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Nonlinear Forces and Omitted Masses: Mass-Spring-Damper Models and Their Model-Form Errors

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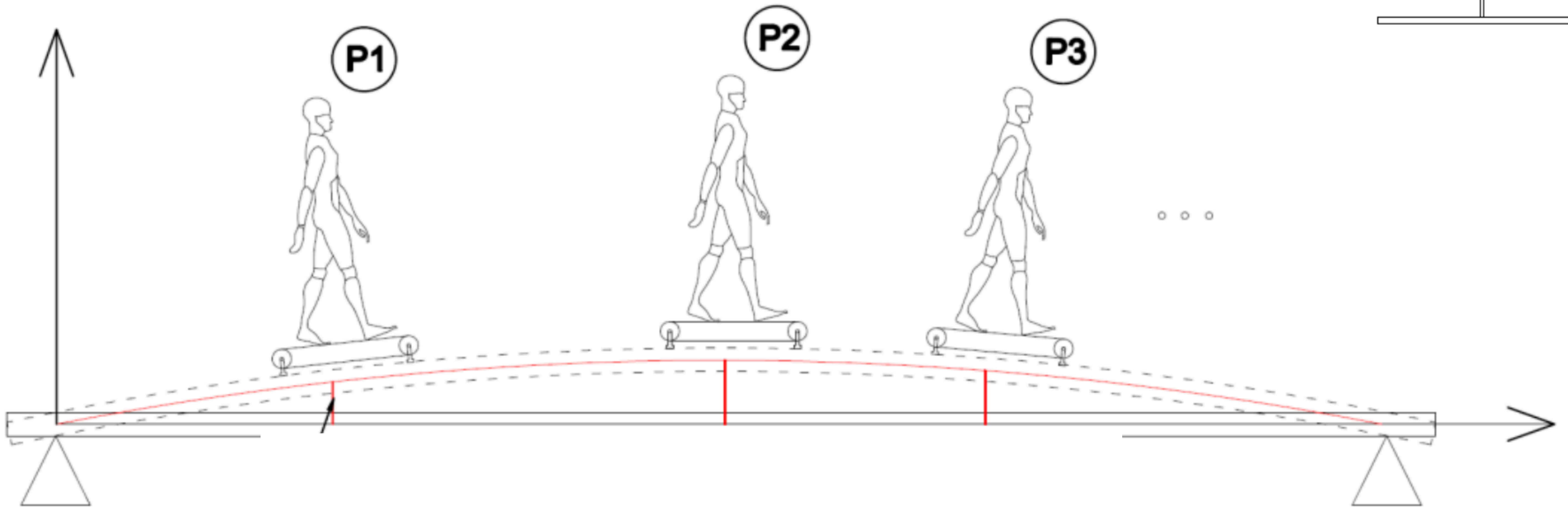
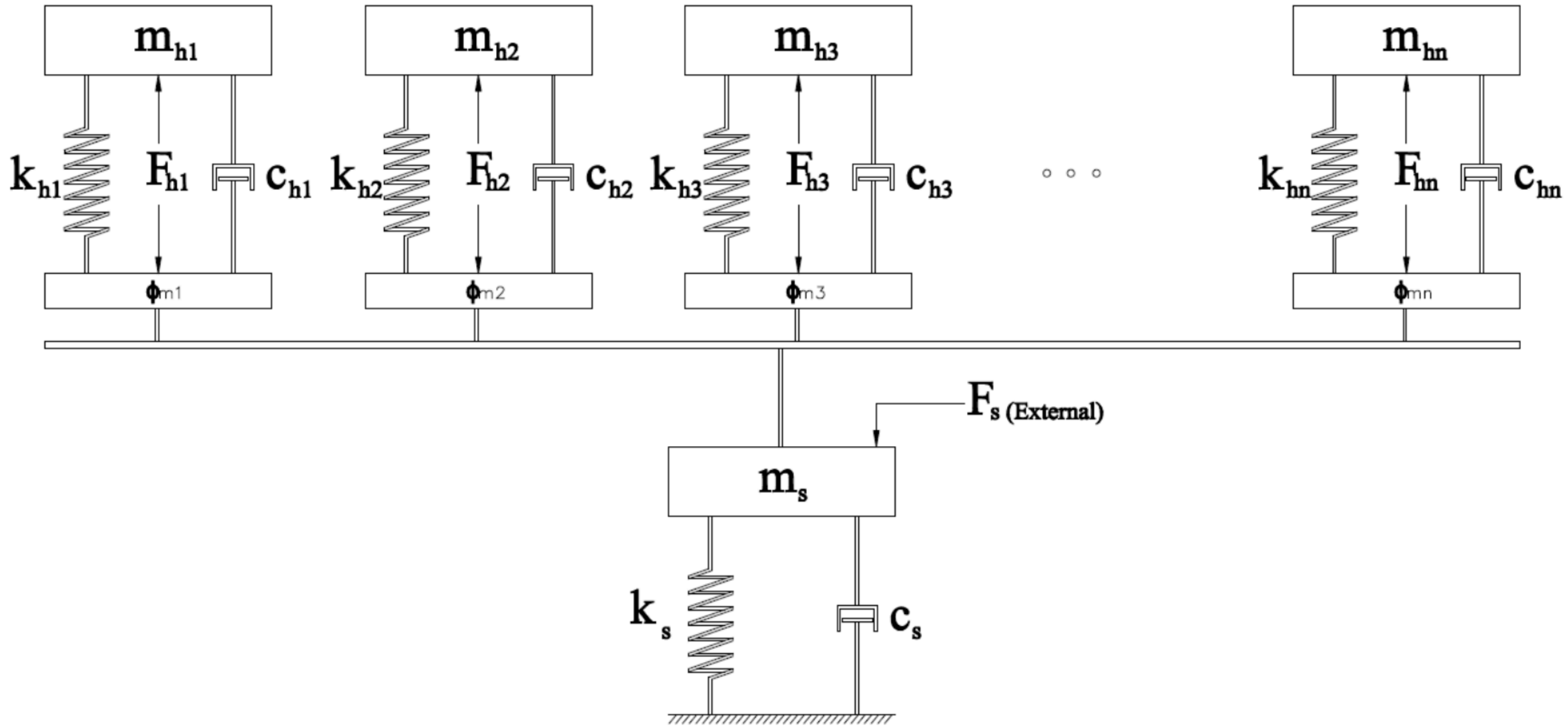
SAND:TODO

