Nucl. Phys. B 1018 (2025) 116987

Contents lists available at ScienceDirect

Nuclear Physics, Section B

journal homepage: www.elsevier.com/locate/nuclphysb

Featured Article

Natural top quark condensation (a redux)

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ARTICLE INFO

Editor: Hong-Jian He

ABSTRACT

The Nambu–Jona-Lasinio (NJL) model involves a pointlike 4-fermion interaction. While it gives a useful description of chiral dynamics (mainly in QCD), it nonetheless omits the crucially important internal wave-function of a two-body bound state, $\phi(r)$. This becomes significant near critical coupling where $\phi(r)$ extends to large distance, leading to dilution and suppression of induced couplings $\propto \phi(0)$, such as the Yukawa and quartic couplings, as well as reduced fine-tuning of a hierarchy. In top quark condensation, where the Brout-Englert-Higgs (BEH) boson is a $\bar{t}t$ bound state and we have a UV completion such as topcolor, we must go beyond the NJL model and include effects of $\phi(r)$. We provide a formulation of this for the BEH boson, and find that it leads to an extended $\phi(r)$, a significantly reduced and natural composite scale of $M_0 \sim 6$ TeV, a successful prediction for the quartic coupling, λ, and fine–tuning that is reduced to a few percent, providing a compelling candidate solution to the naturalness problem of the BEH boson. The theory is testable and the associated new physics may soon emerge at LHC energy scales.

1. Introduction

In the early 1990's we proposed the idea of "top quark condensation," i.e., that the Brout-Englert-Higgs (BEH) boson is composed of top + anti-top quarks [1]–[7]. The minimal model introduced a 4-fermion pointlike interaction, at large mass scale, M_0, amongst third generation quarks,

$$\frac{g_0^2}{M_0^2}[\overline{\psi}_{iL}(x)\psi_R(x)]\,[\overline{\psi}_R(x)\psi_L^i(x)]; \qquad \psi_L^i = \begin{pmatrix} t \\ b \end{pmatrix}_L, \quad \psi_R = t_R, \tag{1}$$

(where i is an electroweak $SU(2)$ index, and [...] = denotes a sum over color). To treat this we deployed the Nambu-Jona-Lasinio (NJL) model [8], and introduced significant renormalization group (RG) improvement [3]. The NJL model led to a composite BEH electroweak isodoublet, described by a local field, $H^i(x) \sim [\overline{\psi}_R(x)\psi_L^i(x)]$. When tuned to the known electroweak scale, $v_{weak} = 175$ GeV, the theory predicted the top quark and BEH boson masses.

The minimal top-condensation theory was one of the earliest composite BEH models, and was "philosophically successful" in that it non-trivially tied together unrelated parameters of the Standard Model (SM). However, the explicit predictions of the model were ultimately ruled out by subsequent experiment: $m_{top} \approx 220$ GeV, (cf. 175 GeV, experiment) and $m_{BEH} \approx 260$ GeV (cf. 125 GeV,

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https://doi.org/10.1016/j.nuclphysb.2025.116987
Received 15 April 2025; Accepted 13 June 2025

Available online 20 June 2025

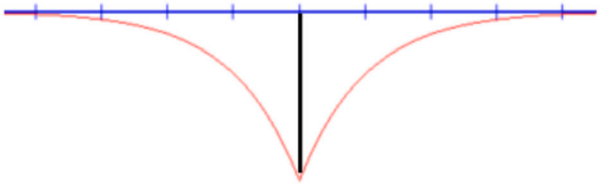

Fig. 1. Dirac δ-function potential and its extended wave-function.

experiment). The predicted m_{BEH} corresponded to a SM quartic coupling prediction of $\lambda \approx 1$, (cf. $\lambda \approx 0.25$, experiment), while the predicted m_{top} corresponded to a SM Yukawa coupling prediction of $g_{top} \equiv g_Y \approx 1.25$, (cf. $g_Y \approx 1.00$, experiment).

Moreover, to accommodate the electroweak scale, the theory required ultra-large $M_0 \sim 10^{15}$ GeV, which implied a drastic fine–tuning of the coupling g_0^2 to its critical value g_c^2,

$$\left(1 - \frac{g_0^2}{g_c^2}\right) \sim \frac{v_{weak}^2}{M_0^2} \sim 10^{-26}\,(!) \tag{2}$$

So, the top condensation theory was directly testable by experiment and it evidently failed.

The NJL model was used in top condensation since it is concise, manifestly Lorentz invariant and provides a guide to the chiral symmetry breaking dynamics seen in QCD. There it leads to useful results, where the fundamental chiral current quarks dynamically become the heavy constituent quarks, yielding the chiral–constituent quark model [9]. The NJL model builds the pseudoscalar mesons and σ field by "integrating out" the light quarks which, in a sense, imitates quark confinement. It does well at explaining the Gasser-Leutwyler coefficients and other parameters [10][11] and it remarkably predicted a universal "chiral mass gap" in all heavy-light quark bound states, leading to long lived heavy-strange resonances, such as the $D_s(2317)$, [12].

However, in the case of top condensation one has a bound state of non-confined constituents. Hence the bound state couples to free unbound fermions that have the same quantum numbers as the constituents. Here we encounter fundamental physical limitations of the NJL model:

- the NJL model is an effective pointlike 4-fermion interaction associated with a "large" mass scale M_0, and the resulting bound states emerge as *pointlike* fields with mass $\mu^2 < M_0^2$;
- in the NJL model the binding mechanism is entirely driven by quantum loop effects, while we see in nature that binding readily occurs semiclassically without quantum loops, such as the hydrogen atom;
- Mainly, the NJL model lacks an internal wave-function $\phi(r)$. The inclusion of $\phi(r)$ has significant impact upon the conclusions drawn from the model.

In the case of the hydrogen atom, before turning on the Coulomb interaction, there are open scattering states involving free protons and electrons.[1] As the interaction is turned on the lowest energy scattering states flow to become the bound states, while most scattering states remain unbound. The dynamics is governed by the non-relativistic semiclassical (tree level) Schrödinger equation [13], leading to normalizable yet spatially extended wave-functions on the scale $(\alpha m_e)^{-1}$. The atom is described naturally in a configuration space picture. Quantum loop effects (such as the Lamb shift) are higher order corrections to this mostly semiclassical phenomenon.

In the NJL model the picture is substantially different. There is no semiclassical binding producing an extended bound state. Rather, the bound state is described by a local effective field, $\Phi(x)$, with its properties arising from quantum loops. The loops integrate out the constituent fermions from the large mass scale of the interaction, M_0, down to an IR cut-off μ (e.g., $M_0 \sim 1$ GeV and $\mu \sim f_\pi \sim 100$ MeV in QCD). When the discussion is formulated in momentum space, treated in the large N_{color} limit, bound states appear as poles in the S-matrix upon summing towers of fermion loop diagrams. With a large hierarchy, $M_0/\mu \gg 1$, there are also large logarithms, and the sum of loop diagrams is best handled by using an effective action and the renormalization group (RG). $\Phi(x)$ has only the minimal dynamical degrees of freedom of a pointlike field. Hence, the NJL model leads to a pointlike field theory description of a bound state, with boundary conditions on the RG running of its couplings at the scale M_0.

Since the old top condensation theory used the pointlike NJL model it therefore omitted an internal bound state wave-function, $\phi(r)$. The formulation of $\phi(r)$, in a UV completion of the NJL model, has been developed in a recent work [14][15]. The omission of $\phi(r)$ is not expected to significantly affect QCD applications of the NJL model, due to confinement of quarks which would presumably cut off any wave-function spreading. However, if the NJL coupling constant is near its critical value for a non-pointlike and non-confining theory, then the low energy effective theory is approximately conformal. This implies that the internal wave-function $\phi(r)$ spreads significantly into empty space.

An extended wave function occurs for any localized potential with small eigenvalue for the Hamiltonian. The Dirac δ-function potential in $1+1$ dimensions provides a typical example (Fig. 1). The Schrödinger equation is,

[1] We refer to compact bound state wave-functions as "normalizable." Open free particle scattering states, that require typically "box normalization," are referred to as "non-normalizable," requiring introduction of some IR cut-off on the wave-functions.

$$-\frac{1}{2m}\frac{d^2}{dx^2}\phi(x) + V(x)\phi(x) = E\phi(x), \qquad \text{where}, \qquad V(x) = -\alpha\delta(x). \tag{3}$$

The bound state solution ($-\infty < x < \infty$) is $\phi(x) = -\sqrt{\alpha m}\exp(-\alpha m|x|)$, with eigenvalue $E = -\alpha^2 m/2$. The bound state exists for any $\alpha > 0$, with eigenvalue $E < 0$ (analogous to the Coulomb potential in $1+3$ dimensions). Hence the critical coupling, the value of α at which $E = 0$, is $\alpha = 0$, and the external wave-function then coincides with the lowest energy $1+1$, non-normalizable, scattering state, $\phi(x) \to (constant)$. Note that the transition from bound to unbound at $\alpha = 0$ is discontinuous (non-analytic) since the normalization, is finite (compact) for $\alpha > 0$ and divergent (non-normalizable) for $\alpha < 0$ [16]. This illustrates the general result that a near–critical bound state in a localized potential must always be an extended object, due to approximate scale invariance external to the potential.

There are many models of bound states with internal structure, such as [17][18][19] to name a few. We prefer, however, to focus on the well-defined NJL model [8] and generalize it to a non-pointlike theory. To understand the internal $\phi(r)$ in field theory we must first ask exactly how the local pointlike 4-fermion interaction is generated as the limit of a bilocal interaction, $V(x,y) \to V(x)$. With a bilocal interaction we must then replace the pointlike $H(x)$ by a *bilocal field* $H(x,y)$. In the NJL model the natural candidate for this is "topcolor" [5] where the non–pointlike interaction arises from the exchange of a massive gluon-like object, of mass M_0 and coupling g_0, called a "coloron" [7], [20]. In the large M_0 limit the interaction recovers the pointlike NJL form, but the bilocal nature of $H(x,y)$ is then established and remains so even in the pointlike $V(x,y) \to V(x)$ limit!

This requires a general formulation of a bilocal field theory. The starting point for this begins with old ideas of Yukawa [21] of multilocal fields. We can modify and extend Yukawa's bilocal fields to an action formalism. We then have a non-pointlike UV description of the physics as a generalization of the NJL model which we can rely upon for intuition. The semiclassical binding interaction is enhanced by a factor of N_c in analogy to BCS superconductivity [22]. The formalism then leads to a Schrödinger-Klein Gordon (SKG) equation that determines $\phi(r)$ with eigenvalue μ^2. Above critical coupling the eigenvalue, μ^2, becomes negative and spontaneous symmetry breaking (SSB) occurs.

In the symmetric phase of the model (before SSB) we find that bound states will form semiclassically, $(\hbar)^0$, similar to the hydrogen atom, but relativistically in a configuration space picture. The resulting bound state near criticality, where the mass $|\mu|$ is small compared to the composite scale (the coloron mass M_0), will have an extended "tail" in its rest frame $\phi(r) \sim e^{-|\mu|r}/r$ where $r = |(\vec{x}-\vec{y})/2|$.

The major implication is that the wave-function spreading causes a significant "dilution" of $\phi(0) \sim \sqrt{|\mu|/M_0}$. The resulting top quark Yukawa coupling, $g_Y \propto \phi(0)$, and quartic coupling, $\lambda \propto |\phi(0)|^4$, are then determined by $\phi(0)$, with its power law suppression, rather than the relatively slow RG evolution in the NJL model. By fitting the top quark Yukawa coupling to its known value $g_Y \approx 1$, and $\mu^2 = -(88)$ GeV^2, the Lagrangian mass of the BEH boson in the symmetric phase of the SM, we can then readily determine M_0. The implied scale of compositeness of $H(x,y)$, i.e., the mass of the coloron, is then significantly reduced compared to the NJL model and we obtain a result: $M_0 \sim 6$ TeV(!). The SM parameters (Lagrangian BEH mass, $-\mu^2$, electroweak VEV, v_{weak}, Yukawa coupling, g_Y, and (remarkably) the quartic coupling, λ) all become concordant with experiment. Moreover, the fine–tuning of the model is vastly reduced to a few %. The major prediction is the existence of a QCD color octet of colorons with mass $M_0 \sim 6$ TeV that may be accessible to the LHC.[2]

A core issue of a bilocal (or multilocal) description is the "relative time" problem [24]. In a two body bound state each particle carries its own clock, hence we have times t_1 and t_2, therefore we have the "average time" $(t_1+t_2)/2$ and the "relative time" (t_1-t_2). This is endemic to non-relativistic, as well as relativistic systems. In the center of mass frame, which is the rest frame of the bound state (often called the "barycentric frame") the relative time drops out of the kinetic terms. We can then integrate out the relative time and the interaction becomes a static potential in the rest frame. This requires a normalization of the bilocal field kinetic terms to establish the relevant normalized currents and charges when relative time is removed. The pointlike NJL theory avoids the relative time problem because it simplifies the interaction to a single point in spacetime, but one then misses the extended wave function $\phi(r)$. For a relativistic system the reduction is done with Lorentz invariant constraints. While one loses *manifest* Lorentz invariance in the rest frame of the bound state, the overall Lorentz invariance of the theory is maintained, a procedure akin to gauge fixing in a gauge invariant field theory (see Appendices B and C for further discussion.)

Mainly, we propose in the present paper a new version of the top condensation idea, a "redux," which relies on a "topcolor" interaction that generates the UV completion of the NJL scheme and provides the binding mechanism through a bilocal interaction $V(x,y)$. Though topcolor was previously introduced in the 1990's, much of its structure carries over for us presently [5], [6]. Here we are invoking it as the primary binding mechanism of the BEH boson (replacing, e.g., "technicolor," rather than "assisting technicolor").

We begin with a quick summary of key features in the old top condensation NJL model. We then give a simple example of a bilocal formulation of the non-relativistic hydrogen atom, which illustrates the formal issues and the problem of relative time. To provide orientation, we then follow with a lightning summary of the composite BEH theory. The full technical details, some of which we think are rather stunning, are then given in the bulk of the paper.

[2] See Appendix D for a summary of the symmetric and broken phase parameters of the SM. We emphasize that we obtain results *in the semiclassical limit.* We have not yet completed analysis of the quantum loop corrections to this [23], which may be significant, so we quote $M_0 \sim 6$ TeV with potential uncertainties.

1.1. Nambu–Jona-Lasinio model application to top condensation

We will rely heavily on intuition from the NJL model, so we provide this quick summary (more details appear in [14], [15]). The "old" NJL model of top condensation assumes chiral fermions, with $N_c = 3$ "colors" and a pointlike 4-fermion interaction, hence we have:

$$S_{NJL} = \int d^4x \left(i[\overline{\psi}_L(x)\not{D}_L\psi_L(x)] + i[\overline{\psi}_R(x)\,\not{D}_R\psi_R(x)] + \frac{g_0^2}{M_0^2}[\overline{\psi}_{iL}(x)\psi_R(x)]\,[\overline{\psi}_R(x)\psi_L^i(x)] \right), \tag{4}$$

where i is an isospin index, [..] implies color singlet combination, and $\psi_{R,L} = (1 \pm \gamma^5)\psi/2$.

The NJL interaction is invariant under $SU(3)_{QCD} \times SU(2) \times U(1)$ gauge symmetry. The fields and covariant derivatives are defined in the standard model (for simplicity we don't display the color indices on the quark fields):

$$\psi_L^i = \frac{(1-\gamma^5)}{2}\begin{pmatrix} t \\ b \end{pmatrix}, \qquad \psi_R = \frac{(1+\gamma^5)}{2}t,$$

$$D_{L\mu} = \partial_\mu - ig_2 W_\mu^A \frac{\tau^A}{2} - ig_1 B_\mu \frac{Y_L}{2} - ig_3 \mathcal{G}^A \frac{\chi^A}{2}, \qquad D_{R\mu} = \partial_\mu - ig_1 B_\mu \frac{Y_R}{2} - ig_3 \mathcal{G}^A \frac{\chi^A}{2}, \tag{5}$$

where the QCD gluons are $\mathcal{G}^A$, the weak hypercharges $Y_L = 1/3$, $Y_{tR} = 4/3$, and $Y_{bR} = -2/3$, and the electric charges are as usual: $Q = I_3 + \frac{Y}{2}$ (e.g., for the b_L quark, $Q = -\frac{1}{2} + \frac{1}{2}\frac{1}{3} = -\frac{1}{3}$, while for b_R, $Q = 0 - \frac{1}{2}\frac{2}{3} = -\frac{1}{3}$, etc.).

