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Notes on the non-dimensionalization of the MHD Boussinesq convection equations

J. Pratt
Astronomy and Astrophysics Analytics Group, Physics Division

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The objective of these notes is to demonstrate that the formulation of the MHD Boussinesq convection equations that follows the conventions of the turbulence community used in our previous work [Pratt et al., 2013, 2020, 2017], is consistent and compatible with the formulation from the convection community used in Busse and Pesch [2006].

1 | The MHD Boussinesq convection equations

In Gaussian units the complete set of governing equations are [as written in Biskamp, 2003]:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + (\boldsymbol{V}\cdot\nabla)\rho &= 0 \,, && (1.1)\\
\rho\frac{\partial \boldsymbol{V}}{\partial t} + \rho(\boldsymbol{V}\cdot\nabla)\boldsymbol{V} - \frac{1}{c}(\boldsymbol{J}\times\boldsymbol{B}) &= -\nabla P + \rho\boldsymbol{g} + \mu\nabla^2\boldsymbol{V} \,, && (1.2)\\
\frac{\partial \boldsymbol{B}}{\partial t} - \nabla\times(\boldsymbol{V}\times\boldsymbol{B}) &= \eta\nabla^2\boldsymbol{B} \,, && (1.3)\\
\frac{\partial T}{\partial t} + (\boldsymbol{V}\cdot\nabla)T &= \kappa\nabla^2 T \,, && (1.4)\\
\nabla\times\boldsymbol{B} &= \frac{4\pi}{c}\boldsymbol{J} \,, && (1.5)\\
\nabla\cdot\boldsymbol{B} &= 0 \,. && (1.6)
\end{aligned}
$$

Here ρ is the density, $\boldsymbol{V}$ is the velocity vector, $\boldsymbol{B}$ is the magnetic field vector, $\boldsymbol{J}$ is the electrical current vector, T is the temperature, P is the thermal pressure, $\boldsymbol{g}$ is the acceleration due to gravity, and c is the speed of light. We split each variable into mean and fluctuating quantities:

$$
\begin{aligned}
\rho &= \rho_0 + \delta\rho \,, && (1.7)\\
\boldsymbol{V} &= \boldsymbol{V}_0 + \boldsymbol{v} \,, && (1.8)\\
\boldsymbol{B} &= \boldsymbol{B}_0 + \boldsymbol{b} \,, && (1.9)\\
T &= T_0 + \boldsymbol{\theta} \,, && (1.10)\\
\boldsymbol{P} &= \boldsymbol{P}_0 + \boldsymbol{p} \,, && (1.11)\\
\boldsymbol{J} &= \boldsymbol{J}_0 + \boldsymbol{j} \,. && (1.12)
\end{aligned}
$$

Generally we assume no static velocity or magnetic features $\boldsymbol{V}_0 = 0$, $\boldsymbol{B}_0 = 0$. This does not remove the possibility that a mean velocity or magnetic field may be present or evolve, but it is treated as a fluctuating quantity. Using these definitions, we derive the governing equations for the fluctuations

$$
\begin{aligned}
\rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} &= -\nabla\bar{P} + \rho\boldsymbol{g} + \frac{1}{4\pi}(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + \mu\nabla^2\boldsymbol{v} \,, && (1.13)\\
\frac{\partial \boldsymbol{b}}{\partial t} - \nabla\times(\boldsymbol{v}\times\boldsymbol{b}) &= \eta\nabla^2\boldsymbol{b} \,, && (1.14)\\
\frac{\partial \theta}{\partial t} + (\boldsymbol{v}\cdot\nabla)(T_0+\theta) &= \kappa\nabla^2\theta \,, && (1.15)\\
\nabla\cdot\boldsymbol{v} &= 0 \,. && (1.16)
\end{aligned}
$$

Here we have used a vector identity to obtain the magnetic contribution to the momentum equation (1.13):

$$\frac{1}{c}\boldsymbol{j}\times\boldsymbol{b} = -\frac{1}{8\pi}\nabla b^2 + \frac{1}{4\pi}\boldsymbol{b}\cdot\nabla\boldsymbol{b}\ . \tag{1.17}$$