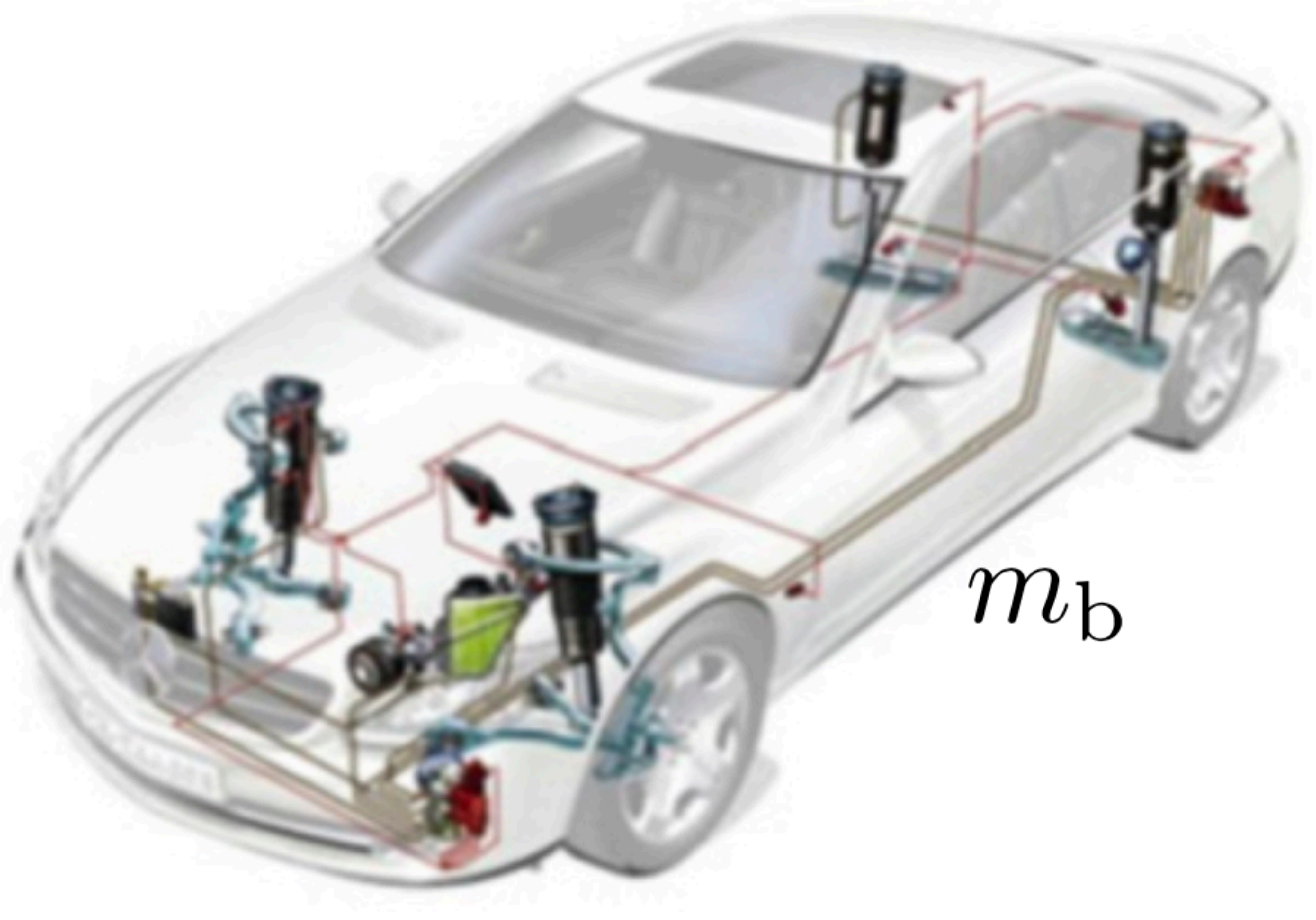
Mass-spring-damper models approximate the impact load on bridges