An equivalent form of the interaction can be written by introducing an auxiliary isodoublet field $H^i(x)$:

$$S_{NJL} = \int d^4x \left(i[\overline{\psi}_L(x)\not{D}_L\psi_L(x)] + i[\overline{\psi}_R(x)\not{D}_R\psi_R(x)] - M_0^2 H^\dagger(x)H(x) + (g_0[\overline{\psi}_{iL}(x)\psi_R(x)]H^i(x) + h.c.) \right). \tag{6}$$

The "equation of motion" for $H^i(x)$ is then:

$$M_0^2 H^i(x) = g_0[\overline{\psi}_R(x)\psi_L^i(x)]. \tag{7}$$

$H(x)$ will become the bound state field. Note that $H(x)$ is a pointlike field since the 4-fermion interaction is pointlike.

Following Wilson [25] we view eqs. (6), (7) as the effective action at the high scale $m = M_0$. We integrate out the fermions to obtain the effective action for the bound state field $H(x)$ at a lower scale $m \ll M_0$:

$$S_\mu = \int d^4x \left(i[\overline{\psi}_L\not{D}_L\psi_L] + i[\overline{\psi}_R\not{D}_R\psi_R] + Z D_{H\mu}\Phi^\dagger D_H^\mu\Phi - \mu^2 H^\dagger H - \frac{\lambda}{2}(H^\dagger H)^2 + (g_0[\overline{\psi}_{iL}\psi_R]H^i(x) + h.c.) \right), \tag{8}$$

where,

$$\mu^2 = M_0^2 - \frac{g_0^2 N_c}{8\pi^2}M_0^2, \qquad Z = \frac{g_0^2 N_c}{8\pi^2}\ln(M_0/m), \qquad \lambda = \frac{g_0^4 N_c}{4\pi^2}\ln(M_0/m). \tag{9}$$

We see, from Feynman loops, that $H(x)$ acquires a kinetic term with the covariant derivative,

$$D_{H\mu} = \partial_\mu - ig_2 W_\mu^A \frac{\tau^A}{2} - ig_1 B_\mu \frac{Y_H}{2}, \tag{10}$$

where the gluons cancel, and the weak hypercharge becomes $Y_H = -1$, apropos the BEH boson of the SM.

In particular, note the behavior of the composite BEH boson mass, μ^2, of eq. (9) due to the loop contribution, $-g_0^2 N_c M_0^2/8\pi^2$, (we use a UV cut-off M_0^2 on the fermion loops to imitate a softening of the interaction on scale $m \gg M_0$). The NJL model therefore has a critical value of its coupling, g_c, defined by the vanishing of μ^2:

$$\frac{g_c^2 N_c}{8\pi^2} = 1. \tag{11}$$

We can renormalize, $H \to \sqrt{Z}^{-1}H$, to obtain the full renormalized effective Lagrangian. The notable feature here is that the renormalized couplings evolve logarithmically in the RG "running mass" m:

$$g_Y^2 = \frac{g_0^2}{Z} = \frac{4\pi^2}{N_c\ln(M_0/m)}, \qquad \lambda_r = \frac{\lambda}{Z^2} = \frac{16\pi^2}{N_c\ln(M_0/m)}. \tag{12}$$

These are the solutions to the RG equations in the large N_c limit, keeping only fermion loops, [4]. Eq (12) implies the renormalized couplings have Landau poles, i.e., $(g_Y^2(m), \lambda_r(m))$ blow up logarithmically as $m \to M_0$. This defines "compositeness boundary conditions" on H for the RG running. We can then use the full RG equations, including QCD and electroweak interactions, to obtain precise low energy predictions [4]. In particular, the Yukawa coupling g_Y approaches the IR fixed point value [26]. Results are shown in Fig. 2 and Table 1.

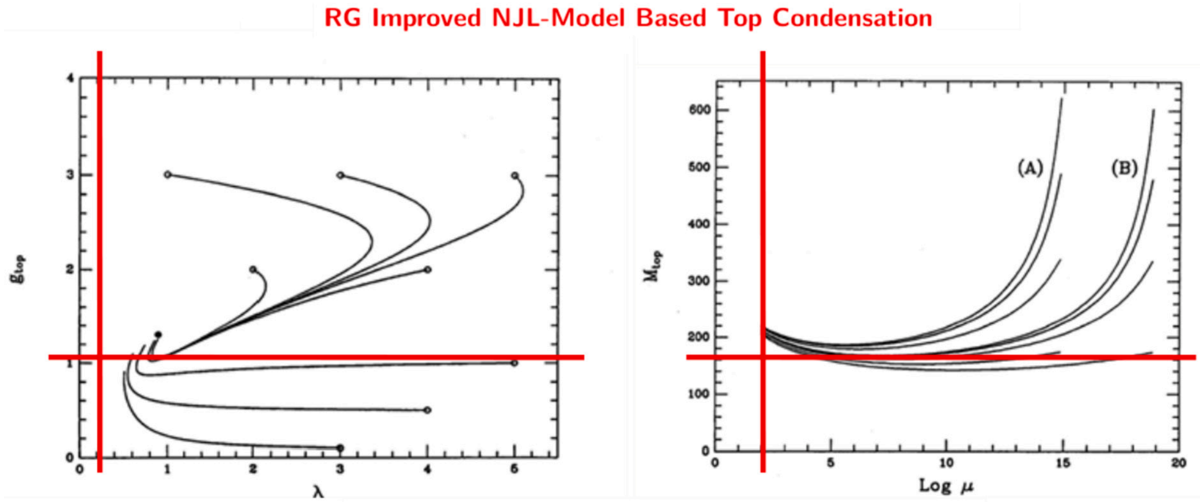

Fig. 2. Figure on left shows the full joint RG running of $g_{top} \equiv g_{Y(ukawa)}$ and λ flowing from initial values at $M_0 = 10^{15}$ GeV to v_{weak}. Right figure shows full running of effective top quark mass and RG fixed point [26]. Solid (red) lines indicate experimental values.

Table 1
Results for the top quark mass, m_{top}, determined by RG running from Landau pole in g_Y at M_0, to v_{weak}.

M_0 GeV	10^{19}	10^{15}	10^{11}	10^7	10^5
m_t (GeV) Fermion Loops	144	165	200	277	380
m_t (GeV) Planar QCD	245	262	288	349	432
m_t (GeV) Full RG	218	229	248	293	360
m_{BEH} (GeV) Full RG	239	256	285	354	455

For super-critical NJL coupling, $g^2 > g_c^2$, we see that the (renormalized) Lagrangian mass, $\mu_r^2 = \mu^2/Z < 0$, implying there will be a vacuum instability. The effective action, with the induced quartic coupling $\sim \lambda_r(\Phi^\dagger\Phi)^2$ term, yields the usual sombrero potential, and the $SU(2)\times U(1)$ symmetry is spontaneously broken, and the neutral component of the BEH field H acquires a VEV,

$$\langle H^0 \rangle = v_{weak} = \frac{|\mu_r|}{\sqrt{\lambda_r}}. \tag{13}$$

The top quark then acquires mass $m_t = g_Y v_{weak}$ (for light quark and lepton masses and mixings we would rely upon higher dimension "Extended Technicolor" (ETC) operators [27]).

The solutions for the NJL based top quark mass are shown in Table 1. At the time the model was proposed there were upper bounds on the top quark and BEH boson masses of order several hundred GeV. We see that, to obtain a top quark mass $m_t \lesssim 230$ GeV, we require very large M_0 due to the slow running of the RG and its fixed point. With a choice of e.g., of $M_0 \sim 10^{15}$ GeV, we obtain from eq. (9):

$$\mu_r^2 \sim M_0^2\left(1 - \frac{g_0^2 N_c}{8\pi^2}\right) = M_0^2\left(1 - \frac{g_0^2}{g_c^2}\right). \tag{14}$$

We see that small BEH mass, μ_r^2, mandates the fine–tuning of g_0^2/g_c^2 at the level of $\sim 10^{-26}$:

$$\frac{\delta g_0^2}{g_c^2} \sim \frac{|\mu_r^2|}{M_0^2} \sim 10^{-26}\ ! \tag{15}$$

While the top condensation theory was directly testable by experiment, it evidently failed. However, the difficulties with the old top condensation theory stem from the limitations of the Nambu–Jona-Lasinio model.

1.2. The hydrogen atom as a bilocal field theory

As a warm-up example we presently give a derivation of the well-known Schrödinger equation for the hydrogen atom, using bilocal field techniques that we develop for a composite BEH boson. This illustrates issues that will arise subsequently for the problem of bound states of pairs of chiral fermions and our generalization of the NJL model.

We can represent the hydrogen atom in terms of proton and electron fields (scalars for present purposes), $\psi_p(y)$ and $\psi_e(x)$, where (x^μ, y^μ) are 4-vectors. We then introduce a bilocal field describing a proton-electron pair,

$$\Phi(x,y) = \psi_p(y)\psi_e(x). \tag{16}$$

Since we typically define normalizations as $\int d^3x|\psi|^2 = 1$, therefore the ψ fields have mass-dimension $1/\sqrt{V} \sim M^{3/2}$. We will presently assume Φ has mass dimension $\sim M^3$.

A dimensionless bilocal action can be written as:

$$\int d^4x\, d^4y \left(iZ\Phi^\dagger \frac{\partial}{\partial x^0}\Phi + iZ\Phi^\dagger \frac{\partial}{\partial y^0}\Phi - \frac{1}{2m_p} Z|\partial_{\vec{y}}\Phi|^2 - \frac{1}{2m_e} Z|\partial_{\vec{x}}\Phi|^2 - e^2 D(x-y)|\Phi|^2 \right). \tag{17}$$

While this is non-relativistic we have kept the 4-volumes associated with each coordinate (x,y). We have included a normalization factor, Z, of mass dimension $\sim M$ on the bilocal kinetic terms which will become clear momentarily. The interaction is generated by a single photon exchange between the non-relativistic charge densities of the proton and electron, and using eq. (16):

$$e^2|\psi_e(x)|^2|\psi_p(y)|^2 D(x-y) = -e^2|\Phi(x,y)|^2 \int \frac{d^4q}{(2\pi)^4} \frac{e^{iq_\mu(x-y)^\mu}}{q_0^2 - \vec{q}^2 + i\epsilon}. \tag{18}$$

It is useful to go to coordinates where the heavy proton is located at X^μ and the electron at $X^\mu + \rho^\mu$ as,

$$X = y; \quad \rho = x - y; \quad \text{then, } \partial_x = \partial_\rho; \quad \partial_y = \partial_X - \partial_\rho; \quad \text{with Jacobian: } \left|\frac{\partial(X,\rho)}{\partial(x,y)}\right| = 1, \tag{19}$$

and the action becomes, with $\Phi'(X,\rho) = \Phi(X+\rho, X)$,

$$\int dX^0 d^3X\, d\rho^0 d^3\rho \left(iZ\Phi'^\dagger \frac{\partial}{\partial \rho^0}\Phi' + iZ\Phi'^\dagger \left(\frac{\partial}{\partial X^0} - \frac{\partial}{\partial \rho^0} \right)\Phi' - \frac{1}{2m_p} Z|(\partial_{\vec{X}} - \partial_{\vec{\rho}})\Phi'|^2 - \frac{1}{2m_e} Z|\partial_{\vec{\rho}}\Phi'|^2 - e^2 D(\rho)|\Phi'|^2 \right). \tag{20}$$

We see in eq. (20) that the derivative $\partial/\partial\rho^0$ cancels. This is also central to a relativistic formalism: ρ^0 is the "relative time" and is seen to drop out of the action in the rest frame. The only time degree of freedom then carried by the system is X^0 and we can therefore integrate out ρ^0. This reveals the purpose of the normalization factor Z, and we define:

$$\int d\rho^0\, Z = 1. \tag{21}$$

Z acts only on the kinetic terms and is needed to insure these are canonical, e.g., that they generate properly normalized Noether currents, etc. Note that we could alternatively define $Z = \delta(\rho^0)$.

We therefore assume that Φ has no dependence upon ρ^0 (this can be done in the relativistic case with Lorentz invariant constraints). We can then integrate over ρ^0 in the interaction which then becomes the standard Coulomb form,

$$e^2\int d\rho^0 D_F(\rho)|\Phi'(\vec{\rho})|^2 = -e^2\int d\rho^0 \frac{d^4q}{(2\pi)^4}\frac{1}{q^2} e^{iq_\mu\rho^\mu}|\Phi'(\vec{\rho})|^2 = e^2\int \frac{d^3q}{(2\pi)^3}\frac{1}{\vec{q}^{\,2}} e^{-i\vec{q}\cdot\vec{\rho}}|\Phi'(\vec{\rho})|^2 = \frac{e^2}{4\pi|\vec{\rho}|}|\Phi'(\vec{\rho})|^2. \tag{22}$$

We can then approximate the proton as infinitely heavy, since the proton kinetic term is suppressed as $1/m_p \sim 0$. In the rest frame the proton is a zero-momentum plane wave with "volume normalization," $\psi_p(x) = e^{iEX^0}/\sqrt{V}$ and we can integrate it out with $\int d^3X = V$. We then have $\Phi'(X,\rho) \to e^{-iEX^0}\psi_e(\vec{\rho})$, where the "clock time" is $X^0 = t$, and we have a static electron wave-function $\psi_e(\vec{\rho})$. The bilocal action then becomes the usual single particle form for the electron:

$$\int dt\, d^3\rho \left(E|\psi_e(\vec{\rho}\,)|^2 - \frac{1}{2m_e}|\partial_{\vec{\rho}}\psi_e(\vec{\rho}\,)|^2 + \frac{\alpha}{|\vec{\rho}\,|}|\psi_e(\vec{\rho}\,)|^2 \right). \tag{23}$$

In spherical coordinates, integrating by parts and extremalizing the action gives the Schrödinger equation,

$$-\frac{1}{2m_e}\nabla^2_{\vec{\rho}}\psi_e(\vec{\rho}\,) - \frac{\alpha}{|\vec{\rho}\,|}\psi_e(\vec{\rho}\,) = E\psi_e(\vec{\rho}\,), \tag{24}$$

where E is the eigenvalue, and $\alpha = e^2/4\pi$.

This illustrates that the natural starting point for a two-body bound state is a bilocal field, $\Phi(x,y)$, and schematically anticipates the treatment we will use below for pairs of chiral fermions [14]. To give another less trivial example, we give a relativisitic bilocal field theory composed of scalar fields in ref. ([15]). We now turn to the UV completion and bilocalization of the NJL model.

1.3. *Outline of a bilocal BEH boson theory*

We presently give a brief summary of our theory of a composite BEH boson arising in "natural" top quark condensation. This will illustrate the principles of the construction and one will see similarities with the simple hydrogen atom example given above. This preliminary summary omits the technical details that follow in the bulk of the paper.

In the NJL top condensation model the pointlike 4-fermion effective interaction of eq. (1) can be viewed as a Fierz rearrangement of a color-current interaction (the Fierz rearrangement is derived in Appendix C of [14]):

$$-\frac{g^2}{M_0^2}[\overline{\psi}_{iL}\psi_R][\overline{\psi}_R\psi_L^i] = \frac{g^2}{M_0^2}(\overline{\psi}_{iL}\gamma_\mu\frac{\chi^A}{2}\psi_L^i)(\overline{\psi}_R\gamma^\mu\frac{\chi^A}{2}\psi_R) + O(1/N_c), \tag{25}$$

where $N_c = 3$ is the number of colors. This is exactly the form (including the sign) induced by a massive color octet vector boson exchange, and leads to topcolor models [5].

In topcolor, QCD is embedded into an $SU(3)_1 \times SU(3)_2$ gauge group at higher energies. The second (weaker) $SU(3)_2$ gauge interactions acts upon the first and second generation quarks while the (stronger) $SU(3)_1$ interaction acts upon the third generation and drives the formation of the BEH bound state, H. Additional dynamics is also incorporated to disallow the formation of a second BEH boson, H', containing b_R, (usually achieved by introducing a heavy Z' so the $\overline{\psi}_L b_R$ channel is subcritical; this can be accommodated in extension of the present minimal model). In the following minimal model we simply omit the b_R quark from the binding dynamics.

The gauge structure at high energies (ignoring any Z' interactions) is therefore [5]:

$$SU(3)_1 \times SU(3)_2 \times SU(2)_L \times U(1)_Y \to SU(3)_{QCD} \times SU(2)_L \times U(1)_Y. \tag{26}$$

The SM fermions are assigned to $(SU(3)_1, SU(3)_2, SU(2), Y)$, where $Q = I_3 + \frac{Y}{2}$, as follows:

$$\begin{aligned}
&(t,b)_L \sim (3,1,2,1/3) \qquad (t)_R \sim (3,1,1,4/3) \qquad [(b)_R \sim (3,1,1,-2/3)] \\
&(\nu_\tau,\tau)_L \sim (1,1,2,-1) \qquad (\tau)_R \sim (1,1,1,-2) \\
&(u,d)_L,\ \ (c,s)_L \ \sim (1,3,2,1/3) \qquad (u)_R,\ (c)_R \sim (1,3,0,4/3) \\
&(\nu_e,e)_L,\ (\nu_\mu,\mu)_L \sim (1,1,2,-1) \qquad (\ell)_R,\ (\mu)_R \sim (1,1,1,-2).
\end{aligned} \tag{27}$$

The $SU(3)_1 \times SU(3)_2$ extended color interaction is broken to the diagonal $SU(3)_{QCD}$ (this is described elsewhere [5][7]) leading to the massive octet of "colorons," $\mathcal{G}'^A_\mu$, and the massless octet of the gluons of QCD, $\mathcal{G}^A_\mu$.