In eq. (1.17) the first term on the right hand side is the magnetic pressure, which has been included in the pressure term of eq. (1.13). In eq. (1.13) we have not yet used the Boussinesq approximation and the density is not yet split into mean and fluctuating parts. In the Boussinesq approximation we assume that $\delta\rho = 0$ except for the buoyancy force, which is expressed in the gravity term of the momentum eq. (1.13). This approximation amounts to:

$$\frac{\delta\rho}{\rho_0} = -\alpha\theta\ , \tag{1.18}$$

where α is the volume thermal expansion coefficient. Without loss of generality, we also assume that the gravity vector has the form $\boldsymbol{g} = -g\hat{z}$. We then have the correct momentum equation (1.13) for the Boussinesq MHD equations

$$\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \quad = \quad -\nabla\Pi + \alpha\theta g\hat{z} + \frac{1}{4\pi\rho_0}(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + \nu\nabla^2\boldsymbol{v}\ . \tag{1.19}$$

Here Π is a tensor that includes pressure, magnetic pressure, the pressure due to the gravitational force, etc. The kinematic viscosity is $\nu = \mu/\rho_0$.

2 | Working backward from the equations in Busse and Pesch [2006]

The non-dimensional equations used by Busse and Pesch [2006] to study convection are:

$$\frac{\partial\boldsymbol{v'}}{\partial t'} \quad + \quad (\boldsymbol{v'}\cdot\nabla')\boldsymbol{v'} = -\nabla'\pi' + \nabla'^2\boldsymbol{v'} + \boldsymbol{B'}\cdot\nabla'\boldsymbol{B'} + \theta'\hat{z}\ , \tag{2.1}$$

$$\mathsf{Pr}\ \frac{\partial\theta'}{\partial t'} \quad + \quad \mathsf{Pr}\ (\boldsymbol{v'}\cdot\nabla')\theta' = \mathsf{Ra}\ v'_z + \nabla'^2\theta'\ . \tag{2.2}$$

These equations have been non-dimensionalized using the Prandtl number Pr and Rayleigh number Ra in the following way

$$t' = t\nu/L^2\ , \tag{2.3}$$

$$\boldsymbol{r'} = \boldsymbol{r}/L\ , \tag{2.4}$$

$$\boldsymbol{v'} = \boldsymbol{v}L/\nu\ , \tag{2.5}$$

$$\theta' = \theta/(\Delta T\mathsf{Pr}/\mathsf{Ra}) = \theta\alpha g L^3/\nu^2\ . \tag{2.6}$$

The prime variables are non-dimensional, and the non-primed variables have Gaussian units. Here ν is the kinematic viscosity, and L is the height of a convecting layer, i.e. the length scale of the temperature gradient in a periodic system. ΔT is the change in temperature across the convecting layer. To calculate the derivatives we apply the chain rule

$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} = \frac{L^2}{\nu}\frac{\partial}{\partial t}\ , \tag{2.7}$$

$$\nabla' = L\nabla\ , \tag{2.8}$$

$$\nabla'^2 = L^2\nabla^2\ . \tag{2.9}$$

For the sake of comparison, we then seek to convert eqs. (2.10)-(2.2) to Gaussian units. We find

$$\frac{\partial\boldsymbol{v}}{\partial t} \quad + \quad (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{\nu}{L}\nabla\pi + \nu\nabla^2\boldsymbol{v} + \frac{\nu^2}{L^2}\boldsymbol{B'}\cdot\nabla\boldsymbol{B'} + \alpha g\theta\hat{z}\ , \tag{2.10}$$

$$\frac{\partial\theta}{\partial t} \quad + \quad (\boldsymbol{v}\cdot\nabla)\theta = v_z\ \frac{\Delta T}{L} + \kappa\nabla^2\theta\ . \tag{2.11}$$