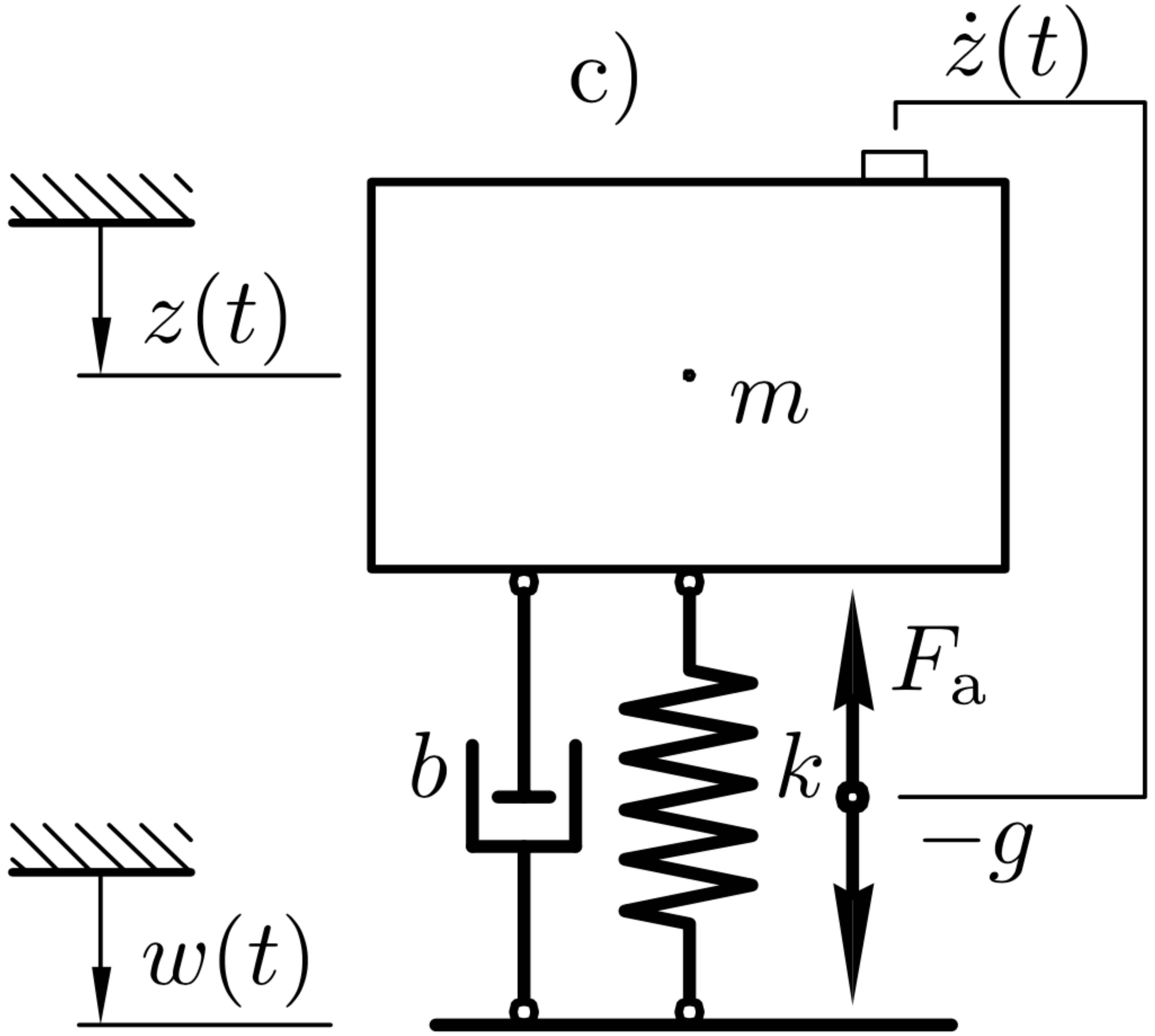




[1] E. SHAHABPOOR, A. PAVIC, AND V. RACIC, *Identification of mass–spring–damper model of walking humans*, in Structures, vol. 5, Elsevier, 2016, pp. 233–246.

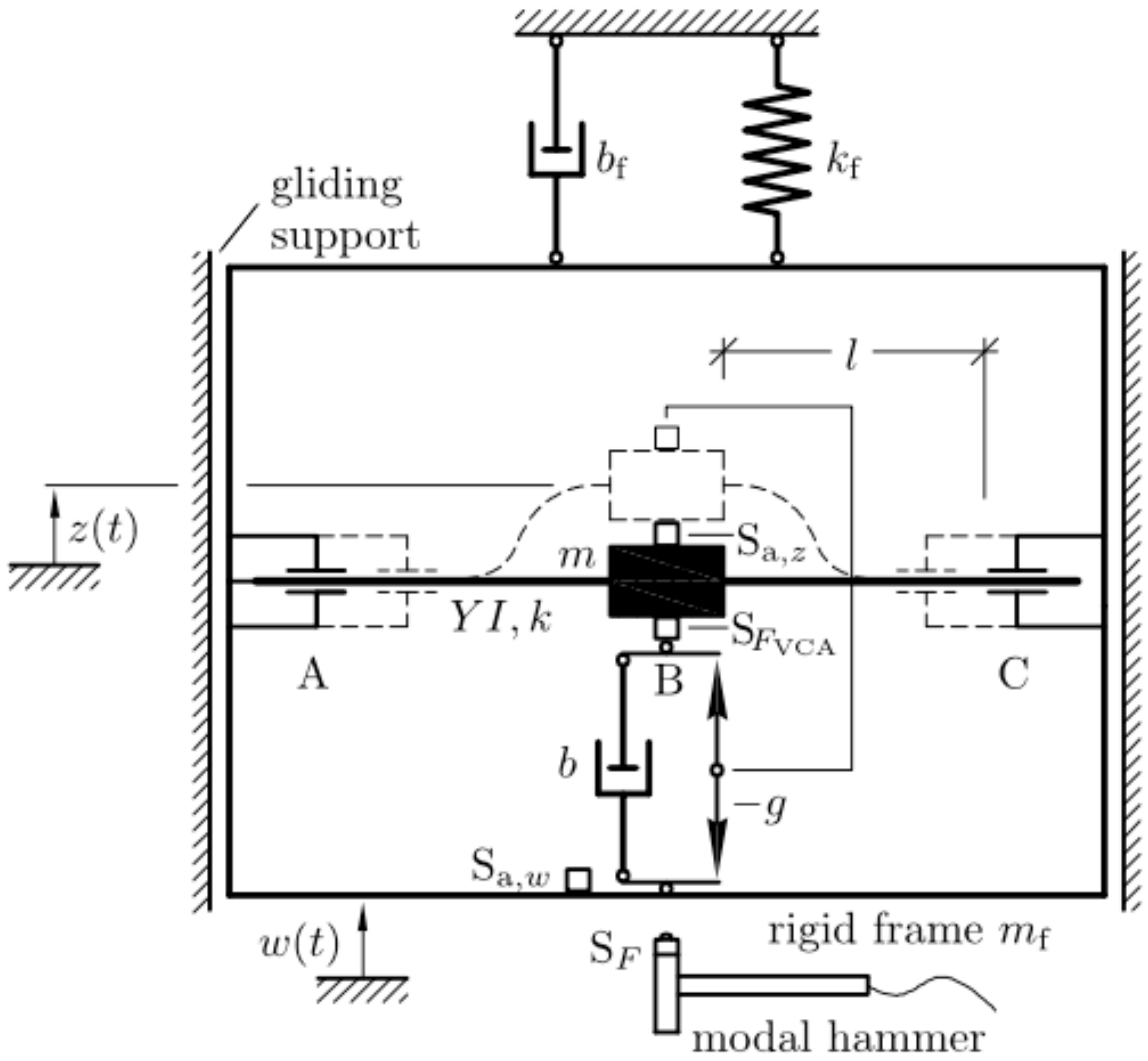
A one-mass oscillator estimates a driving car's vertical dynamic behavior