Integrating out the heavy colorons and Fierz rearranging gives a bilocal interaction:

$$S' = g_0^2 \int d^4x d^4y\, [\overline{\psi}_{iL}(x)\psi_R(y)] D_F(x-y) [\overline{\psi}_R(y)\psi^i_L(x)]. \tag{28}$$

We therefore introduce a color singlet bilocal BEH field of mass dimension 1, (analogous to eqs. (7), (16):

$$\sqrt{N_c} M_0^2 H^i(x,y) = [\overline{\psi}_R(x)\psi^i_L(y)], \tag{29}$$

(here (i,j) are electroweak indices, [..] denotes color indices summed, and H^i is conventionally color normalized, as in Section 2.1 below). The fields appearing on the *rhs* of eq. (29) are those that will form the bound state when the interaction is turned on, generally the low momentum scattering states. The interaction then becomes bilocal,

$$g_0^2 M_0^4 N_c \int d^4x d^4y\, D_F(x-y)|H(x,y)|^2, \quad \text{where,} \quad D_F(x-y) = -\int \frac{1}{q^2 - M_0^2} e^{iq(x-y)} \frac{d^4q}{(2\pi)^4}, \tag{30}$$

where, due to a color singlet normalization of H, an N_c enhancement occurs in analogy to BCS theory [22] (as detailed in Section 2). The interaction also generates Yukawa couplings of H to fields that remain free fermions,

$$g_0^2 M_0^2 \sqrt{N_c} \int d^4x d^4y\, D_F(x-y) [\overline{\psi}^i_L(y)\psi_R(x)]_f H_i(x,y) + h.c. \tag{31}$$

We can then construct the Lorentz invariant action that yields the equations of motion by variation:

$$S_K = M_0^4 \int d^4x d^4y \left(Z|D^\dagger_R H(x,y)|^2 + Z|D_L H(x,y)|^2 + g_0^2 N_c D_F(x-y)|H(x,y)|^2 \right), \tag{32}$$

where the covariant derivatives are as defined in the NJL model:

$$D_{L\mu} = \frac{\partial}{\partial y_\mu} - ig_2 W^A_\mu(y)\frac{\tau^A}{2} - ig_1 B_\mu(y)\frac{Y_L}{2}; \qquad D^\dagger_{R\mu} = \frac{\partial}{\partial x_\mu} + ig_1 B_\mu(x)\frac{Y_R}{2}. \tag{33}$$

Note that D_L ($D^\dagger_R$) acts at coordinate y (x), and $D^\dagger_R$ acts on $\bar{t}_R$, hence the sign flip in the gauge field terms (note the derivative $D^\dagger$ acts in the forward direction as we have written the kinetic term eq. (32)).

We now pass to barycentric coordinates,

$$X^\mu = \frac{x^\mu + y^\mu}{2}, \quad r^\mu = \frac{x^\mu - y^\mu}{2}, \quad \partial_x = \frac{1}{2}(\partial_X + \partial_r), \quad \partial_y = \frac{1}{2}(\partial_X - \partial_r) \ \text{ with Jacobian: } \left|\frac{\partial(X,\rho)}{\partial(x,y)}\right|^{-1} \equiv J = 2^4. \tag{34}$$

We then use Wilson lines to "pull-back" the gauge couplings from (x,y) to the center X. This is done by field redefinitions (as discussed in Appendix A):

$$H(x,y) \to W^\dagger_R(X,x) W_L(X,y) H(X)\phi(r), \tag{35}$$

where we have also made a factorized ansatz for the H^i field following Yukawa [21]:

$$H^i(x,y) \to \sqrt{2/J}\ H^i(X)\phi(r), \qquad \text{where } \phi \text{ is normalized as,} \qquad Z M_0^4 \int d^4r |\phi(r)|^2 = 1, \tag{36}$$

and ϕ is therefore dimensionless. This leads to the action:

$$=\int d^4X \left(|D_H H(X)|^2 + |H(X)|^2 M_0^4 \int d^4r \left(Z|\partial_{r^\mu}\phi(r)|^2 + 2g_0^2 N_c D_F(2r)|\phi(r))|^2 \right)\right), \tag{37}$$

where the $H^i(X)$ kinetic term is now canonical, and the covariant derivative is:

$$D_{H\mu} = \frac{\partial}{\partial X^\mu} - ig_2 W_\mu^A(X)\frac{\tau^A}{2} - ig_1 B_\mu(X)\frac{Y_H}{2}. \tag{38}$$

We see that the "pull-back" of the Wilson lines has moved all the electroweak gauging of the bilocal field to the "center," X, and changes the covariant derivative to the precise form apropos the BEH boson, (and the gluons have canceled). The BEH boson has therefore become a "hedge-hog" configuration of radiating Wilson lines. The internal wave-function $\phi(r)$ is now a complex scalar that carries no gauge charges.

We note that the action is Lorentz invariant (see Appendix B) and can therefore be evaluated in any frame. If we consider a pair of massless particles of 4-momenta, p_1^μ and p_2^μ, we have total momentum, $P^\mu = (p_1^\mu + p_2^\mu)$, and relative momentum, $Q^\mu = (p_1^\mu - p_2^\mu)$, where $P_\mu Q^\mu = p_1^2 - p_2^2 = 0$. This implies that there is always a rest frame in which $P^\mu = (P_0, 0)$ and $Q^\mu = (0, -\vec{q})$. In the rest frame we see that $Q^0 = 0$, therefore the relative time, r^0, drops out of the kinetic terms.

Hence, we can assume $\phi(r) \to \phi(\vec{r})$ is a "static field," with no dependence on r^0. This converts the $\partial_{r^\mu}^2 \to -\partial_{\vec{r}}^2$. We then define Z by $ZM_0\int dr^0 = 1$ in analogy to eq. (21), (or $Z \to \delta(M_0\omega^\mu r_\mu)$, where $\omega_\mu = P_\mu/\sqrt{P^2}$ is a timelike unit 4-vector). The ϕ normalization becomes:

$$1 = \int d^3r\, M_0^3\, |\phi(\vec{r})|^2. \tag{39}$$

The coloron exchange potential when integrated over r^0 then becomes a static Yukawa potential. Including the Yukawa interaction and a loop generated quartic term (see Section 5), we obtain the effective action for the composite BEH field,

$$S = \int d^4X \left(|D_H H(X)|^2 + |H(X)|^2\, M_0^3 \int d^3r \left(-|\partial_{\vec{r}}\phi(r)|^2 + g_0^2 N_c M_0 \frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}|\phi(\vec{r})|^2 \right)\right.$$
$$\left. -\frac{\lambda}{2}(H^\dagger H)^2 - g_Y\left([\overline{\psi}_{iL}(X)t_R(X)]_f H^i(X) + h.c.\right)\right), \tag{40}$$

where the subscript f denotes free unbound fermions. The internal field $\phi(r)$ is "nested" within the action for a conventional pointlike BEH boson, $H(X)$. The static ϕ field has Hamiltonian:

$$\mathcal{M} = M_0^3 \int d^3r \left(|\partial_{\vec{r}}\phi(r)|^2 - g_0^2 N_c M_0 \frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}|\phi(\vec{r})|^2 \right). \tag{41}$$

Extremalization of these yields the Schrödinger-Klein-Gordon (SKG) equation for ϕ with the eigenvalue μ^2:

$$-\nabla^2\phi - g_0^2 N_c M_0 \frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}\phi(r) = \mu^2\phi. \tag{42}$$

We find that the SKG equation has a critical coupling, g_c, for which $\mu^2 = 0$, that is very close to the quantum NJL critical coupling (see Sections 4.2 and 4.6). When $g_0 > g_c$ the eigenvalue μ^2 becomes negative, $= -|\mu|^2$. In such a solution the action for $H(X)$ then becomes the familiar,

$$S = \int d^4X \left(|D_H H(X)|^2 + |\mu|^2|H(X)|^2 - \frac{\lambda}{2}(H^\dagger H)^2 - g_Y\left([\overline{\psi}_L(X)t_R(X)]_f H(X) + h.c.\right)\right), \tag{43}$$

with the "sombrero potential":

$$-|\mu|^2|H(X)|^2 + \frac{\lambda}{2}(H^\dagger H)^2. \tag{44}$$

The technical details of this derivation are many, and together with the analysis of the results, are given in the remainder of this paper. The results of the present scheme are:

- The solution of the SKG equation for $\phi(r)$ indeed extends to large distances, $\phi(r) \sim e^{-|\mu|r}/r$ where $|\mu| < M_0$ near critical coupling. This dilutes the value of $\phi(0) \sim \sqrt{|\mu|/M_0}$. We find that the Yukawa coupling $g_Y \propto \phi(0)$ and $\lambda \propto g_Y^4 \propto |\phi(0)|^4$ (derived below). This has profound effects on the theory compared to the pointlike NJL model.
- Inputting the known value of the Lagrangian mass of the BEH boson in the symmetric phase, which is $-|\mu|^2 = -(88)^2$ GeV^2, we find that **the scale M_0 is now $M_0 \approx 6$ TeV** (cf, no longer the nonsensical 10^{15} GeV in the NJL model).
- Moreover, the quartic coupling, λ, is now determined at loop level by RG running from M_0, with boundary condition $\lambda = 0$ (not a Landau pole!) down to $|\mu| \sim 88$ GeV. This yields, at one loop, $\lambda \approx 0.23$, whereas the standard model determines $\lambda \approx 0.25$, hence remarkable agreement is obtained (whereas in the NJl model we had $\lambda \sim 1$).

- The degree of fine-tuning of the theory is also suppressed by $\phi(0)$ in a subtle way. Rather than the naive result one would expect from the NJL model, $\delta g_0^2/g_c^2 \sim |\mu|^2/M_0^2 \sim 10^{-4}$, we now obtain a linear relation: $\delta g_0^2/g_c^2 \sim |\mu|/M_0 \sim 1\%$.

This concludes a lightning summary to give the reader a sense of this approach and what it yields. We now descend into the technical details.

2. Semiclassical non-pointlike generalization of the NJL model

2.1. Bilocal fields

We now consider in greater detail the formalism for a semiclassical approach to binding in a non-confining theory of chiral fermions in analogy to our brief sketch of the hydrogen atom above.

In the limit of shutting off an interaction, a bound state is just a two-body scattering state, such as a product of a free electron and free proton wave-functions in the case of hydrogen. For chiral fermions this can be described by a complex bilocal field $\Phi^A_B(x,y)$,

$$M^2\Phi^A_B(x,y) = \overline{\psi}^A_R(x)\psi_{BL}(y), \tag{45}$$

(where (A,B) arbitrary unsummed color and flavor indices for more general $G_L \times G_R$ chiral group). Note that here we have implicitly defined Φ as a mass dimension-1 field, like a scalar, and the mass prefactor, M^2, will be elaborated below. Φ represents a "bosonization" of the pair of chiral fermions, as is done in writing chiral Lagrangians, such as the Σ-model. M is *a priori* arbitrary, but will be determined dynamically. Φ can in principle describe arbitrary pairs of fermions, including bound states or open scattering states.

Eq. (45) has a formal similarity to the factorized auxiliary field of the NJL model in eq. (6), however, $\Phi^A_B(x,y)$ is now a distinct physical free field. Unlike the auxiliary field in the NJL model it's kinetic term is not induced by loops, and it will have a free field kinetic term. We'll presently restrict ourselves to a single flavor, hence a $U(1)_L \times U(1)_R$ flavor symmetry, and $(A,B) \to (a,b)$ are $SU(N_c)$ color indices (this can be readily extended to $G_L \times G_R$ flavor group).

In the UV completion (coloron) model of the next section, we will see that only the color singlet field forms a bound state of a pair of chiral fermions. With $SU(N_c)$ color indices, (a,b), the field $\Phi^a_b(X,r)$ is a complex matrix that transforms as a product of $SU(N_c)$ representations, $\overline{N}_c \times N_c$, and therefore decomposes into a singlet plus an adjoint representation. We designate the color singlet bilocal field as Φ^0 and conventionally normalize it as,

$$\Phi^a_b(x,y) = \frac{1}{\sqrt{N_c}}\delta^a_b\Phi^0(x,y). \tag{46}$$

The conventional normalization allows canonically normalized kinetic terms, $\mathrm{Tr}[\partial\Phi^\dagger\partial\Phi] = \partial\Phi^{0\dagger}\partial\Phi^0$. Note $\mathrm{Tr}\,\Phi = \Phi^a_a(x,y) = \sqrt{N_c}\Phi^0(x,y)$.

We can have both bound and unbound free fermionic two-body scattering states (these are modes that will remain free after the interaction is turned on), and we will denote the free fermion pair by $\overline{\psi}^A_R(x)\psi_{BL}(y)_f$ with subscript f. We can therefore consider a quantum state consisting of superposition of a to-be-bound state and to-remain-free fermions written as,

$$\overline{\psi}^A_R(x)\psi_{LB}(y) = \overline{\psi}^A_R(x)\psi_{LB}(y)_f + M^2\Phi^A_B(x,y). \tag{47}$$

Technically, the components of this are orthogonal,

$$\int d^4x d^4y\, \overline{\psi}^A_R(x)\psi_{LB}(y)_f\Phi^{\dagger\, A'}_{B'}(x,y) = 0, \tag{48}$$

which would become relevant when we do perturbation theory. Since only the color singlet binds, we can rewrite eq. (47) containing free fields and the color singlet bound state of eq. (46),

$$\overline{\psi}^a_L(x)\psi_{bR}(y) \to \overline{\psi}^a_L(x)\psi_{bR}(y)_f + M^2\frac{\delta^a_b}{\sqrt{N_c}}\Phi^0(x,y). \tag{49}$$

2.2. The coloron model

As described above the pointlike NJL model can be viewed as the limit of a physical theory with a bilocal interaction. The primary example is the "coloron model" [5,7,20]. The coloron is a perturbative, massive gauge boson, a massive analogue of the gluon, arising in a local $SU(N_c)$ gauge theory broken to a global $SU(N_c)$.

We integrate out the massive coloron to generate a single particle exchange potential that defines the model. This leads to a bilocal current-current form:

$$S' = -g_0^2\int d^4x d^4y\, [\overline{\psi}_L(x)\,\gamma_\mu T^A\psi_L(x)]D^{\mu\nu}(x-y)[\overline{\psi}_R(y)\,\gamma_\nu T^A\psi_R(y)], \tag{50}$$

where $T^A = T^{Ab}_a$ are generators of $SU(N_c)$, and color indices are contracted within brackets [...].

The coloron propagator in Feynman gauge is:

$$D_{\mu\nu}(x-y) = g_{\mu\nu} D_F(x-y); \qquad D_F(x-y) = -\int \frac{1}{q^2 - M_0^2} e^{iq(x-y)} \frac{d^4q}{(2\pi)^4}. \tag{51}$$

A Fierz rearrangement of the interaction to leading order in $1/N_c$ leads to a potential:

$$S' = g_0^2 \int d^4x d^4y\, [\overline{\psi}_L(x)\psi_R(y)] D_F(x-y) [\overline{\psi}_R(y)\psi_L(x)], \tag{52}$$

(the Fierz rearrangement is given explicitly in Appendix C of ref. [14]).

S' of eq. (52) is the most attractive channel and leading in large N_c. Hence, we replace the pointlike 4-fermion interaction with the non-pointlike S' of eq. (52). Note that if we suppress the q^2 term in the denominator, of eq. (51) we have,

$$D_F(x-y) \to \frac{1}{M_0^2} \delta^4(x-y), \tag{53}$$

and we recover the pointlike NJL model interaction, corresponding to the large M_0^2 limit.