Here $v_z\ \Delta T/L$ is the approximation of $v\cdot\nabla T_0$ when the gradient is purely in the z-direction, and therefore eq. (2.11) is identical to eq. (1.15). We compare eq. (2.10) to eq. (1.19); for these equations to be the same, the magnetic field terms must be equal

$$\frac{1}{4\pi\rho_0}(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} \equiv \frac{\nu}{L^2}\boldsymbol{B'}\cdot\nabla\boldsymbol{B'}\ . \tag{2.12}$$

Thus some non-dimensionalization must be taken into account for the magnetic field in the equations of Busse and Pesch [2006]. We therefore define:

$$B' = B/b_0 \; . \tag{2.13}$$

From eq. (2.10) we can derive the form of the unknown factor b_0

$$\frac{1}{4\pi\rho_0} = \frac{\nu^2}{L^2} b_0^2 \; . \tag{2.14}$$

Solving we find

$$b_0 = \frac{L}{\nu\sqrt{4\pi\rho_0}} \equiv \frac{v_0}{\sqrt{4\pi\rho_0}} \; . \tag{2.15}$$

Using this non-dimensionalization of the magnetic field, the MHD convection equations of Busse and Pesch [2006] are consistent with the standard equations in Gaussian units.
The equation for the evolution of the magnetic field, using this non-dimensionalization, can be derived from the Gaussian eq.

$$\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = \eta \nabla^2 \boldsymbol{B} \; . \tag{2.16}$$

This produces:

$$\frac{\nu b_0}{L^2}\frac{\partial \boldsymbol{B'}}{\partial t'} - \frac{\nu b_0}{L^2}\nabla' \times (\boldsymbol{v'} \times \boldsymbol{B'}) = \frac{\eta b_0}{L^2}\nabla'^2 \boldsymbol{B'} \; , \tag{2.17}$$

$$\mathsf{Pr_M}\frac{\partial \boldsymbol{B'}}{\partial t'} - \mathsf{Pr_M}\nabla' \times (\boldsymbol{v'} \times \boldsymbol{B'}) = \nabla'^2 \boldsymbol{B'} \; . \tag{2.18}$$

The final equation here uses the magnetic Prandel number $\mathsf{Pr_M} = \nu/\eta$ to non-dimensionalize the induction equation.

3 | The formulation of the equations used in `MHDT`

It is convenient to evolve the vorticity field, rather than the momentum equation in eq. (1.19). In the vorticity equation the pressure term vanishes because the curl of a gradient is zero. The MHD turbulence code `MHDT`[1], used in Pratt et al. [2013, 2020, 2017] and many other works, thus evolves the vorticity. The Boussinesq MHD equations including the vorticity fluctuation equation are

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\boldsymbol{v} \times \omega + \frac{1}{c\rho_0}\boldsymbol{j} \times \boldsymbol{b}) = \alpha g \nabla\theta \times \hat{z} + \nu\nabla^2\boldsymbol{\omega} \; , \tag{3.1}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{b}) = \eta\nabla^2\boldsymbol{b} \; , \tag{3.2}$$

$$\frac{\partial \theta}{\partial t} + (\boldsymbol{v}\cdot\nabla)(T_0 + \theta) = \kappa\nabla^2\theta \; . \tag{3.3}$$

These equations are identical to eqs. (1.13)-(1.15) except that the current has not been eliminated, and we have taken the curl of the momentum equation to produce a vorticity equation.
We non-dimensionalize eqs. (3.1)-(3.3) using a length and time scale associated with the temperature gradient

$$L = T_0/\nabla T_0 \; , \tag{3.4}$$

$$t_{\mathsf{b}} = (\alpha g \nabla T_0)^{-1/2} \; . \tag{3.5}$$

In our periodic system L is conceptually the same as the length scale used in the Busse and Pesch [2006] equations. The time scale is different; it is related to the buoyancy force, rather than the viscosity. The

[1] The `MHDT` code, a classical workhorse originating in the turbulence community, is focused on producing the highest Reynolds number flows for the least expense; it is a pseudo-spectral code with low-storage explicit time integration.