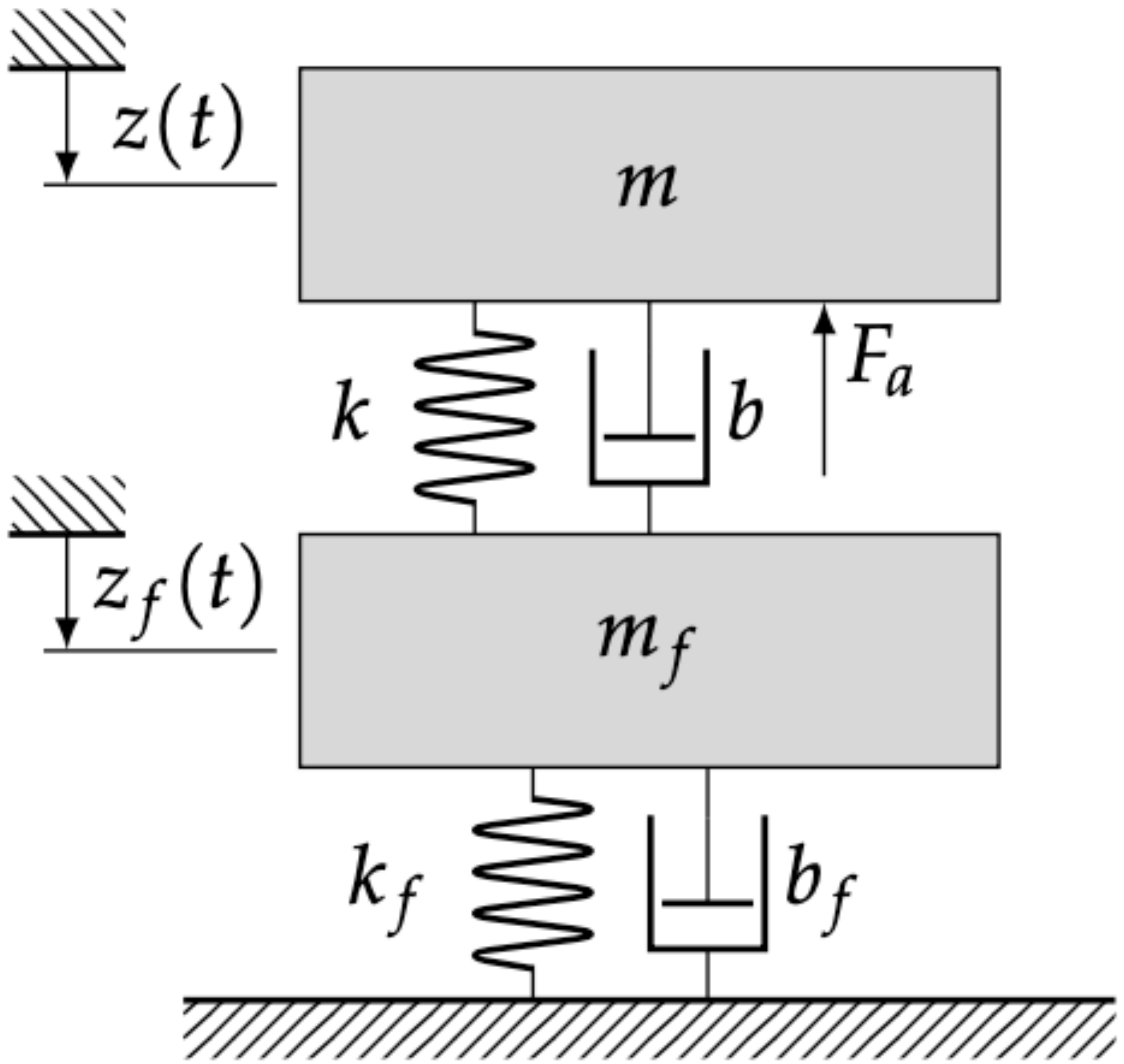




[2] R. PLATZ, *Comprehensive testing environment to evaluate approaches in uncertainty quantification for passive and active vibration isolation*, Model Validation and Uncertainty Quantification, Volume 3, (2023), p. 97