Now, substitute eq. (49) into eq. (52) to obtain,

$$\begin{aligned} S' \longrightarrow\; & g_0^2 \int d^4x\, d^4y\, [\overline{\psi}_L(x)\psi_R(y)]_f D_F(x-y) [\overline{\psi}_R(y)\psi_L(x)]_f \\ & + g_0^2 \sqrt{N_c} M^2 \int d^4x\, d^4y\, [\overline{\psi}_L(x)\psi_R(y)]_f D_F(x-y)\, \Phi^0(x,y) + h.c. \\ & + g_0^2 N_c M^4 \int d^4x\, d^4y\, \Phi^{0\dagger}(x,y)\, D_F(x-y)\, \Phi^0(x,y). \end{aligned} \tag{54}$$

The leading (first) term S' of eq. (54) is the unbound 4-fermion scattering interaction and has the structure of the NJL interaction in the limit of eq. (53) and identifies g_0 as the analogue of the NJL coupling constant. The second term, $\sim g_0^2\sqrt{N_c}[\psi^\dagger\psi] D_F \Phi^0 + h.c.$, has the form of the Yukawa interaction between the bound state Φ^0 and the free fermion scattering states. Note the appearance of the color factors, $\sqrt{N_c}$ and N_c, in the second and third lines respectively. The third term $\sim D_F|\,\mathrm{Tr}\,\Phi|^2 \propto N_c$, is the potential that makes the semiclassical bound state, with the N_c enhancement analogous to a BCS superconductor [22].

In this scheme, the mass scale, M, is ultimately inherited by the bound state Φ from the coloron interaction which introduces the scale M_0. The prefactor, M, in eq. (45), should be viewed part of the wave-function of Φ. It is *a priori* arbitrary, but we can swap it for a dimensionless parameter ϵ as:

$$M = \epsilon M_0. \tag{55}$$

The theory, like the NJL model, is viewed as having a maximum cut-off mass scale, M_0, hence $\epsilon \leq 1$ (since, for $q^2 \gg M_0^2$, the interaction turns off as $1/q^2$). In a variational calculation of the effective potential (in Section 4.1 below) we will see that ϵ is determined as a minimum. We find that $\epsilon = 1$ extremalizes the SKG effective potential in a bound state with super critical coupling $g_0^2 > g_c^2$, and generates a negative eigenvalue μ^2. Furthermore, we find that $\epsilon = 0$ is the extremal value for the subcritical case. Hence in the subcritical case, Φ, as a stable bound state, disappears and we are left with only unbound fermions, while the bound state with negative μ^2 will lead to spontaneous symmetry breaking. Note ϵ rescales the coupling constant $\sim g_0^2\epsilon$ and the maximal strength of the interaction for a given g_0^2 corresponds to $\epsilon \to 1$.

2.3. Fake chiral instability: the need for the dynamical internal wave function

Consider the pointlike limit of eq. (54) and the semiclassical fields in eq. (53), where we take a pointlike limit of the potential and of the wave-function, to replace $\Phi^0(x,y) \to \Phi^0(x)$, and obtain,[3]

$$S' \to \int d^4x \left(\frac{g_0^2}{M_0^2} [\overline{\psi}_L\psi_R]_f [\overline{\psi}_R\psi_L]_f + \widehat{M}^2 \Phi^{0\dagger}\Phi^0 + g_0^2 \epsilon \sqrt{N_c} ([\overline{\psi}_L\psi_R]_f \Phi^0 + h.c.) \right), \tag{56}$$

where $\widehat{M}^2 = g_0^2 N_c M^2$. Eq. (56) contains a wrong-sign ("tachyonic") mass term, implying a potential $\sim -\widehat{M}^2|\Phi|^2$. This appears to generate spontaneous symmetry breaking for any values of the underlying parameters M_0 and g_0 and the vacuum implodes. A chiral vacuum instability is apparently an immediate, large effect of introducing eq. (49)!

Such a conclusion is obviously physically incorrect. In naively replacing $\Phi(x,y)$ with $\Phi(x)$ we have neglected the kinetic term of the internal wave-function, $|\partial_r\Phi|^2$ where $r = (x-y)/2$. This opposes the instability like a repulsive interaction and will stabilize the vacuum in weak coupling. This is similar to the stabilization of the classical hydrogen atom by the Schrödinger wave-function. A

[3] $\Phi(x)$ is analogous to, but *should not be confused with the factorized NJL interaction.*, eq. (6), where M_0^2 is a right-sign non-tachyonic mass. Here Φ is not a pure auxiliary field, but rather is physical.

chiral instability can occur through competition of the repulsive internal wave-function kinetic term and the attractive potential, but will require a sufficiently large coupling, $g_0^2 > g_c^2$, to drive it, and a quartic coupling to stabilize the vacuum.

We therefore must consider the internal dynamics of the non-pointlike bound state. We note that this in the spirit of ref. [28], as these authors were essentially arguing for a bare kinetic term of the factorized NJL model of eq. (6). We are presently arguing for the necessity of all bare kinetic terms of the bilocal field $\Phi(x, y)$.

3. Relativistic bilocal fields

We begin by again noting the all–important kinematics of a two particle massless fermion state. For a pair of massless particles of 4-momenta p_1 and p_2, we have $p_1^2 = p_2^2 = 0$, and we can have two-body plane waves, $\Phi(x, y) \sim \exp(ip_1 x + ip_2 y)$. We pass to the total momentum $P = (p_1 + p_2)$ and relative momentum $Q = (p_1 - p_2)$, and the plane waves become $\exp(iPX + iQr)$ where we define "barycentric coordinates,"

$$X^\mu = \frac{x^\mu + y^\mu}{2}, \qquad r^\mu = \frac{x^\mu - y^\mu}{2}, \qquad \partial_x = \frac{1}{2}(\partial_X + \partial_r), \qquad \partial_y = \frac{1}{2}(\partial_X - \partial_r). \tag{57}$$

Note that $P_\mu Q^\mu = p_1^2 - p_2^2 = 0$. This implies that there is always a rest frame in which $P_\mu = (P_0, 0)$ and $Q_\mu = (0, \vec{q})$. Hence, in the rest frame the dependence upon $\vec{X}$ and, in particular due to the vanishing of Q^0 the relative time, r^0, drop out. If the particles are constituents of a bound state then this is the rest frame of the composite particle.

To proceed we require the generalized kinetic term of $\Phi(x, y)$ viewed as a bilocal field with an internal wave-function coordinate, r^μ. A free particle scattering state, $\Phi(x, y)$, composed of massless particles, will satisfy [21]:

$$\partial_x^2\Phi(x, y) + \partial_y^2\Phi(x, y) = 0 \qquad \text{or equivalently,} \qquad \frac{1}{2}\partial_X^2\Phi'(X, r) + \frac{1}{2}\partial_r^2\Phi'(X, r) = 0, \tag{58}$$

where $\Phi'(X, r) = \Phi(X - r, X + r)$.

The bilocal field $\Phi(x, y) = \overline{\psi}_R(x)\psi_L(y)$ represents a "bosonization" of the pair of chiral fermions, as in chiral Lagrangians. The equations of motion follow from the square of the free particle Dirac equations, $(\not\partial)_x^2\psi_R(x) = 0$ and $(\not\partial)_y^2\psi_L(y) = 0$. Note that we have chosen to describe the separation as r, which is the radius, where $2r = \rho \equiv (x - y)$ denotes the separation of the particles. The choice of r leads to more symmetrical expressions in r and X, and (somewhat) suppresses inconvenient factors of 2.

We can construct an action that yields the equations of motion by variation:

$$\begin{aligned} S_K &= M^4 \int d^4x d^4y \left(Z|\partial_x\Phi|^2 + Z|\partial_y\Phi|^2 + g_0^2 N_c D_F(x - y)|\Phi(x, y)|^2 \right) \\ &= \frac{1}{2} J M^4 \int d^4X d^4r \left(Z|\partial_X\Phi'|^2 + Z|\partial_r\Phi'|^2 + 2g_0^2 N_c D_F(2r)|\Phi'(2r)|^2 \right), \end{aligned} \tag{59}$$

where $J = |\partial(x, y)/\partial(X, r)| = 2^4$ is the Jacobian in passing from (x^μ, y^ν) to the barycentric coordinates (X^μ, r^ν); the factor $\frac{1}{2}\times$ comes from the derivatives, $(\partial_x^2 + \partial_y^2) \to \frac{1}{2}(\partial_X^2 + \partial_r^2)$. The normalization factor Z is necessary in using bilocal fields to remove relative time, to yield properly normalized Noether currents, and maintain canonical kinetic terms (as in the hydrogen atom example of Section 1.2 above).

Following Yukawa [21], consider the factorized ansatz of $\Phi'(X, r)$:

$$\sqrt{J/2}\,\Phi'(X, r) = \chi(X)\phi(r). \tag{60}$$

The action with the factorized field becomes,

$$S = M^4 \int d^4X d^4r \left(Z|\phi(r)|^2|\partial_X\chi(X)|^2 + Z|\chi(X)|^2|\partial_r\phi(r)|^2 + 2g_0^2 N_c D_F(2r)|\chi(X)\phi(r)|^2 \right). \tag{61}$$

A canonical normalization of the $\chi(X)$ kinetic term (which is the requirement of a normalized Noether current, such as $i\chi^\dagger \frac{\overleftrightarrow{\partial}}{\partial X^\mu}\chi$ [14]) dictates a normalization constraint on $\phi(r)$. Define the Lorentz invariant normalization:

$$1 = ZM^4 \int d^4r\, |\phi(r)|^2. \tag{62}$$

Following the elementary two-body kinematics, where the relative time disappears in the rest frame, then $\phi(r) \to \phi(\vec{r})$ becomes a static field that has no dependence upon r^0. We can then define Z,[4].

$$1 = ZM \int dr^0 \equiv ZMT = \epsilon Z M_0 T, \tag{63}$$

which, in turn, dictates a normalization for $\phi(\vec{r})$:

[4] Note: A more general discussion, of Lorentz invariant constraints, can be found in Section 2.5 of ref. ([14]). See also Appendix B

$$1 = M^3 \int d^3r\, |\phi(\vec{r})|^2. \tag{64}$$

The condition $1 = \epsilon Z M_0 T$ removes the relative time, $T = \int dr^0$, from the kinetic terms. Note that $\phi(\vec{r})$ is then dimensionless with eq. (64). The action becomes,

$$S = \int d^4X \left(|\partial_X \chi(X)|^2 + |\chi(X)|^2 M^3 \int d^3r \left(-|\partial_{\vec{r}}\phi(\vec{r})|^2 + \int dr^0\, 2g_0^2 N_c M D_F(2r^\mu)|\phi(\vec{r})\,|^2 \right)\right). \tag{65}$$

(Note $|\partial_r\phi|^2 = |\partial_{r^0}\phi|^2 - |\partial_{\vec{r}}\phi|^2$). We finally integrate over r^0 in the interaction term:

$$\int dr^0 D_F(2r) = -\int dr^0 \frac{d^4q}{(2\pi)^4}\frac{1}{q^2 - M_0^2} e^{2iq_\mu r^\mu} = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\vec{q}^{\,2} + M_0^2} e^{2iq_\mu r^\mu} = -\frac{1}{2}V_0(2|\vec{r}|). \tag{66}$$

The $\vec{q}$ momentum integral yields the familiar Yukawa potential (where $2r$ is the separation of the particles),

$$V_0(2r) = -\frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}. \tag{67}$$

The action then becomes,

$$S = \int d^4X \left(|\partial_X \chi(X)|^2 + |\chi(X)|^2 M^3 \int d^3r \left(-|\partial_{\vec{r}}\phi(\vec{r})|^2 + g_0^2 N_c M \frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}|\phi(\vec{r})|^2 \right)\right). \tag{68}$$

Note that, in the limit of suppressing the $\vec{q}^{\,2}$ in the denominators of the integrands of eq. (66), we obtain the large M_0^2 limit of the potential (using $J = 2^4$, and $\delta^3(\vec{r}) = (4\pi r^2)^{-1}\delta(r)$):

$$V_0(2r) \to -\frac{1}{M_0^2}\delta^3(2\vec{r}) = -\frac{2}{JM_0^2}\delta^3(\vec{r}) = -\frac{1}{2\pi J M_0^2 r^2}\delta(r). \tag{69}$$

We emphasize that the theory remains Lorentz invariant, albeit not manifestly so (see Appendix B). Note that it is convenient to write things as, in the format:

$$S = \int d^4X \left(|\partial_X \chi|^2 - |\chi|^2 \mathcal{M}^2 \right)$$
$$\mu^2 = \mathcal{M}^2 \equiv M^3 \int d^3r \left(|\partial_{\vec{r}}\phi|^2 - g_0^2 N_c M \frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}|\phi|^2 \right), \tag{70}$$

where μ^2 is the eigenvalue of the "Hamiltonian," $\mathcal{M}^2$. We see that the action of ϕ is "nested" within the action of χ, where χ controls the motion of the collective state and ϕ describes the internal relative motion of the constituents. $\mathcal{M}^2$ is a Hamiltonian for the static internal wave-function $\phi(\vec{r})$.

Extremalizing $\mathcal{M}^2$ with respect to $\phi(r)$ implies the Schrödinger–Klein–Gordon (SKG) equation for an s-wave ground state and its eigenvalue, μ^2:

$$-\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\phi(r) - g_0^2 N_c M \frac{e^{-2M_0 r}}{8\pi r}\phi(r) = \mu^2\phi(r) \tag{71}$$

We then see that $\mathcal{M}^2 = \mu^2$ is then the physical mass of the bound state. The χ action in any frame is manifestly Lorentz invariant (where we include a quartic term, which will be developed below):

$$S = \int d^4X \left(|\partial_X \chi(X)|^2 - \mu^2|\chi(X)|^2 - \frac{\lambda}{2}|\chi(X)|^4 \right). \tag{72}$$

The Yukawa potential has a critical coupling, $g_0 = g_c$, where the eigenvalue is then $\mu = 0$. For $g_0 > g_c$ then $\mu^2 < 0$, and we have spontaneous symmetry breaking.

3.1. A simple example:

As an aside we note that we can represent a two body open scattering state as a bilocal wave-function with $\phi(\vec{r}) = N\exp(2i\vec{Q}\cdot\vec{r})$. $\phi(r)$ is then "non-normalizable" (requiring box normalization) and is then $M^3N^2\int d^3r|\phi(\vec{r})|^2 = 1$. In the center of mass rest frame action becomes,

$$S_K = V_3 \int dX^0 \left(|\partial_0\chi(X^0)|^2 - 4\vec{Q}^{\,2}|\chi(X^0)|^2 \right) \tag{73}$$

This is a state described by $\chi(X)$ which satisfies the equation of motion,

$$\partial_0^2\chi + \mu^2\chi = 0 \qquad \mu^2 = 4\vec{Q}^{\,2}. \tag{74}$$

This is a zero 3-momentum two body scattering state of invariant mass $2|\vec{Q}| = \mu$, with conventional volume normalization $\sim V^{-3/2}$. Technically, the experimentalists' "invariant mass" of a two body state is not a mass at all; a mass appears in the trace of the stress tensor and massless particles have a vanishing trace. In theories like QCD the mass scale is set by the "trace anomaly," and presumably the scale, M_0, would emerge in similar fashion in the coloron theory.

3.2. The induced bound state Yukawa interaction

The Yukawa interaction of the bound state with the free scattering state fermions is now induced from the second term, S'_Y, in eq. (54). We have, noting eqs. (54), (60):

$$\begin{aligned} S'_Y &= g_0^2\sqrt{N_c}M^2\int d^4x d^4y\,[\overline{\psi}_L(x)\psi_R(y)]_f\,D_F(x-y)\,\Phi^0(x,y)+h.c. \\ &= \sqrt{2N_cJ}\,g_0^2\epsilon^2M_0^2\int d^4X d^4r\,[\overline{\psi}_L(X+r)\psi_R(X-r)]_f\,D_F(2r)\,\chi(X)\phi(\vec{r})+h.c.. \end{aligned} \tag{75}$$

Consider the pointlike limit of the potential, eq. (53), $D_F(2r)\to(JM_0^2)^{-1}\delta^4(r)$:

$$S'_Y \to \sqrt{2N_c/J}\,g_0^2\epsilon^2\int d^4X\,[\overline{\psi}_L(X)\psi_R(X)]_f\,\chi(X)\phi(0)+h.c.. \tag{76}$$

We therefore see that the induced Yukawa coupling to the field $\chi(x)$ in the pointlike limit *of the potential* (which is a reasonable low energy approximation) is:

$$g_Y = \hat{g}_Y\phi(0) \qquad \text{where,} \qquad \hat{g}_Y \equiv g_0^2\epsilon^2\sqrt{2N_c/J} = g_0^2\epsilon^2\sqrt{3/8}. \tag{77}$$

We emphasize that this is a significant result and fundamentally different than the NJL model result. We have taken the pointlike limit of the potential as in the NJL model, but obtain a result that is dependent crucially upon the non-pointlike internal wave-function $\propto \phi(0)$. The implication is that a strong coupling, g_0^2, can produce, in principle, a small Yukawa coupling if $\phi(0)\ll 1$. In the usual pointlike NJL model the induced Yukawa coupling runs to smaller values in the IR, but it does so only logarithmically, via the RG. Here the behavior of $\phi(0)$ is a suppression of g_Y that will be seen, in the next section, to be power-law $\sim \sqrt{|\mu|/M_0}$ for small μ, near the critical coupling.

