False) \cap (\beta, False) &\coloneqq (\alpha \cup \beta, False)\end{aligned} \tag{5}$$

and the subset identities analogous to (Equation 2) with:

$$\begin{aligned}(\alpha, True) \subseteq (\beta, True) &\coloneqq \alpha \subseteq \alpha\\ (\beta, True) \subseteq (\alpha, False) &\coloneqq \alpha \cap \beta == \emptyset\\ (\alpha, False) \subseteq (\beta, True) &\coloneqq False\\ (\alpha, False) \subseteq (\beta, False) &\coloneqq \beta \subseteq \alpha\ ,\end{aligned} \tag{6}$$

for $\alpha \in 2^{\Omega\times\mathbb{R}^2}$, $\beta \in 2^{\Omega\times\mathbb{R}^2}$. To cast category-interval focal elements into an open world, the two morphisms

$$\begin{aligned}\text{open} : 2^{\Omega\times\mathbb{R}^2} &\to 2^{\Omega\times\mathbb{R}^2} \times \mathbb{B}\\ \text{open}(\alpha) &= (\alpha, False),\end{aligned}$$

and

$$\begin{aligned}\text{closed} : 2^{\Omega\times\mathbb{R}^2} &\to 2^{\Omega\times\mathbb{R}^2} \times \mathbb{B}\\ \text{closed}(\alpha) &= (\alpha, True)\ ,\end{aligned}$$

allow us to specify either an open or closed casting. Whereas in Section 3 an open focal element was conceptually thought of as the “complement" or “not" of the set of propositions, this is not the case with category-intervals. Here, an open focal element is a designation that it could be any additional unlisted category in addition to those explicitly stated. In this setting an infeasible category must be explicitly identified as such with a conflict interval, $(\omega, [\emptyset])$. Due to this conceptual difference, we introduce a different shorthand for the open and closed morphisms than in Section 3,

$$+\alpha \coloneqq \text{open}(\alpha) = (\alpha, False)$$
$$\alpha \coloneqq \text{closed}(\alpha) = (\alpha, True)\ .$$

The + operator casts the category-interval focal element to an open category-interval focal element representing the contents of the focal element plus any additional unlisted categories. The lack of the + operator implicitly indicates that the category-interval focal element should be cast to a closed category-interval focal element representing just the contents of the focal element. This implicit casting should always be clear with context.

Example 32. *Let*

$$\alpha \in 2^{\Omega\times\mathbb{R}^2}$$
$$\beta \in 2^{\Omega\times\mathbb{R}^2} \times \mathbb{B}$$
$$\gamma \in 2^{\Omega\times\mathbb{R}^2} \times \mathbb{B}$$
$$\delta \in 2^{\Omega\times\mathbb{R}^2} \times \mathbb{B}$$

with

$$\alpha = \{(RA, [1,3]), (SN, [2,4])\}$$
$$\beta = \{(RA, [1,3]), (SN, [2,4])\}$$
$$\gamma = +\{(RA, [1,3]), (SN, [2,4])\}$$
$$\delta = +\{(RA, [1,3]), (SN, [2,4]), (GR, [\emptyset])\}\ .$$

Here, α represents the hypothesis that it will either rain between 1 and 3 inches or snow between 2 and 4 inches. The focal element β has an implicit casting to a closed focal element, and represents an identical notion to α. The elements γ and δ are both explictly cast as open focal elements. The element γ represents the hypothesis that it will rain between 1 and 3 inches, snow between 2 and 4 inches, or that any other form of precipitation will occur. The element δ represents that it will rain between 1 and 3 inches, snow between 2 and 4 inches, or that any other form of precipitation besides hail will occur.

6.2 Open-World Category-Interval Mass Functions

Open-world category-interval focal elements can be directly used to construct open-world category-interval mass functions.