A two mass oscillator approximates the vibration isolation experiment

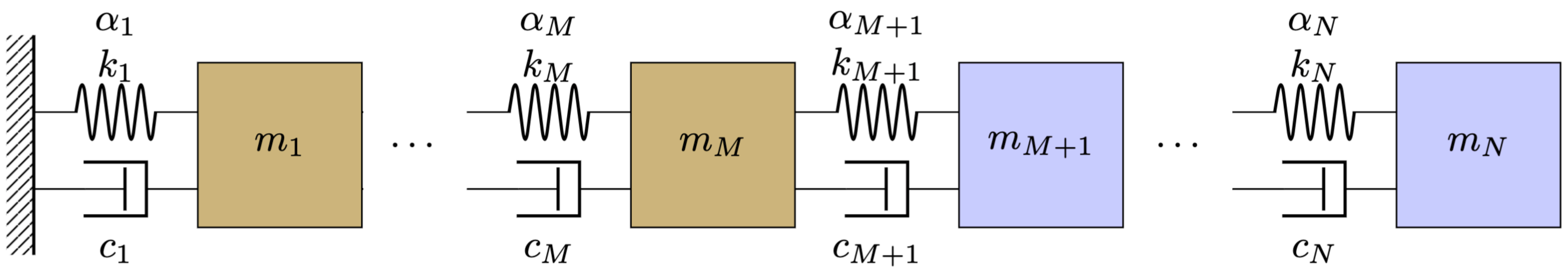




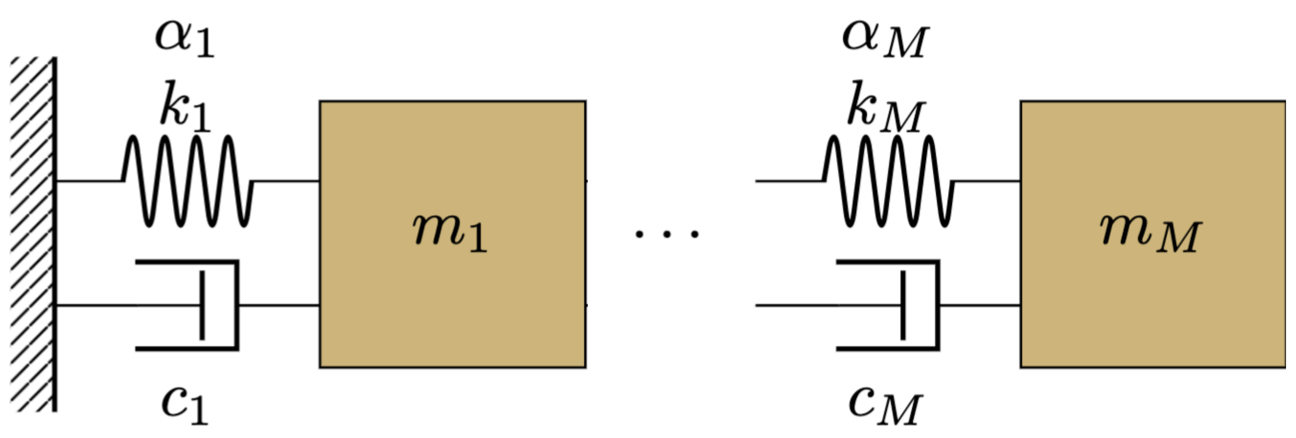
[2] R. PLATZ, Comprehensive testing environment to evaluate approaches in uncertainty quantification for passive and active vibration isolation, Model Validation and Uncertainty Quantification, Volume 3, (2023), p. 97

How do model-form errors arise?