non-dimensional variables used in MHDT are:

$$t' = t/t_b \ , \tag{3.6}$$

$$\boldsymbol{r'} = \boldsymbol{r}/L \ , \tag{3.7}$$

$$\boldsymbol{v'} = \boldsymbol{v} t_b/L \ , \tag{3.8}$$

$$\boldsymbol{b'} = \boldsymbol{b}/b_0 \ , \tag{3.9}$$

$$\boldsymbol{j'} = \boldsymbol{j} L/b_0 \ , \tag{3.10}$$

$$\theta' = \theta/(L\nabla T_0) \ . \tag{3.11}$$

Because the time scale used to non-dimensionalize these equations is different from the one used by Busse and Pesch [2006], the velocity and temperature field non-dimensionalizations are commensurately adjusted. The normalization factor b_0 for the magnetic field is left undetermined at this point and will be discussed below. For the derivatives we use the conversion

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = \frac{1}{t_b}\frac{\partial}{\partial t'} \ , \tag{3.12}$$

$$\nabla = \frac{1}{L}\nabla' \ , \tag{3.13}$$

$$\nabla^2 = \frac{1}{L^2}\nabla'^2 \ . \tag{3.14}$$

Using these non-dimensional quantities, and dropping the prime notation, the equations become:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\boldsymbol{v} \times \omega + \frac{1}{c\rho_0}(b_0/v_0)^2 \frac{c}{4\pi}(\nabla \times \boldsymbol{b}) \times \boldsymbol{b}) = \nabla\theta \times \hat{z} + \tilde{\nu}\nabla^2\boldsymbol{\omega} \ , \tag{3.15}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{b}) = \tilde{\eta}\nabla^2\boldsymbol{b} \ , \tag{3.16}$$

$$\frac{\partial \theta}{\partial t} + (\boldsymbol{v}\cdot\nabla)\theta = \tilde{\kappa}\nabla^2\theta + v_z\nabla_z T_0 \ . \tag{3.17}$$

In the non-dimensionalizated eqs. (3.15)-(3.17) the new non-dimensional dissipative parameters are defined using tildes. These are

$$\tilde{\nu} = \left(\frac{\mathsf{Pr}}{\mathsf{Ra}}\right)^{1/2} , \tag{3.18}$$

$$\tilde{\eta} = \left(\frac{\mathsf{Pr}}{\mathsf{Ra}\ \mathsf{Pr}_m^2}\right)^{1/2} , \tag{3.19}$$

$$\tilde{\kappa} = \left(\frac{1}{\mathsf{Pr}\ \mathsf{Ra}}\right)^{1/2} . \tag{3.20}$$

We have not yet addressed the normalization factor b_0 for the magnetic field. In the vorticity eq. (3.15) there is a pre-factor on the magnetic field contribution. We select b_0 so that this pre-factor is one, i.e.

$$b_0 = v_0\sqrt{4\pi\rho_0} \ . \tag{3.21}$$

So we find that the non-dimensionalization of the magnetic field is the same as Busse and Pesch [2006] except for the time scale used to define v_0. This choice of normalization for the magnetic field has sometimes been called Alfvénic units, because it conceptually associates the Alfvén velocity with v_0.

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4 | References

FH Busse and W Pesch. Thermal convection in a twisted horizontal magnetic field. *Geophysical and Astrophysical Fluid Dynamics*, 100(2):139–150, 2006.

J Pratt, A Busse, and W-C Müller. Fluctuation dynamo amplified by intermittent shear bursts in convectively driven magnetohydrodynamic turbulence. *Astronomy & Astrophysics*, 557:A76, 2013.

Jane Pratt, Angela Busse, and Wolf-Christian Müller. Lagrangian statistics of heat transfer in homogeneous turbulence driven by Boussinesq convection. *Fluids*, 5(3):127, 2020.

Jane Pratt, Angela Busse, Wolf-Christian Müller, Nicholas W Watkins, and Sandra C Chapman. Extreme-value statistics from Lagrangian convex hull analysis for homogeneous turbulent Boussinesq convection and MHD convection. *New Journal of Physics*, 19(6):065006, 2017.

Dieter Biskamp. *Magnetohydrodynamic turbulence.* Cambridge University Press, 2003.