Definition 27. *An open world category-interval mass function is a function from open world category-interval focal elements to the real numbers*

$$\mathbf{m} : 2^{\Omega\times\mathbb{R}^2} \times \mathbb{B} \to \mathbb{R},$$

which assigns mass to the open world category-interval focal elements such that

$$\mathbf{m}(\emptyset) = 0$$
$$0 \le \mathbf{m}(x) \le 1 \text{ for } x \in 2^{\Omega\times\mathbb{R}^2} \times \mathbb{B}$$
$$\sum_{x\in 2^{\Omega\times\mathbb{R}^2}\times\mathbb{B}} \mathbf{m}(x) = 1\ .$$

This generalization allows for immediate extensions of the marginal belief and plausibility functions. The marginal pignistic can only be computed on mass functions where all focal elements are closed. A open-world category-interval mass function can be closed to the categories in Ω with,

$$m \oplus \{(\{(\omega, [-\infty, \infty]) : \omega \in \Omega\}, True) : 1\} ,$$

at which point Definition 25 can be employed as all focal elements are closed. For cumulative functions we analyze one category $\omega \in \Omega$ at a time by applying the cumulative definitions in Section 4 to

$$m \oplus \{(\{(\omega, [-\infty, \infty])\}, True) : 1.0\} .$$

Example 33. *A meteorologist, creates the following open world mass function*

$$m = \{\{(RA, [1, 2]), (GR, [3, 4])\} : 0.4, +\{(RA, [1, 3]), (SN, [2, 5])\} : 0.6\} .$$

We can close this mass function on $\Omega = \{RA, SN, GR\}$ with

$$m_\Omega = m \oplus \{\{(RA, [-\infty, \infty]), (SN, [-\infty, \infty]), (GR, [-\infty, \infty])\} : 1\}$$

where

$$m_\Omega[\{(RA, [1, 2]), (GR, [3, 4])\}] = 0.4$$
$$m_\Omega[\{(RA, [1, 3]), (SN, [2, 5]), (GR, [-\infty, \infty])\}] = 0.6 .$$

If we consider another frame of discernment, $\Omega' = \{RA, GR, PL\}$, we can close m on Ω' with

$$m_{\Omega'} = m \oplus \{\{(RA, [-\infty, \infty]), (GR, [-\infty, \infty]), (PL, [-\infty, \infty])\} : 1\},$$

where

$$m_\Omega[\{(RA, [1, 2]), (GR, [3, 4])\}] = 0.4$$
$$m_\Omega[\{(RA, [1, 3]), (GR, [-\infty, \infty]), (PL, [-\infty, \infty])\}] = 0.6 .$$

Example 34. *Let*

$$m_1[+\{(RA, [1, 2]), (SN, [2, 5])\}] = 0.4$$
$$m_1[+\{(RA, [0, 3]), (SN, [3, 4])\}] = 0.6$$

and

$$m_2[+\{(RA, [0, 1]), (SN, [4, 7])\}] = 0.3$$
$$m_2[+\{(RA, [0, 1]), (SN, [5, 8])\}] = 0.7.$$

Then, $m = m_1 \oplus m_2$ is computed:

$$m[+\{(RA, [\emptyset]), (SN, [4, 5])\}] = 0.12$$
$$m[+\{(RA, [0, 1]), (SN, [\emptyset])\}] = 0.6$$
$$m[+\{(RA, [\emptyset]), (SN, [\emptyset])\}] = 0.28 .$$

Categories with Continuous Variables

There are three open category-interval focal elements with nonzero mass. The element

$$+\{(RA, [\emptyset]), (SN, [4, 5])\},$$

which has a mass of 0.12, represents snow between 4 and 5 inches, or any other category of precipitation other than rain. In other words, 12% of the evidence supports this hypothesis, and is unable to distinguish between these possibilities. Similarly, 60% of the evidence is assigned to

$$+\{(RA, [0, 1]), (SN, [\emptyset])\}.$$

This suggests that some evidence supports, but is unable to distinguish between, rain between 0 and 1 inches, or any other type of precipitation other than snow. Lastly, 28% of the evidence supports the possibility of it doing something other than raining or snowing, but cannot predict an amount for any of the unknown types of precipitation. The marginal and cumulative belief, pignistic, and plausibility can be seen in Figure 6-1 with $\Omega = \{RA, SN, GR\}$. *We can see that even though hail (GR) is not specified in either* m_1 *nor* m_2, *the marginalized pignistic value for this precipitation type increases for* $m_1 \oplus m_2$. *This ability to quantify the likelihood of a type of precipitation not previously considered allows for the inference of an unknown category.*