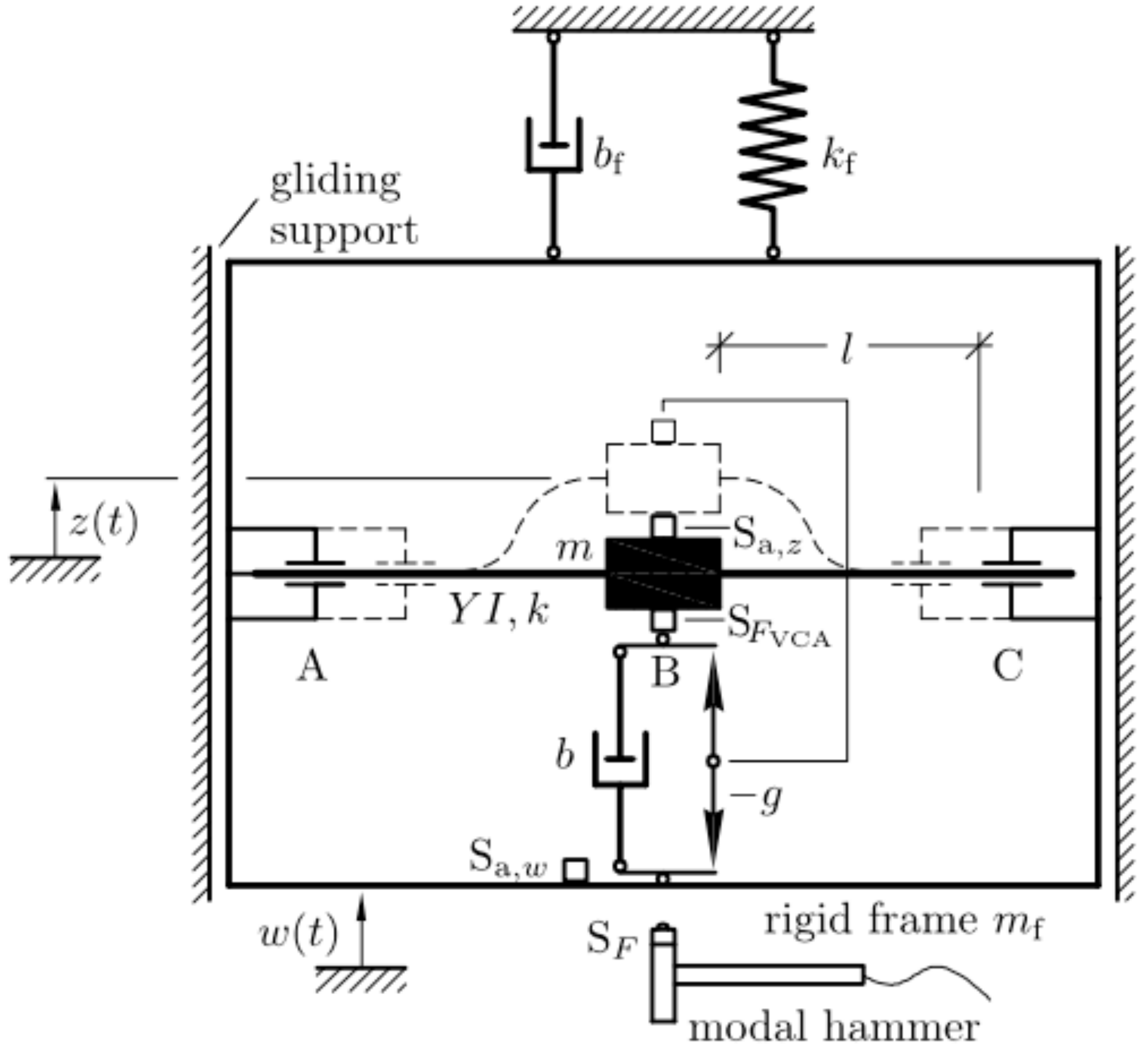
Case 1



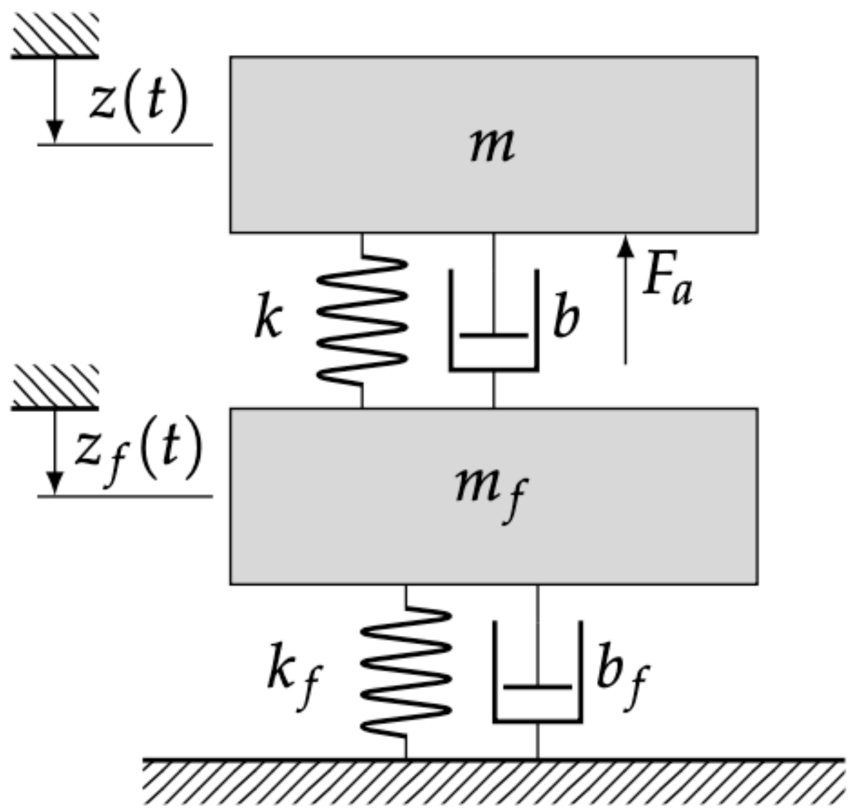
Omitting masses →



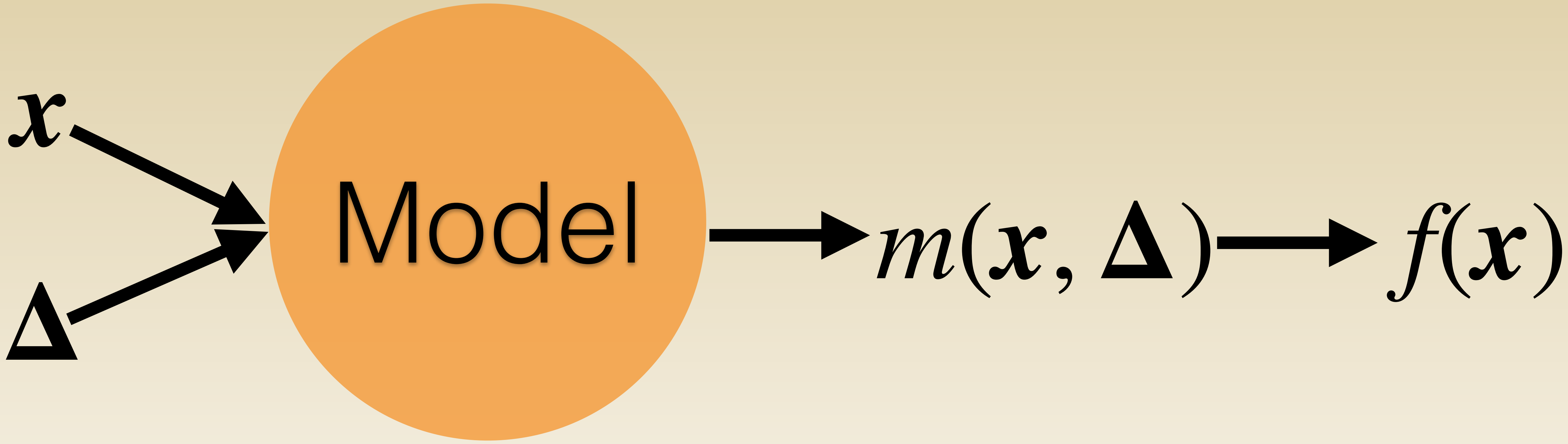
Case 2



Empirical force representations →



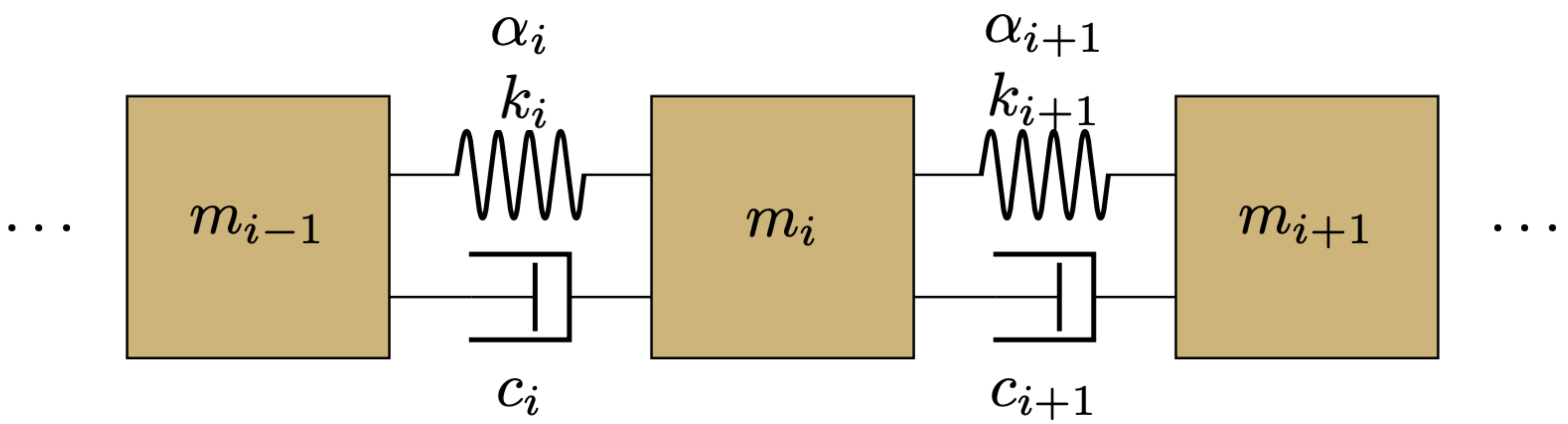
Embedded model enrichments reduce the model-form error and enable extrapolative predictions