4. The Schrödinger-Klein-Gordon (SKG) equation

While formally similar to the non-relativisitic Schrödinger equation, the SKG equation, eq. (71), has key physical differences:

- the potential has dimension (mass)^2, rather than energy;
- the eigenvalue describes resonances for positive μ^2;
- a negative eigenvalue, $-|\mu|^2$, implies vacuum instability and spontaneous symmetry breaking.

Mainly the Hamiltonian, $\mathcal{M}^2$ of eq. (70), which generates the SKG equation, is amenable to variational calculations as we show below. We presently give some examples of solutions and stress some subtleties, though much can be done to refine and extend this discussion. The solutions allow the computation of the induced top quark Yukawa coupling of the bound state to free fermions, g_Y, via the wave-function at the origin, $\phi(0)$, from which one can extract M_0, the mass scale of the potential (i.e. the coloron mass).

A negative eigenvalue of the Schrödinger equation defines our conventional view of a non-relativistic bound state. However, in the relativistic case, for a pair of chiral fermions, the SKG equation with a bound state solution implies a negative μ^2. This is, of course, the behavior of Σ-model in QCD and the BEH boson in the standard model and requires additional physics to stabilize the vacuum, such as quartic interactions. Hence, the general result is that a scalar bound state of massless chiral fermions in the symmetric (unbroken) phase must either be an unstable resonance (subcritical coupling and positive μ^2), which decays rapidly to its constituents, or tachyonic (supercritical coupling, negative μ^2) leading to a chiral instability of the vacuum.

4.1. Variational calculation determining ϵ

We defined $M=\epsilon M_0$, as the mass scale in the ansatz, introducing the parameter ϵ. The largest mass scale at which the static potential approximation is applicable is $M=M_0$, hence $\epsilon\le 1$. ϵ is seen to multiply the underlying coupling constant, $\tilde{g}_0^2=\epsilon g_0^2$. The largest value of $\tilde{g}_0^2$, is therefore g_0^2, hence $\epsilon=1$ implies the smallest possible critical value of the underlying coupling g_0^2. We view ϵ as part of the wave-function ansatz, and allow the variational calculation of the bound state mass to determine ϵ by minimization of the Hamiltonian. By "Hamiltonian" we mean $\mathcal{M}^2$ of eq. (70).

A solution to the SKG equation for the eigenvalue can be approximated by a variational calculation. For the present estimate we assume an ansatz consisting of a Hydrogenic wave-function, $\widetilde{\phi}(r)=Ae^{-Mr}$, with $M=\epsilon M_0$ and ϵ as the variational parameter, and M_0 is the scale of the Yukawa potential. Of course, this will not be a precise description near criticality, where the eigenvalue μ^2 is small, because $\widetilde{\phi}(r)$ lacks the large distance tail $\propto e^{-|\mu|r}/r$, however, it conveniently illustrates how $\epsilon\to 1$ is established dynamically for the bound state.

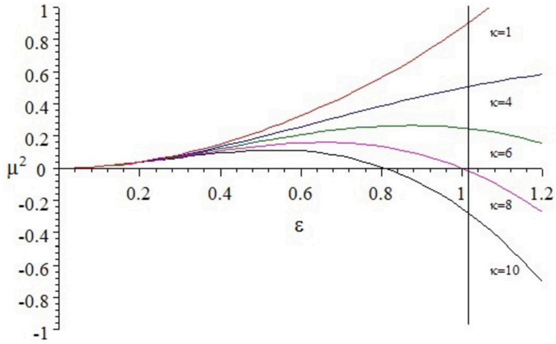

Fig. 3. $\mathcal{M}^2 = \mu^2$ of eq. (79) is plotted vs. ϵ, with $M_0 = 1$ for values of $\kappa = g_0^2 N_c/4\pi = (1, 4, 6, 8, 10)$. The critical coupling is the value of κ for which a massless bound state first occurs, i.e., $\mathcal{M}^2 = \mu^2 = 0$. This happens for the $\kappa = 8$ curve, which intersects the $\epsilon = 1$ vertical line and has $\mu^2 = 0$. Hence $\kappa_c = 8$ is the critical coupling with this ansatz.

The normalization condition for the ansatz is defined in eq. (64),

$$1 = 4\pi A^2 \epsilon^3 M_0^3 \int_0^\infty e^{-2\epsilon M_0 r} r^2 dr; \qquad \text{hence,} \qquad A^2 = \frac{1}{\pi}. \tag{78}$$

The Hamiltonian $\mathcal{M}^2$ and eigenvalue μ^2 of eq. (70) with $M = \epsilon M_0$ is therefore,

$$\mathcal{M}^2 = \epsilon^3 M_0^3 \int d^3r \left(|\partial_{\vec{r}}\phi|^2 + g_0^2 N_c \epsilon M_0 V_0(2r) |\phi|^2 \right) \equiv \mu^2, \tag{79}$$

where, $V_0(2r) = -e^{-2M_0 r}/8\pi r$, as in eq. (67), with the fixed coloron mass M_0 (no ϵ factor is present in $V_0(2r)$). In what follows we will use a definition of the coupling,

$$\kappa = \frac{g_0^2 N_c}{4\pi} \qquad \kappa_{c_{NJL}} = 2\pi, \tag{80}$$

(where we quote the NJL critical value of $\kappa_{c_{NJL}}$). We then compute the eigenvalue $\mu^2 = \mathcal{M}^2$ as a function of ϵ and κ:

$$\begin{aligned}\mathcal{M}^2 &= A^2 \epsilon^3 M_0^3 \int d^3r \left(|\partial_r e^{-\epsilon M_0 r}|^2 - \frac{\kappa \epsilon M_0}{2} \frac{e^{-2M_0 r}}{r} |e^{-\epsilon M_0 r}|^2 \right) \\ &= M_0^2 \left(\epsilon^2 - \frac{\kappa \epsilon^4}{2(1+\epsilon)^2} \right).\end{aligned} \tag{81}$$

In Fig. 3 we plot a family of curves of $\mathcal{M}^2 = \mu^2$ for various values of κ as function of ϵ. We see that the extremal (smallest) value for positive μ^2 corresponds to $\mu^2 = 0$ and occurs for any $\kappa < 8$. On the other hand, for $\kappa > 8$ (purple curve) the extremal, most negative value of $\mu^2 = \mathcal{M}^2$, occurs when $\epsilon \to 1$. Hence, the critical coupling for this ansatz is $\kappa_c = 8$ and we then have,

$$\frac{\kappa_c}{2\pi} = \frac{g_c^2 N_c}{8\pi^2} = \frac{4}{\pi} = 1.27, \tag{82}$$

compared to the NJL critical value 1.00 (reflecting the crudeness of the ansatz lacking the large distance tail).

However, we see an interesting result: We find that $\epsilon = 1$ is the true minimum for any $g_0^2 > g_c^2$ where $g_c^2 = 4\pi\kappa_c/N_c$ is the critical coloron coupling. In this case $\mu^2 < 0$ and we will have spontaneous symmetry breaking. For $g_0^2 < g_c^2$, then $\mu^2 > 0$, and there is a true minimum at $\epsilon = 0$. However, for subcritical couplings $g_0^2 < g_c^2$, in the range $8 \gtrsim \kappa \gtrsim 6$, we see from the Fig. 3 that there are a quasi-stable minima at $\epsilon = 1$, with $\mu^2 > 0$. Here we expect to have resonances which decay to free fermion pairs. We believe the quasi-stable minimum reflects the presence of such resonances, where the wave-function with $\epsilon \sim 1$ would tunnel through the $\epsilon < 1$ barrier to reach the true minimum $\epsilon = 0$. As $g_0^2 \ll g_c^2$, roughly $\kappa \lesssim 6$, the quasi-stable minimum disappears, and the true minimum is at $\epsilon = 0$, and no bound state resonances forms.

The variational result for g_c^2 with this ansatz gives a false value for the normalized trial wave-function at the origin for critical coupling, $\phi(0) = 1/\sqrt{\pi}$. The reason is, of course, that the ansatz does not include a $e^{-|\mu|r}/r$ tail at large r, which significantly affects the normalized $\phi(0)$. Below we do refined calculations that demonstrate the effects of the large distance tail of $\phi(r)$.

4.2. Exact criticality of the Yukawa potential

The coloron model furnishes a direct UV completion of the NJL model. It leads to an SKG potential of the Yukawa form which has a critical coupling, $g_0^2 = g_c^2$. The critical coupling is that value of g_0^2 for which the eigenvalue μ^2 is zero. We wish to determine g_c^2 exactly.

The criticality of the Yukawa potential in the non-relativistic Schrödinger equation is widely discussed in the literature in the context of "screening" (see [30] and references therein). The non-relativistic Schrödinger equation $r = |\vec{r}|$ is:

$$-\nabla^2\psi - 2m_e\alpha\frac{e^{-\mu r}}{r}\psi = 2m_e E, \tag{83}$$

with m_e the electron mass, and eigenvalue $E = 0$ occurs for a critical screening with $\mu = \mu_c$. A numerical analysis yields, [30],

$$\mu_c = 1.19061\ \alpha m_e. \tag{84}$$

For the spherical SKG equation in the coloron model eq. (71) we have from the Hamiltonian,

$$-\nabla^2\phi(r) - g_0^2 N_c M_0\frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}\phi(r) = \mu^2\phi(r), \tag{85}$$

where we assume $\epsilon = 1$ as determined by the variational calculation above.

We can obtain the critical coloron model coupling constant by comparing, eq. (83) and eq. (85). We have,

$$2m_e\alpha \to g_0^2 N_c M_0/8\pi, \qquad\qquad \mu_c \to 2M_0, \tag{86}$$

then substituting into eq. (84), $2M_0 = 1.19061(g_0^2 N_c M_0/16\pi)$, and therefore,

$$\left.\frac{g_0^2 N_c}{8\pi^2}\right|_c = \frac{4}{(1.19061)\pi} = 1.06940. \tag{87}$$

By comparison, the loop level NJL critical value of eq. (9) is,

$$\left.\frac{g_c^2 N_c}{8\pi^2}\right|_{NJLc} = 1.00. \tag{88}$$

Hence, we see that the NJL quantum critical coupling has a remarkably similar numerical value to the classical critical coupling. (It is beyond the scope of the present paper to understand why these are not identically equal!)

4.3. *Spherical potential well*

To compute the induced Yukawa coupling, g_Y, we then need to calculate $\phi(0)$. For a convenient solvable potential problem and warm-up exercise, we turn to the spherical potential well,

$$\frac{e^{-2M_0 r}}{8\pi r} \longrightarrow \frac{\lambda}{8\pi}M_0\theta(1 - M_0 r). \tag{89}$$

We match the integral over the spherical potential well to the integral over the Yukawa potential, which determines $\lambda = 3/4$. Once matched, the spherical well can be used as an approximation to the Yukawa potential. Moreover, the formula $g_Y = g_0^2\sqrt{2N/J}\ \langle\phi(0)\rangle$ of eq. (77) is modified by replacing $\phi(0)$ by its volume average $\langle\phi(0)\rangle$ over the well, $r < M_0^{-1}$. The SKG equation for the potential well is:

$$-\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\phi(r) - \frac{3g_0^2 N M_0^2}{32\pi}\theta(1 - M_0 r)\phi(r) = \mu^2\phi(r), \tag{90}$$

with solution:

$$\phi(r) = \phi_1(r)\theta(1 - M_0 r) + \phi_2(r)\theta(M_0 r - 1) \qquad \text{where,}$$
$$\phi_1(r) = \frac{A\sin(kr)}{r} \qquad \phi_2(r) = \frac{Be^{-|\mu|(r - M_0^{-1})}}{r}. \tag{91}$$

Continuity at $M_0 r = 1$ then implies,

$$A\sin(kM_0^{-1}) = B; \qquad Ak\cos(kM_0^{-1}) = -B\,|\mu|\,. \tag{92}$$

Critical coupling implies $|\mu| = 0$, hence, $\cos(kM_0^{-1}) = 0$, which yields $k = \frac{\pi}{2}M_0$ and $A = -B$. Therefore, we obtain the critical coupling for the spherical well,

$$\left(\frac{\pi}{2}M_0\right)^2 - \frac{3g_c^2 N_c}{32\pi}M_0^2 = 0, \qquad \text{hence,} \qquad \frac{g_c^2 N_c}{8\pi^2} = \left(\frac{\pi}{3}\right) = 1.0472. \tag{93}$$

This is close to the NJL result (= 1.0), or to Yukawa (= 1.06940). Note that at critical coupling we expect resonances at $\mu_r \approx \frac{\pi}{2}\sqrt{N^2 + 2N}M_0$, for $N = 1, 2, \ldots$

The normalization of the ansatz of eq. (64) determines the coefficient, A, and is dominated by the tail of $\phi(r)$ for small μ:

$$1 = 4\pi A^2 M_0^3 \int r^2 dr\, |\phi(r)|^2 \approx 4\pi A^2 M_0^3 \int_0^\infty \frac{e^{-2|\mu| r}}{r^2} r^2 dr = \frac{2\pi A^2}{|\mu|}(M_0)^3$$

$$A \approx \frac{1}{M_0}\left(\frac{\mu}{2\pi M_0}\right)^{1/2} = \frac{0.398\,94}{M_0}\left(\frac{\mu}{M_0}\right)^{1/2}. \tag{94}$$

The wave-function at the origin is technically given by,

$$\phi(0) = \left(\frac{A\sin(kr)}{r}\right)_{r\to 0} = Ak = \frac{\pi}{2} A M_0 = \left(\frac{\pi\mu}{8M_0}\right)^{1/2}. \tag{95}$$

We see that $\phi(0)$ is therefore suppressed as $\sim \left(|\mu|/M_0\right)^{1/2}$. However, low momentum fermions in the Yukawa interaction would experience the volume average of the wave-function in the well, (recall above $k = \pi M_0/2$):

$$\langle\phi(0)\rangle = 4\pi \mathcal{N}^{-1} \int_0^{M_0^{-1}} \left(\frac{A\sin(kr)}{r}\right) r^2 dr \quad \text{where,} \quad \mathcal{N} = \frac{4}{3}\pi M_0^{-3}$$

$$\text{hence,} \quad \langle\phi(0)\rangle = \frac{6\sqrt{2}}{\pi^{5/2}}\sqrt{|\mu|/M_0}. \tag{96}$$

We can then compute M_0 from an input $|\mu|$ and the value of the Yukawa coupling using eq. (77):

$$1 \approx g_Y = g_c^2\sqrt{2N_c/J}\langle\phi(0)\rangle = \left(\frac{\pi}{3}\right)\frac{8\pi^2}{N_c}\sqrt{3/8}\langle\phi(0)\rangle \approx 8.1867\sqrt{|\mu|/M_0}. \tag{97}$$

Applying this to top quark condensation we input the (symmetric phase) BEH boson Lagrangian mass, $|\mu| = 88$ GeV, to obtain:

$$M_0 \approx 5.9 \text{ TeV}. \tag{98}$$

The "skeletal solution" of Section 4.5 below yields a similar $M_0 \sim 6$ TeV result. Note that if we had used $\phi(0)$, in the potential well, which is much larger than the average $\langle\phi(0)\rangle$ as given by eq. (95), we would obtain $M_0 \approx 10$ TeV.

4.4. Significantly reduced fine tuning due to dilution

The spherical potential well also illustrates a key feature of the fine–tuning and the nature of the phase transition. If we consider the Hamiltonian we have:

$$\mu^2 = 4\pi M_0^3\left(\int_0^{M_0^{-1}} r^2 dr\left((\partial_r\phi_1)^2 - \frac{3g_0^2 N M_0^2}{32\pi}\phi_1^2\right) + \int_{M_0^{-1}}^\infty r^2 dr (\partial_r\phi_2)^2\right)$$

$$= 4\pi M_0^3 A^2\left(\int_0^{M_0^{-1}} r^2 dr\left(\frac{\pi^2}{4}M_0^2 - \frac{3N_c g_c^2}{32\pi}M_0^2\right)\frac{\sin^2(kr)}{r^2} + \int_{M_0^{-1}}^\infty r^2 dr \mu^2 \frac{e^{-|2\mu|(r-M_0^{-1})}}{r^2}\right)$$

$$= |\mu|\left(M_0\left(\frac{\pi^2}{4} - \frac{3N_c g_c^2}{32\pi}\right) + |\mu|\right). \tag{99}$$

Here we see that the critical behavior is significantly modified with respect to the NJL model. In the NJL model the critical behavior in g_0^2 near its critical value g_c^2 is analogous to that of a second order phase transition,

$$\mu^2 \sim M_0^2\left(1 - \frac{g_0^2}{g_c^2}\right), \tag{100}$$

which implies significant fine–tuning to obtain a large hierarchy,

$$\frac{\delta g_0^2}{g_c^2} \sim \frac{\mu^2}{M_0^2}. \tag{101}$$

However, in the present framework we see from eq. (99) that, *for a negative* μ^2, the large distance tail and its dilution effect (where $\phi(0) \propto \sqrt{|\mu|/M_0}$) modify this relationship as:

$$\mu^2 \sim \phi(0)^2\left(M_0^2 - \frac{g_0^2}{g_c^2}M_0^2\right) \sim \frac{|\mu|}{M_0}\left(M_0^2 - \frac{g_0^2}{g_c^2}M_0^2\right)$$
$$\rightarrow \frac{\mu^2}{|\mu|} \sim M_0\left(1 - \frac{g_0^2}{g_c^2}\right). \tag{102}$$

We thus obtain a linear relationship between μ and M_0 near criticality,

$$\frac{\delta g_0^2}{g_c^2} \sim \frac{|\mu|}{M_0}. \tag{103}$$

Due to the dilution effect we can tolerate significant departures from criticality $g_c^2 + \mathcal{O}\left(\mu/M_0\right)$ and we will still have an expectation value of Hamiltonian of $\sim \mathcal{O}(\mu^2)$. The specification of the critical coupling is therefore made much less stringent by the large distance tail and dilution effect of the wave-function.