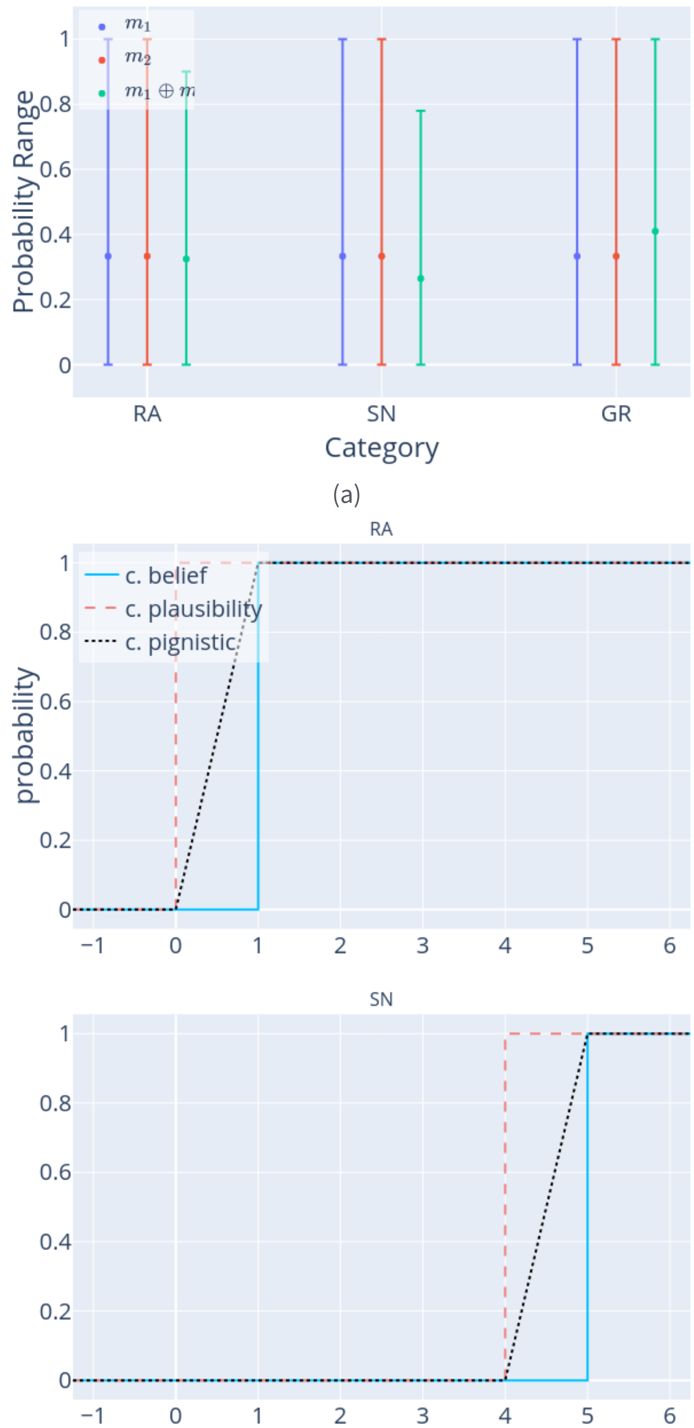


Figure 6-1: Panel a: The marginalized belief, pignistic, and plausibility for Example 34. Panel b: The cumulative belief, pignistic, and plausibility curves for rain and snow from the same example.

7 Conclusions

In this document, we have provided an overview of the traditional closed-world DST framework. We then extend this to an open-world implementation, where all outcomes are not required to be known a priori. This open-world implementation is capable of handling intervals, which is useful if a continuous variable is trying to be quantified.

A METAR Weather Codes

Table A-1 shows some selected METAR codes for different types of precipitation. These are taken from `https://www.weather.gov/media/wrh/mesowest/metar_decode_key.pdf`.

Precipitation	METAR code
DZ	drizzle
GR	hail (graupel)
GS	small hail/ice pellets
IC	ice crystals or in-cloud lightning
PL	ice pellets
RA	rain
SN	snow
SG	snow grains
UP	unknown precipitation

Table A-1: Selected METAR weather codes.

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