- Δ is physics-informed $\rightarrow$ extrapolative
- Quantified uncertainties informed by Bayesian calibration
- Computational cost:

  low-fidelity model $\lessapprox$ enriched model $\ll$ high-fidelity model

Case 1: forming mass-spring-damper models



Newton's Second Law of Motion:

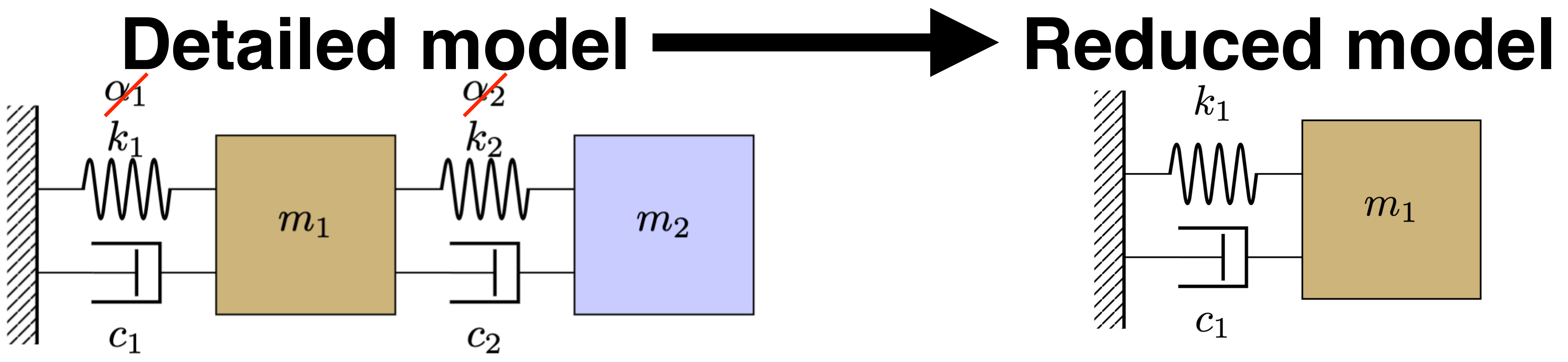
$$\mathrm{diag}(\boldsymbol{m})\ddot{\boldsymbol{x}}(t) = \sum \boldsymbol{f}$$
$$= \boldsymbol{f}_d(t) + \boldsymbol{f}_s(t)$$

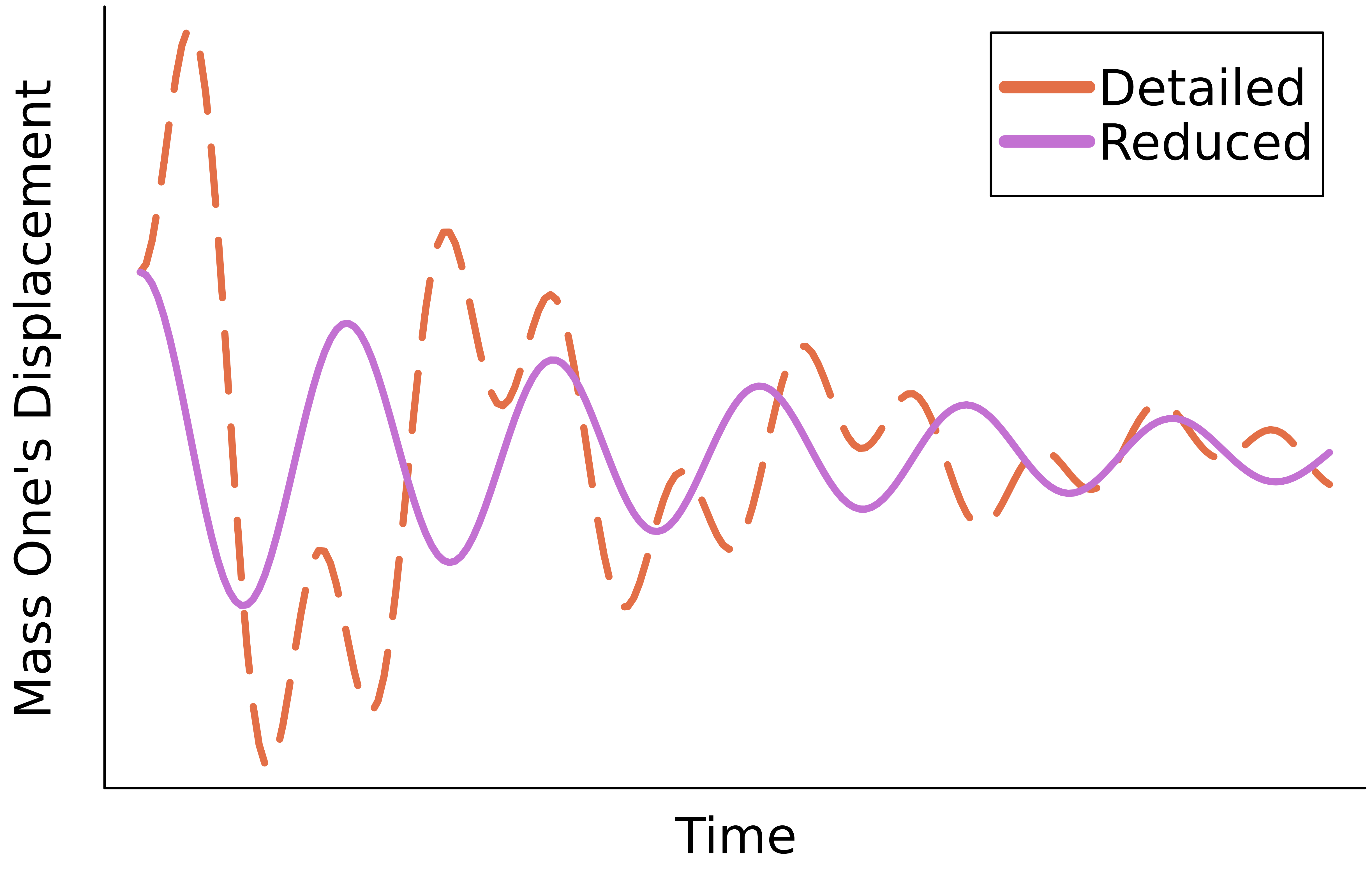
Damping ⟶ $f_{d,i}(t) = -c_i \dot{x}_i(t)$