This result can be checked by perturbing the potential well solution around the critical coupling value. We have for $g_0^2 = g_c^2 + \delta g^2$ and $k \rightarrow k + \delta k$,

$$A\sin((k+\delta k)M_0^{-1}) = B; \qquad A(k+\delta k)\cos((k+\delta k)M_0^{-1}) = -B\,|\mu|\,. \tag{104}$$

Expanding in δk and using $kM_0^{-1} = \pi/2$,

$$B = A; \qquad B = A\frac{k\delta k}{M_0|\mu|} \qquad \text{hence,} \qquad \delta k = \frac{2|\mu|}{\pi}. \tag{105}$$

The potential well yields:

$$\left(\frac{\pi}{2}M_0 + \delta k\right)^2 - \frac{3N_c(g_c^2+\delta g^2)}{32\pi}M_0^2 = \mu^2 \qquad \text{hence,} \qquad \frac{\pi\delta k}{M_0} - \frac{3N_c(\delta g^2)}{32\pi} = \mathcal{O}(\mu^2/M_0^2) \approx 0, \tag{106}$$

so we obtain (using eq. (93)):

$$\frac{\delta g^2}{g_c^2} = \frac{4\delta k}{\pi M_0} = \frac{8}{\pi^2}\frac{|\mu|}{M_0} + \mathcal{O}(\mu^2/M_0^2), \tag{107}$$

confirming the linear relation between $\delta g^2/g_c^2$ and $|\mu|$.

We have illustrated the bound state with the spherical well potential since it is simple and can be readily verified. Note that, if we had used the NJL criterion for fine–tuning, we would have $\delta g_0^2/g_c^2 \sim \mathcal{O}(10^{-4})$. The linear relationship yields tuning at the reduced few % level, due to dilution from the tail of the internal wave-function.

4.5. Skeletal solution of the SKG equation

We can give a very simple illustrative result that faithfully reproduces the mass scale M_0 as determined from low energy inputs. Here we assume the short distance potential is given by the δ−function limit of the Yukawa potential, and $\phi(0)$.

We can approximate the Yukawa potential by a δ-function at short distance using eq. (69) (recall $J = 16$ is the Jacobian passing from (x, y) to (X, r)):

$$\frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|} \rightarrow \frac{1}{M_0^2}\delta^3(2\vec{r}) = \frac{2}{JM_0^2}\delta^3(\vec{r}). \tag{108}$$

Consider the form of the potential that appears in the coloron model,

$$-\nabla^2\phi - g_0^2 N_c M_0\frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}\phi(r) \longrightarrow -\nabla^2\phi - \frac{2g_0^2N_c}{JM_0}\delta^3(\vec{r})\phi(0) = \mu^2\phi \tag{109}$$

Using $\nabla^2(1/r) = -4\pi\delta^3(\vec{r})$, the large distance solution is then,

$$\phi(r) = \frac{ce^{-|\mu|r}}{r}; \quad \text{where,} \quad c = \frac{g_0^2N_c}{2\pi JM_0}\phi(0). \tag{110}$$

At critical coupling $g_0^2 = g_c^2$, and then $\mu^2 = \eta^2 \approx 0$, where η is an infinitesimal infrared cut-off mass scale that is necessary to give $\phi(r)$ a finite normalization:

$$1 = M_0^3 4\pi\int_0^\infty r^2 dr\,\frac{c^2}{r^2}e^{-2|\eta|r} = 4\pi\frac{M_0^3}{2|\eta|}\left(\frac{g_c^2N_c}{2\pi JM_0}\phi(0)\right)^2, \tag{111}$$

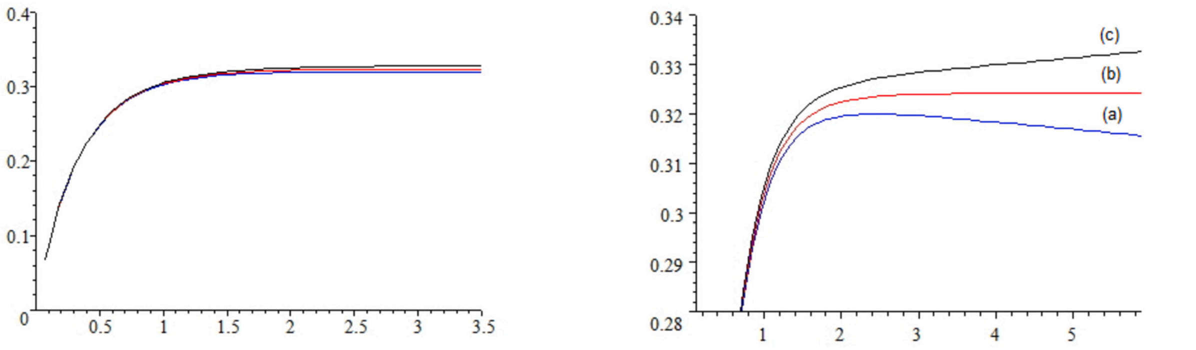

Fig. 4. Numerical solutions to eq. (115) for $g_0^2 N_c/8\pi = c$, where: (a) (blue) $c = 3.37$ (b) (red) $c = 3.36$; (c) (black) $c = 3.35$. We see that the curve (b) flattens, corresponding to $|\mu| = 0$ and $\phi(r) \sim u(\infty)/r$ critical coupling, with $u(\infty) = 0.3240$, (see ref. [30]).

hence, with $g_c^2 N/8\pi^2 = 1$,

$$\phi(0) = \frac{2\sqrt{2}}{\pi^{3/2}}\sqrt{\frac{|\eta|}{M_0}} \sim 0, \tag{112}$$

and we see that $\phi(0)$ is diluted to zero in the critical coupling case as $\sim \sqrt{\eta} \to 0$.

If the coupling is chosen to be supercritical, $g_0^2 = g_c^2 + \delta g_0^2$, then the mass μ^2 is physical and the normalization becomes,

$$\phi(0) = \frac{2\sqrt{2}}{\pi^{3/2}}\sqrt{\frac{|\mu|}{M_0}} = 0.50795\sqrt{\frac{|\mu|}{M_0}}. \tag{113}$$

If we assume we are close to critical coupling, $g_0^2 N_c \approx 8\pi^2$, we can approximate the result for the Yukawa coupling and use as input $|\mu| = 88$ GeV:

$$1 \approx g_Y = \frac{8\pi^2}{3}\sqrt{\frac{2N_c}{J}}\,\phi(0) = 16.12\,\phi(0) \approx 8.187\sqrt{\frac{|\mu|}{M_0}}; \quad \text{therefore,} \quad M_0 = 5.9\ \text{TeV}. \tag{114}$$

The skeletal model tells us nothing about what determines the critical coupling, g_c^2. This is determined by the short distance solution inside the potential and its matching onto the large distance solution as we saw in the previous spherical well example. However, it is sufficient to assume the critical behavior which leads to the approximate scale invariance (small $|\mu|$) of the large distance solution to obtain the relation between g_Y and M_0.

4.6. Numerical integration the SKG equation

Central to this theory is determining the mass M_0 given the Yukawa coupling of the top quark, $g_Y \equiv g_{top} \approx 1$ which, via the wave-function $\phi(r)$, determines $|\mu|/M_0$, where $|\mu|$ is the symmetric phase BEH mass, $|\mu| \sim 88$ GeV. In preceding estimates we considered the spherical well solution, or we took the short-distance ($\delta-$function) limit of the potential, while allowing the wave-function spread as $\phi(r) \sim e^{-|\mu|}/r$. We then related g_Y to $\phi(0) \propto \sqrt{|\mu|/M_0}$. However, we can obtain a complete numerical solution of the extended wave-function to the SKG equation with the Yukawa potential, we can then relate g_Y to $\phi(r)$ integrated over the potential, without taking the $\delta-$function limit.

The SKG equation (71) with $M_0 = 1$ takes the form in $u(r)$ where $u(r) = r\phi(r)$:

$$u''(r) + \frac{g^2 N_c}{8\pi r} e^{-2r} u(r) = -\mu^2 u(r) \tag{115}$$

It is easy to obtain reasonable approximate numerical solutions, following ref. [30]. We use numerical inputs $g_c^2 N_c/8\pi = \pi f = \pi \times 1.06940 \approx 3.36$, where $g_c^2 N_c/8\pi^2 = 1.06940$ is the exact critical coupling of eq. (87). The critical coupling corresponds to the solution $|\mu| = 0$, for which $u(r) \to$(constant), as $r \to \infty$. We find numerically in the critical case that $u(r) \to 0.3240$ as $r \to \infty$. The numerical solutions are shown in Fig. 4. The resulting numerical solutions can be approximately fit by the function $u_{fit}(r)$,

$$u_{fit}(r) = 0.3240\left(\left(1 - \left(1 - \frac{r}{R_0}\right)^p\right)\theta(R_0 - r) + e^{-|\mu|(r-R_0)}\,\theta(r - R_0)\right) \tag{116}$$

where $R_0 = 1.8(M_0)^{-1}$ and $\mu = 0$ in the exact critical case. The best "visual fit" is obtained for $p = 4$ as shown in Fig. 5, curve (b).

For low momentum fermions interacting in the potential with the bound state we have $[\overline{\psi}_L(X+r)\psi_R(X-r)]_f \approx [\overline{\psi}_L(X)\psi_R(X]_f$. The Yukawa coupling, using $g_0^2 N/8\pi^2 = f$, and setting $M_0 = 1$, is then given by eq. (75):

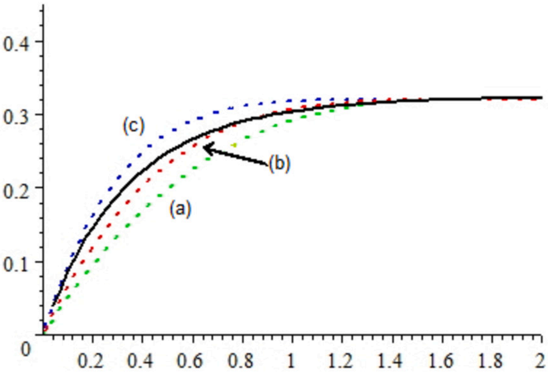

Fig. 5. Convenient fit to the numerical solution by the function of eq. (116), where (a) $p = 3$; (b) $p = 4$; (c) $p = 6$. We use the best fit, with $p = 4$ and with $u(\infty) = 0.3240$.

$$g_Y = \sqrt{2N_c J} g_0^2 \int d^4 r D_F(2r)\phi(r) = \sqrt{\frac{N_c J}{2}} \frac{8\pi^2 f}{N_c} \int_0^\infty 4\pi \frac{e^{-2r}}{8\pi r}\phi(r) r^2 dr = \sqrt{\frac{N_c J}{2}} \frac{\pi f}{N_c} \int_0^\infty 4\pi \left(\frac{e^{-2r}}{r}\phi(r)\right) r^2 dr. \tag{117}$$

We evaluate the integrals of $\phi(r) = Au(r)/r$ over the potential, (with normalization factor A),

$$(4\pi A)\left(\int_0^{R_0} \left(\frac{0.324\left(1-(1-r/R_0)^4\right)}{r}\right)\frac{e^{-2r}}{r} r^2 dr + \int_{R_0}^\infty \left(\frac{0.324}{r} e^{-|\mu|(r-R_0)}\right)\frac{e^{-2r}}{r} r^2 dr\right) = 1.1412 \times A. \tag{118}$$

The second term is insensitive to μ for $|\mu| \ll M_0$. The normalization, as usual, is dominated by the large distance tail:

$$1 \approx A^2 \left(4\pi \int_{R_0}^\infty \left(\frac{0.324}{r} e^{-|\mu|(r-R_0)}\right)^2 r^2 dr\right) = 0.65948 \frac{A^2}{|\mu|^2} \quad \text{hence,} \quad A = 1.2313\sqrt{\frac{|\mu|}{M_0}}, \tag{119}$$

(here we restore M_0; note that previously we defined $R_0 = 1.8(M_0)^{-1}$ and we have set $M_0 = 1$, hence M_0 reemerges in the above ratio $|\mu|/M_0$).

For the Yukawa coupling, g_Y in eq. (117), we have the prefactor, (using $N_c = 3$, $J = 16$, $f = 1.0694$):

$$\sqrt{\frac{N_c J}{2}} \frac{\pi f}{N_c} = 5.4862. \tag{120}$$

We thus obtain the value of M_0 upon combining eqs. (118), (119), (120):

$$g_Y = 1 = 5.4862 \times 1.412 \times 1.2313 \sqrt{\frac{|\mu|}{M_0}} = 7.7090 \sqrt{\frac{|\mu|}{M_0}}. \tag{121}$$

Upon inputting $|\mu| = 88$ GeV, we obtain;

$$M_0 = 5.23 \text{ TeV}. \tag{122}$$

The "fit" used here was chosen for convenience and expedience and it slightly underestimates the mass M_0. A better fit can yield a more precise result. However, this roughly confirms the typical estimates, $M_0 \sim 5 - 6$ TeV, we obtained above from the other approximations.

5. Putting it all together: composite Brout-Englert-Higgs boson and natural top quark condensation

We introduced a dynamical electroweak isodoublet bilocal BEH field composed of third generation chiral quarks:

$$M_0^2 H^i(x,y) \sim [\overline{\psi}_R(x)\psi_L^i(y)], \qquad \psi_L^i = \begin{pmatrix} t \\ b \end{pmatrix}_L, \qquad \psi_R = t_R, \tag{123}$$

[...] color contracted, (^{i}) isospin index. As an ansatz we follow Yukawa and assume this can be factorized [21]:

$$H^i(X,r) = H^i(X)\phi(r). \tag{124}$$

Here the primary factor field, $H(X)$, can be viewed as the normal isodoublet BEH field and carries $SU(2)\times U(1)$ electroweak charges. The internal field, $\phi(r)$, is complex and, as a low energy approximation, we can assume it carries no electroweak charges.

Here we have moved the $SU(2)_L \times U(1)_Y$ left-handed gauge charges, located at $y = X + r$, and the $U(1)_Y$ right-handed hypercharge, located at $x = X - r$, to reside at the center of the state, X. This can be viewed as an approximation, but it can be formally arranged introducing Wilson lines internal to the state, as described in Appendix A. The configuration of the BEH boson is then a collection of radial, L (R), Wilson lines extending respectively from X to $y = X - r$ ($x = X - r$), becoming a "hedge-hog" configuration of internal Wilson lines. This leads to the usual electroweak physics and gauge field mass generation of the BEH boson with no complication from r dependence of $\phi(r)$ to leading order in $1/M_0^2$.

The present scheme is "minimal," so we suppress the b_R quark in the following. More realistically, we can assume the entire third generation experiences the full topcolor $SU(3)_{TC} \times U(1)'$ interaction. Since we want to avoid a $\overline{b}_L b_R$ condensate we typically require a Z' boson as in the old topcolor models [5][6]. This supplies a repulsive force in the $\overline{b}_L b_R$ channel. In a more complete topcolor $SU(3)_{TC} \times U(1)'$ scheme we naturally have $SU(2)_L \times SU(2)_R \times U(1)$ with common M_0 for coloron and Z' masses. To have $SU(2)_L \times U(1)$ with only a top condensate we require [7]:

$$g_0^2 + \frac{8}{27} g_{U(1)'}^2 > g_c^2 \qquad g_c^2 > g_0^2 - \frac{4}{27} g_{U(1)'}^2. \tag{125}$$

This is easy to do without fine–tuning and has been discussed in various model papers [5][6][7].

The action for the composite BEH field then becomes,

$$S = \int d^4X \Bigg(|D_H H(X)|^2 + |H(X)|^2 \, M_0^3 \int d^3r \left(-|\partial_{\vec{r}} \phi(r)|^2 + g_0^2 N_c M_0 \frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|} |\phi(\vec{r})|^2 \right) - \frac{\lambda}{2} (H^\dagger H)^2 - g_Y \left([\overline{\psi}_L(X) t_R(X)]_f H(X) + h.c. \right) \Bigg), \tag{126}$$

where the induced BEH-Yukawa coupling, $g_Y = g_{top} \approx 1.0$, is:

$$g_Y \approx g_0^2 \sqrt{2N_c/J} \, \phi(0). \tag{127}$$

The internal wave-function $\phi(\vec{r})$ satisfies the SKG equation (we can neglect small λ corrections here) with eigenvalue μ^2:

$$-\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \phi(r) - g_0^2 N_c M \frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|} \phi(r) = \mu^2 \phi(r). \tag{128}$$

This yields a compact solution, $\phi(\vec{r})$, and the eigenvalue μ^2 is then the physical (mass)^2 of the bound state.

For supercritical coupling, $g_0^2 > g_c^2$ we have negative μ^2 and we then obtain the SM "sombrero potential" for the symmetric phase:

$$V(H) = -|\mu|^2 |H|^2 + \frac{\lambda}{2} |H|^4. \tag{129}$$

Near critical coupling $\phi(r) \sim e^{-|\mu| r}/r$ with $|\mu| \ll M_0$. We see a chiral instability and the field H develops a VEV. We then conclude $v = |\mu|/\sqrt{\lambda}$, which is true for the pointlike BEH boson.

The quartic interaction is determined at loop level and can be obtained in the full bilocal theory. We find the logarithms match with the effective pointlike theory calculation, which is simpler. The Feynman loop with four vertices integrates the loop momenta from μ^2 to M_0^2 as IR and UV cut-offs (we could equally well include μ^2 in the propagator denominator for the IR cut-off with similar results). In the pointlike limit for the potential, we obtain:

$$\frac{\lambda}{2} = \frac{N_c}{8\pi^2} \hat{g}_Y^4 |\phi(0)|^4 \left(\ln\left(\frac{M_0}{\mu}\right) + \mathcal{O}\left(\frac{\mu^2}{M_0^2}\right) \right) \approx \frac{N_c g_Y^4}{8\pi^2} \ln\left(\frac{M_0}{\mu}\right). \tag{130}$$

The log evolution matches the result for the pointlike NJL case, with $g_{top} = g_Y = \hat{g}_Y \phi(0)$. Not surprisingly when we take the pointlike potential limit the loop result of the bilocal theory confirms a pointlike NJL loop calculation with $g_Y \sim \phi(0)$ (see discussion of Feynman loops in Section 4 of ref. ([14])).

In the standard model the quartic coupling term in the Higgs potential is $\frac{\lambda}{2}(H^\dagger H)^2$. Experimentally, in the SM using the value of $m_{BEH} \approx 125$ GeV and $v_{weak} \approx 175$ GeV, we find $\lambda \approx 0.25$. In the old pointlike NJL top condensation model the quartic coupling was determined by the RG with "compositeness boundary conditions," where we obtained (running-down from the Landau pole at M_0 to $|\mu|$), the result $\lambda \sim 1$. This is too large and leads to predicted $m_{BEH} \sim 260$ GeV. Indeed, the quartic coupling is generally problematic for all pointlike NJL based theories of a composite BEH boson, e.g. [31]. However, in the present bilocal scheme, owing to suppression of $g_Y \sim \phi(0)$, the quartic coupling is also suppressed $\propto |\phi(0)|^4$ and is now generated in running, from a value of zero at M_0, down to $|\mu| \sim 88$ GeV, using the induced g_Y. Keeping only the g_Y^4 contribution we now obtain numerically from eq. (130): $\lambda \approx 0.32$.

This result is significantly better than the old NJL model, but we can do better. The prefactor at one loop level should reflect the full RG running of λ, (see, e.g., [29]), yielding;

$$\lambda \approx (g_Y^4 - g_Y^2 \lambda - \lambda^2) \frac{N_c}{4\pi^2} \ln\left(\frac{M_0}{\mu}\right) \approx 0.23 \quad \text{(cf., 0.25 experiment.);} \tag{131}$$

This is in very good agreement with experiment at one loop precision. It also represents a "break-through" in these kinds of models where, as mentioned above, it has generally been problematic to reduce λ much below unity when only RG running is deployed over a large range of scale.

The key result obtained for M_0 comes from the numerical integration of the SKG equation in Section 4.6, and implementing the relation between g_Y and $\sqrt{|\mu|/M_0}$ with $|\mu| = 88$ GeV, the symmetric phase BEH mass. This yields $M_0 = 5.23$ TeV. Comparable results of $\sim 5.0 - 6.0$ TeV are also obtained in the simple skeletal model of Section 4.5, as well as for the matched spherical potential well of Section 4.3 (in the latter case we may have analytical control over the short-distance limit of the solution). We emphasize that, while we are confident of the prediction $M_0 \sim 5 - 6$ TeV, these results come from the present semiclassical analysis. Quantum corrections will be explored elsewhere [23] and may be significant. Due to the linear relationship between g^2 and $|\mu|/M_0$, a consequence of the dilution effect of $\phi(0)$, we see the degree of fine-tuning of the hierarchy is of order $\delta\kappa/\kappa \sim |\mu|/M_0 \sim 1.4\%$ (Section 4.4).

6. Conclusions

A summary of the results of the bilocal theory of the BEH boson is as follows:

- Symmetric phase BEH mass $\mu^2 = -(88)^2$ (GeV)^2 is an input.
- The top-quark-BEH-Yukawa coupling $g_Y \approx 1.0$ is an input.
- Our approximations of bound state solutions in the Yukawa potential then imply $M_0 \approx 6$ TeV.
- The fine–tuning near critical coupling is about $\sim 1.4\%$, significantly relaxed by dilution.
- The theory "predicts" $\lambda \approx 0.23$ very close to experiment ≈ 0.25.
- The BEH mass ~ 125 GeV and weak scale $v_{weak} = 175$ GeV are then obtained a usual.
- The main prediction for the future is an octet of colorons, at a mass scale of order $M_0 \sim 6$ TeV.

The Nambu–Jona-Lasinio model (NJL) is a Lorentz invariant description of a scalar bound state of relativistic chiral fermions in a 4-fermion short-distance potential: $-\frac{g^2}{M_0^2}\overline{\psi}_L\psi_R\overline{\psi}_R\psi_L$. The solution of the NJL model is constrained to be a pointlike effective field theory, $\Phi(x) \sim \overline{\psi}_R(x)\psi_L(x)$, with renormalization group (RG) boundary conditions on its parameters at M_0. At critical coupling, $g_0^2 \to g_c^2 = 8\pi^2/N_c$, the bound state mass $\mu^2 \to 0$. The low energy effective action then approaches a conformal theory. Indeed, the RG parameters, top Yukawa coupling and quartic coupling, approach IR fixed points, (in analogy to a 2nd order phase transition at critical temperature).

The original top quark condensation (composite BEH boson) model was formulated using the NJL model with third generation constituents [1][2][3]. The model gave precise predictions, but the values obtained for m_{top} and m_{BEH} were not in agreement with experiment. Moreover, the model required an absurd degree of fine-tuning. The authors at the time thought these issues may be resolved in future developments of the theory. We believe the present approach constitutes such.

In quantum mechanics a short distance potential (e.g., $\sim -\alpha\delta^3(r)$), with eigenvalue near zero (critical coupling), will always produce a $\sim 1/r$ large distance "tail" wave-function. This is due to scale symmetry outside the potential. The NJL model is constrained to be pointlike and has no internal wave-function, therefore no IR "tail." To obtain an internal wave-function we must extend the pointlike NJL field description of the bound state, $H(x)$, to a bilocal description (similar to that suggested long ago by [21]) $H(x) \to H(x, y) \sim \overline{\psi}_R(x)\psi_L(y)$. We write a factorized ansatz, $H(x, y) \to H(X, r) \sim H(X)\phi(r)$, where $X = (x + y)/2$ and $r = (x - y)/2$, and $\phi(r)$ becomes the internal wave-function of the bound state.

The electroweak gauging by the covariant derivatives can be assumed to act upon $H(X)$, while $\phi(r)$ is then sterile. Formally, we can arrange this by field redefinitions using Wilson lines to "pull-back" the gauge field currents to X. In any case, this affects only the electroweak gauging at large distances and can be viewed as a low energy approximation. It is sufficient to allow the composite BEH boson to develop a VEV in the usual way and supply masses to W and Z gauge bosons of the standard model.

The pointlike NJL interaction is then replaced by a suitable UV completion. Most natural is "topcolor" [5], consisting of a massive octet of "colorons," leading to a single particle exchange potential. The free bilocal fields must be normalized to have well defined currents and charges. This is nontrivial, and requires removing "relative time," after which $\phi(\vec{r})$ becomes a static field with no dependence upon r^0. This yields, in the rest-frame, a Yukawa potential interaction, $-g^2 N_c M_0(\exp(-2M_0 r)/8\pi r)|\phi(\vec{r})|^2$, with a BCS-like enhancement $\propto N_c$ due to color singlet normalization.

The internal wave-function of the bound state then satisfies a semiclassical Schrödinger-Klein-Gordon (SKG) equation with eigenvalue μ^2 in the rest frame:

$$-\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\phi(r) - g_0^2 N_c M \frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}\phi(r) = \mu^2\phi(r) \tag{132}$$

This yields a compact solution, $\phi(\vec{r})$, and the eigenvalue μ^2 is then the physical (mass)^2 of the bound state. The critical coupling, $g_0^2 = g_c^2$, is numerically almost identical to the NJL critical coupling. At critical coupling, $\mu^2 = 0$, while at super-critical coupling, $\mu^2 < 0$, implying spontaneous symmetry breaking. Near critical coupling $\phi(r) \sim e^{-|\mu|r}/r$ with $|\mu| \ll M_0$. The solution can be reliably obtained by numerically integrating the SKG equation.

The Yukawa coupling of the bound state to free fermions is generated by the coloron interaction, and $g_Y \propto \phi(0)$. Due to the extended wave-function tail, $\phi(0) \sim \sqrt{|\mu|/M_0}$, we have significant "dilution," and g_Y is suppressed. The numerical integration of the solution of the SKG equation in the Yukawa potential (Section 4.6), reveals the critical behavior and yields the result, $g_Y \propto \sqrt{|\mu|/M_0}$, and the value of the coloron mass/composite scale, $M_0 \approx 5.23$ TeV. This confirms various estimates we performed in the $\delta-$function limit of the potential, as well as an exactly solvable spherical well approximation, yielding of order $M_0 = 5.0 - 6.0$ TeV.

The BEH boson quartic coupling arises from loops $\sim N_c g_Y^4 \ln(M/\mu)/4\pi^2 \propto (\phi(0))^4$ (Feynman loops are studied in greater detail in [14]). In application to top quark condensation, the dilution effect suppresses g_Y and, inputting $g_Y = g_{top} = 1$, from which we determine the quartic coupling. The result for the quartic coupling at one loop is $\lambda \sim 0.23$, compared to $\lambda \approx 0.25$ experimentally, in excellent agreement.

Remarkably, fine–tuning of the hierarchy, $M_0/|\mu| \gg 1$, is also suppressed by dilution, of order $\mu/M_0 \sim (100\,GeV)/(6\,TeV) \sim$ (few) %, due to an emergent linear relation, $\delta g_c^2/g_c^2 \sim \mu/M_0$. This is shown to be general by the simple spherical potential model.

The "colorons," which mediate the binding interaction, form a QCD color octet. They have a global $SU(3)$ symmetry and a conserved $SU(3)$ current, and must therefore be pair-produced. They will decay in the minimal model to $\bar{t}t$, but in more realistic topcolor models they can also decay to $\bar{b}b$ ([5][6][7] and references therein). They may be accessible to the LHC in the multi-TeV range [32], favoring the third generation in its couplings. It should be straightforward to obtain a lower bound on M_0 from single gluon or gluon fusion producting pairs of the colorons. The colorons will produce excesses in pairs of boosted 2-top $t\bar{t} + t\bar{t}$, events. In an extended model we would expect boosted $t\bar{t} + b\bar{b}$ and $b\bar{b} + b\bar{b}$ excesses to emerge. The more general topcolor models offer many possibilities within the arena of heavy quark flavor physcs.

This theory, if confirmed, solves the "naturalness problem" of the BEH boson in the Standard Model. Many avenues for further theoretical development exist. Notably, a revisitation of the Topcolor Z' and a possible second heavy boson (resonance?) associated with $\bar{b}b$ would be interesting to study in the present formalism. It may also be useful to apply these techniques in QCD, such as heavy-light meson and heavy-heavy-light baryon theory [12].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

I thank Bill Bardeen, Bogdan Dobrescu and Julius Kuti for discussions. I also thank the Fermi Research Alliance, LLC under Contract No. DE-AC02-07CH11359 with the U.S. Department of Energy, Office of Science, Office of High Energy Physics, and The University of Wisconsin Physics Department for an Honorary Fellowship.

Appendix A. Gauging and Wilson lines

We have simplified the electroweak gauging in the above definition of the BEH model, where the covariant derivatives act only upon the center coordinate X in $H(X)$, as opposed to x and y in $H(x,y)$. We can formally arrange this by incorporating "Wilson lines" into the structure of the wave-function.

Consider the $SU(2)_L$ and weak hypercharge $U(1)_Y$ covariant derivatives (we will omit displaying the gluons in the analysis, which ultimately cancel at the X endpoint), where $H \sim \bar{t}_R \psi_L$,

$$D_R^\dagger H(x,y) = \left(\frac{\partial}{\partial x^\mu} + i g_1 B_\mu(x)\frac{Y_R}{2}\right) H(x,y), \qquad D_L H(x,y) = \left(\frac{\partial}{\partial y^\mu} - i g_1 B_\mu \frac{Y_L}{2} - i g_2 W_\mu^A(y)\frac{\tau^A}{2}\right) H(x,y), \tag{A.1}$$

where the weak hypercharges are $[Y,\psi_L] = Y_L\psi_L = (1/3)\psi_L$, $[Y,\psi_R] = Y_R\psi_R = (4/3)\psi_R$, $[Y,H] = [(Y_L - Y_R), H] = Y_H H = (-1)H$. Note that, since $H \sim \bar{t}_R\psi_L$ the action of the derivatives on $H(x,y)$ is $D_L H$, which acts upon ψ_L, and $D_R^\dagger H$, which acts upon $\bar{t}_R$.

We introduce Wilson lines, $H(x,y) \to W_R^\dagger(X,x) W_L(X,y) H(x,y)$ where,

$$W_L(X,y) = P\exp\left(-i g_1 \frac{Y_L}{2}\int_y^X B_\nu(\rho) d\rho^\nu - i g_2 \int_y^X W_\nu^A(\rho)\frac{\tau^A}{2} d\rho^\nu\right)$$
$$W_R^\dagger(X,x) = P\exp\left(+i g_1 \frac{Y_R}{2}\int_x^X B_\nu(\rho) d\rho^\nu\right), \tag{A.2}$$

and P denotes "path ordering" (which is relevant for the non-abelian components, but trivial for $U(1)_Y$). The derivative terms become, with $X = (x+y)/2$ and $r = (x-y)/2$ and noting sign flip of the gauge fields in $D_R^\dagger$:

$$\begin{aligned} D_R^\dagger(W_R^\dagger W_L H(x,y)) &= W_R^\dagger W_L\left(\frac{\partial}{\partial x^\mu} + i g_1 \frac{Y_R}{2}\frac{\partial X_\mu}{\partial x_\nu} B_{R\nu}(X) - i g_1 \frac{Y_L}{2}\frac{\partial X_\mu}{\partial x_\nu} B_{R\nu}(X) - i g_2 \frac{\partial X_\mu}{\partial x_\nu} W_\nu^A(X)\frac{\tau^A}{2}\right) H(x,y) \\ &= W_R^\dagger W_L\left(\frac{\partial}{\partial x^\mu} + i\frac{1}{2} g_1\left(\frac{Y_R}{2} - \frac{Y_L}{2}\right) B_\mu(X) - i\frac{1}{2} g_2 W_\mu^A(X)\frac{\tau^A}{2}\right) H(x,y) \\ &= W_R^\dagger W_L\left(\frac{\partial}{\partial x^\mu} - i\frac{1}{2} g_1 \frac{Y_H}{2} B_\mu(X) - i\frac{1}{2} g_2 W_\mu^A(X)\frac{\tau^A}{2}\right) H(x,y), \end{aligned} \tag{A.3}$$

(note, e.g., $\partial_{x^\mu}(\int_x^X B_\nu(\rho)d\rho^\nu) = \frac{1}{2}B_\mu(X) - B_\mu(x)$, etc.). Likewise we obtain:

$$D_L H = W_R^\dagger W_L \left(\frac{\partial}{\partial y^\mu} - i\frac{1}{2}g_1 \frac{Y_H}{2} B_\mu(X) - i\frac{1}{2}g_2 W_\mu^A(X)\frac{\tau^A}{2} \right) H(x,y). \tag{A.4}$$

Pass to barycentric coordinates and $H'(X,r)) = H(X+r, X-r)$,

$$\begin{aligned} D_R^\dagger(W_R^\dagger W_L)H' &= \frac{1}{2}W_R^\dagger W_L \left(\frac{\partial}{\partial X^\mu} + \frac{\partial}{\partial r^\mu} - ig_1 \frac{Y_H}{2} B_\mu(X) - ig_2 W_\mu^A(X)\frac{\tau^A}{2} \right) H'(X,r), \\ D_L(W_R^\dagger W_L)H' &= \frac{1}{2}W_R^\dagger W_L \left(\frac{\partial}{\partial X^\mu} - \frac{\partial}{\partial r^\mu} - ig_1 \frac{Y_H}{2} B_\mu(X) - ig_2 W_\mu^A(X)\frac{\tau^A}{2} \right) H'(X,r). \end{aligned} \tag{A.5}$$

Note that each term has acquired the overall factor of $\frac{1}{2}$. We assume factorization, $H(x,y) \to \sqrt{2/J}\ H(X)\phi(r)$ and the overall kinetic term action becomes:

$$\int d^4X d^4r \left| \frac{\partial}{\partial X^\mu}H(X) - ig_1 \frac{Y_H}{2} B_\mu(X)H(X) - ig_2 W_\mu^A(X)\frac{\tau^A}{2}H(X) \right|^2 |\phi(r)|^2 + |\partial_r\phi|^2|\chi|^2 \Big), \tag{A.6}$$

showing that the Wilson lines “pull back” the gauge fields to the center X and assemble the component hypercharges into the BEH bound state $Y_H = Y_L - Y_R$. We have omitted consideration of the QCD terms since they vanish for the same reason that they cancel in a local expression, $\partial_x(\overline{\psi}_R(x)\psi_L(x))$.

This redefines the H field with the Wilson lines, and recovers the gauging we proposed above for our composite BEH theory. The symmetry breaking and masses of the gauge fields go through in the usual way.

Appendix B. Lorentz invariance of μ^2 and covariantization

We show that the resulting eigenvalue μ^2 is Lorentz invariant, resulting from the original manifestly Lorentz invariant factorized action of eq. (61),

$$S = M^4\int d^4X d^4r \left(Z|\phi(r)|^2|\partial_X\chi(X)|^2 + Z|\chi(X)|^2|\partial_r\phi(r)|^2 + 2g_0^2N_cD_F(2r)|\chi(X)\phi(r)|^2 \right). \tag{B.1}$$

First, we see that χ can be defined to have canonically normalized $\chi(X)$ kinetic term by introducing a Lagrange multiplier, as:

$$S_1 = \eta_1\left(1 - M^4\int d^4r\ Z|\phi(r)|^2\right)^2, \tag{B.2}$$

and $\delta S_1/\delta\eta_1 = 0$. We define currents (these are discussed in Appendix B of ref. [14]):

$$J_\mu = i[\chi^\dagger(X)\frac{\overleftrightarrow{\partial}}{\partial X^\mu}\chi(X)], \qquad K_\mu = i[\phi^\dagger(r)\frac{\overleftrightarrow{\partial}}{\partial r^\mu}\phi(r)], \tag{B.3}$$

and we can therefore define a timelike unit vector, ω_μ[5]:

$$\omega_\mu = J_\mu/\sqrt{J_\rho J^\rho} \tag{B.4}$$

We can formally implement the constraint that $\phi(r)$ has no dependence upon r^0 by adding to the action a Lagrange multiplier, η, while preserving Lorentz invariance,

$$S_2 = \eta_2\int d^4X d^4r\ M^4|\omega^\mu K_\mu|^2, \tag{B.5}$$

[5] The bilocal currents are,

$$J_\mu^+(X,r) = iZ'\epsilon^4M^4[\chi^\dagger(X)\frac{\overleftrightarrow{\partial}}{\partial X^\mu}\chi(X)]\phi^\dagger(r)\phi(r), \qquad J_\mu^-(X,r) = iZ'M^4[\phi^\dagger(r)\frac{\overleftrightarrow{\partial}}{\partial r^\mu}\phi(r)]\chi^\dagger(X)\chi(X)$$

These can be integrated to form,

$$J_\mu^+(X) = iZM^4[\chi^\dagger(X)\frac{\overleftrightarrow{\partial}}{\partial X^\mu}\chi(X)]\int d^4r\ \phi^\dagger(r)\phi(r), \qquad J_\mu^-(r) = iZM^4[\phi^\dagger(r)\frac{\overleftrightarrow{\partial}}{\partial r^\mu}\phi(r)]\int d^4X\ \chi^\dagger(X)\chi(X)$$

Normalizing

$$1 = ZM^4\int d^4r\ |\phi(r)|^2 = M^3\int d^3r|\phi(r)|^2 \qquad \text{hence,} \qquad J_\mu(X) = i\chi^\dagger(X)\frac{\overleftrightarrow{\partial}}{\partial X^\mu}\chi(X).$$

and $\delta S_\eta/\delta\eta = 0$. The constraint implies that $\omega^\mu\partial_\mu\phi(r) = 0$ where $\omega^\mu \propto P^\mu$ is the timelike 4-momentum of the bound state, hence ϕ has no r^0 dependence in the rest frame. Note that, in practice, we don't need this formality if we simply assume that we are interested only in the solutions in which $\phi(r)$ has no dependence upon r^0.

Now consider the kinetic terms,

$$S = M^4\int d^4X d^4r \left(Z|\phi(r)|^2|\partial_X\chi(X)|^2 + Z|\chi(X)|^2|\partial_r\phi(r)|^2 \right). \tag{B.6}$$

Z can be interpreted as an operator of the form,

$$Z \to \delta(M_0\omega_\mu r^\mu), \tag{B.7}$$

which removes the relative time in the kinetic terms in the rest-frame:

$$S \to M^3\int d^4X d^3r \left(|\phi(\vec{r})|^2|\partial_X\chi(X)|^2 - |\chi(X)|^2|\partial_{\vec{r}}\phi(\vec{r})|^2 \right), \tag{B.8}$$

and the prefactor is now M^3 (Note $|\partial_r\phi|^2 = |\partial^2_{r^0}\phi|^2 - |\partial_{\vec{r}}\phi|^2$ where $\partial_{\vec{r}}$ is the spatial derivative).

If we examine the constraint S_1 we see that,

$$S_1 = \eta_1\left(1 - M_0^4\int d^4r\, Z|\phi(r)|^2\right)^2 \longrightarrow \eta_1\left(1 - M_0^3\int d^3r\, |\phi(\vec{r})|^2\right)^2, \tag{B.9}$$

which enforces the rest frame normalization $1 = M^3\int d^3r|\phi(\vec{r})|^2$. The kinetic terms become:

$$S \to \int d^4X\, |\partial_X\chi(X)|^2 - M_0^3\int d^4X d^3r\, |\chi(X)|^2|\partial_{\vec{r}}\phi(\vec{r})|^2. \tag{B.10}$$

With the timelike unit vector we can define a tensor,

$$W_{\mu\nu} = \omega_\mu\omega_\nu - g_{\mu\nu}, \tag{B.11}$$

hence,

$$(r^0)^2 = (W_{\mu\nu} + g_{\mu\nu})r^\mu r^\nu; \qquad \vec{r}^{\,2} \equiv W_{\mu\nu}r^\mu r^\nu; \qquad \phi(r) \equiv \phi\left(\sqrt{W_{\mu\nu}r^\mu r^\nu}\right); \qquad W^{\mu\nu}\partial_\mu\phi^\dagger\partial_\nu\phi = -|\partial_{\vec{r}}\phi|^2. \tag{B.12}$$

Using $W^{\mu\nu}$, all of the expressions in eq. (B.1) can be made manifestly Lorentz invariant.

Hence, from eqs. (B.5), (B.12) with $r = \sqrt{W_{\mu\nu}r^\mu r^\nu}$, we see that μ^2 is given by the manifestly invariant form:

$$\begin{aligned}&\mu^2 M_0^4\int d^4r\, \delta(M_0\omega_\mu r^\mu)\, |\phi(r)|^2 = M_0^4\int d^4r \left(\delta(M_0\omega_\mu r^\mu)|\partial_r\phi(r)|^2 + 2g_0^2 N_c M D_F(2r^\mu)|\phi(r)\,|^2 \right)\\ &= \mu^2 = M_0^3\int d^3r \left(-\phi^*(\vec{r})\nabla^2_{\vec{r}}\phi(\vec{r}) - g_0^2 N_c M_0\frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}|\phi(\vec{r})|^2 \right)\end{aligned} \tag{B.13}$$

Once calculated in the rest frame it is the same in any frame.

Appendix C. The vacuum as a Bose-Einstein condensate: frame averaging

We have thus far been working in the rest frame of the BEH boson in the symmetric phase. Here the BEH boson has nonzero timelike 4-momentum, since $P^2 = \mu^2$, and $P_\mu = (\mu, \vec{0})$, putting aside the fact that $\mu^2 < 0$ and μ is imaginary. The broken phase state has $\phi(\vec{r})$ with normalizable wave-function and has a definite $\vec{r}$ dependence. We have upon removal of relative time, $P_\mu r^\mu = 0$, hence $\phi(\vec{r})$ has a purely spacelike argument in the particle rest frame. As the field ϕ acquires a VEV, where then P_μ vanishes, what would select the frame for $\phi(\vec{r})$?

In the SM there is no preferred reference frame for the broken phase that defines the vacuum of the standard model. That is, the BEH vacuum expectation value (VEV) in the SM is v_{weak} and is a constant in space-time. Our present theory would lead to a constant v_{weak}, but may contain a nonzero correlator, $\phi(r^\mu)$. If the vacuum of the theory has an internal wave-function with a frame dependent $\phi(\vec{r})$ it would imply Lorentz violation and observable features. Most parameters (Yukawa and quartic couplings) depend only upon $\phi(0)$ in the large M_0 limit, but perhaps unwanted suppressed effects of $\mathcal{O}(\mu^2/M_0^2)$ arise? Presently we will describe a simple proposal, but this is a preliminary "sketch" and will be (or an alternative will be) developed elsewhere [23].

The vacuum is akin to a condensed matter state, such as a BEC (Bose-Einstein Condensate). For any given composite bilocal BEH boson in its rest frame, defined by ω_μ, we have $\chi(X)\phi(r) \to \chi(X)\phi_0(\vec{r}) = \chi(X)\phi_0(\hat{r})$, where $\hat{r} = W_{\mu\nu}\omega^\mu r^\nu$. We propose that the BEC is a coherent state of BEH bosons with arbitrary ω^μ. We should therefore average the vacuum over ω^μ, or "frame average" amplitudes that dpend upon $\phi(\hat{r})$.

We can give various definitions of frame averaging integral. For example, we might integrate over the space-like hyperboloid defined by $\omega^2 = 1$, then define,

$$\langle F(r^2)\rangle = N\int d^4\omega\, \delta(\omega^2 - 1)F(W^{\mu\nu}r_\mu r_\nu) \;\text{ where, }\; N^{-1} = \int d^4\omega\, \delta(\omega^2 - 1). \tag{C.1}$$

Upon averaging $F(r^2)$ we will have $\langle F(r^2)\rangle$ as a function of the invariant $r^\mu r_\mu$. Alternative averaging functions could be defined.

In the Yukawa interaction of eq. (75) we have,

$$S'_Y = \sqrt{2N_c J}\, g_0^2 M_0^2 \int d^4X d^4r\, [\overline{\psi}_L(X+r)\psi_R(X-r)]_f\, D_F(2r)\, \chi(X)\phi(\vec{r}) + h.c.$$
$$= \hat{g}_Y J M_0^2 \int d^4X d^4r\, [\overline{\psi}_L(X)\psi_R(X) - r^\mu r^\nu \frac{\partial}{\partial X^\mu}\overline{\psi}_L(X)\frac{\partial}{\partial X^\nu}\psi_R(X) + ...]_f\, D_F(2r)\, \chi(X)\phi(\vec{r}) + h.c., \tag{C.2}$$

thus we encounter terms in the expansion in the broken phase, such as,

$$= -\hat{g}_Y M_0^2 \int d^4X d^4r\, [\frac{\partial}{\partial X^\mu}\overline{\psi}_L(X)\frac{\partial}{\partial X^\nu}\psi_R(X) + ...]_f\, D_F(2r) v_{weak} r^\mu r^\nu \phi(\vec{r}). \tag{C.3}$$

These terms are suppressed since $D(2r) \to J^{-1}M_0^{-2}\delta^4(r)$ in the pointlike potential limit; but subleading M_0^{-4} effects may remain. If the vacuum was defined in a particular frame with a definite ω_μ then these terms would lead to Lorentz violation. For example, the top quark (or any other fermion such as the electron), propagating through the medium with 4-momentum p^μ would acquire mass corrections $\propto \int_r p^\mu p^\nu r_\mu r_\nu \sim m_e^2 \vec{p}^{\,2}/M_0^2$. However, upon frame averaging the $\int_r \langle \propto p^\mu p^\nu r_\mu r_\nu\rangle \sim p_\mu p^\mu / M_0^2$ is Lorentz invariant.

However, an intriguing possibility arises in that there may be imprinting of the cosmic reference frame at the time the vacuum forms in the early universe. The covariant tensor, eq. (B.11) may pick up a component of $\epsilon' T^c_{\mu\nu}$, the cosmic background stress tensor. We've briefly looked at this quantitatively, following Coleman and Glashow [33], and we were initially surprised that electron vacuum Cerenkov radiation limits are satisfied. From [33] we have for a particle of mass m the modified dispersion relation, $E^2 - (1+\delta)\vec{p}^{\,2} = m^2$, where δ parameterizes Lorentz breaking and would lead to vacuum Cerenkov radiation if present. If the vacuum wave-function has residual $\vec{r}$ dependence then we would have nonzero δ. We find, however, that with $\epsilon' \sim 1$, that δ is sufficiently suppressed for the electron (which gets mass from the BEH boson with $\phi(r)$ via with higher dimension Eichten-Lane operators [27]). We estimate $\delta \sim m_e^2/M_0^2 \sim 10^{-14}$, below the limit $\sim 10^{-12}$ quoted in [33].

However, this is not the whole story since, for the top quark, we expect large δ. This may lead potentially to a large loop induced magnetic $\vec{B}^2$ correction to the electromagnetic kinetic term, the most sensitive probe as identified by Kostelecky [34]. We have not done this calculation, but estimates appear problematic. Hence a vacuum frame averaged condensate appears preferable, with a mechanism for suppressing $\epsilon' \sim 0$.

Appendix D. Summary of the top-BEH subsystem of the standard model

Lagrangian in Symmetric Phase:

$$(DH)^\dagger DH - \mu^2 H^\dagger H - \frac{\lambda}{2}(H^\dagger H)^2 - (g_t[\overline{\psi}_L t_R]H + h.c.) + \overline{\psi}_L \not{D}\psi_L + \bar{t}_R \not{D}\psi_R$$
$$H = \begin{pmatrix} H^0 \\ H^- \end{pmatrix}, \qquad D_\mu = \partial_\mu - ig_2 W^A_\mu \frac{\tau^A}{2} - ig_1 B_\mu \frac{Y}{2}, \qquad \psi^i_L = \left(\frac{1-\gamma^5}{2}\right)\begin{pmatrix} t \\ b \end{pmatrix}, \qquad t_R = \left(\frac{1+\gamma^5}{2}\right)t$$
$$\mu^2 \approx -(88)^2\, GeV^2, \qquad g_t \approx 1, \qquad \lambda \approx 0.25, \qquad Q = \frac{\tau^3}{2} + \frac{Y}{2} \qquad [\overline{\psi}\psi] = \overline{\psi}^a\psi_a \text{ denotes color sum.} \tag{D.1}$$

Lagrangian in Broken Phase:

$$\frac{1}{2}(\partial h)^2 - \frac{1}{2}m^2 h^2 - m_t[\bar{t}t] - \frac{g_t}{\sqrt{2}}[\bar{t}t]h - \frac{\lambda}{8}(h)^4 ...$$
$$H = \begin{pmatrix} v_{weak} + \frac{h}{\sqrt{2}} + i\phi^0 \\ \phi^- \end{pmatrix} \qquad \phi^0, \phi^\pm \text{ massless Nambu-Goldstone modes “eaten” by } Z^0 \text{ and } W^\pm$$
$$m^2 \approx (125)^2\, GeV^2 \qquad v_{weak} = \frac{\mu}{\sqrt{\lambda}} \approx 175\, GeV \qquad m_t = g_t v_{weak} \qquad \lambda \approx 0.25. \tag{D.2}$$

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