Spring ⟶ $$f_{s,i}(t) = \Bigg[\underbrace{-k_i(x_i(t) - x_{i-1}(t)) + k_{i+1}(x_{i+1}(t) - x_i(t))}_{\text{Hooke's law}}\Bigg]\Bigg(1 + \underbrace{\alpha_i(x_{i+1}(t) - x_{i-1}(t))}_{\text{nonlinear term}}\Bigg)$$

The reduced model is formed by subsampling the detailed model

Omitting masses causes error in the reduced model



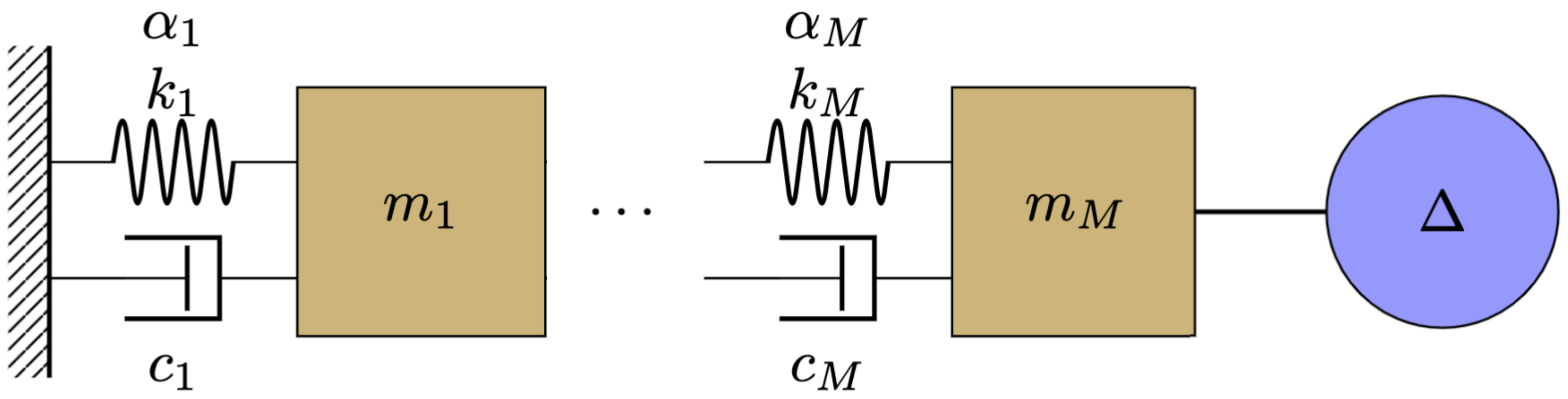


An enrichment operator is added to the reduced model to form the enriched models

$$\tilde{x}_{M+1}(t) = a\exp(-\beta t)\cos(\omega t)$$

The enrichment operator approximates the movement of mass $M+1$ with a simple oscillator

Enriched model



$$\tilde{x}_{M+1}(t) = a\exp(-\beta t)\cos(\omega t)$$

$$\ddot{x}_M(t) = \frac{1}{m_M}\left\{-c_M\dot{x}(t) + \left[-k_M\left(x_M(t) - x_{M-1}(t)\right) + \delta(\tilde{x}_{M+1}(t) - x_M(t))\right]\left(1 + \alpha_i(\tilde{x}_{M+1}(t) - x_{M-1}(t)\right)\right\}$$

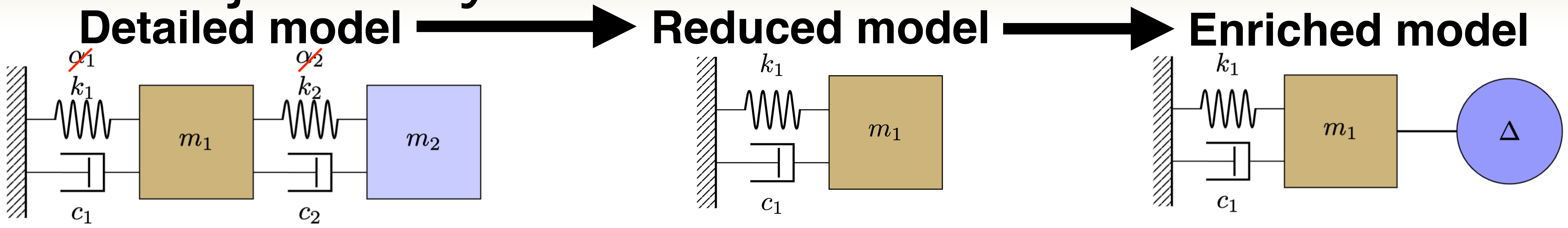
- Model parameters: $\boldsymbol{\theta} = \left(\delta, a, \beta, \omega\right)$

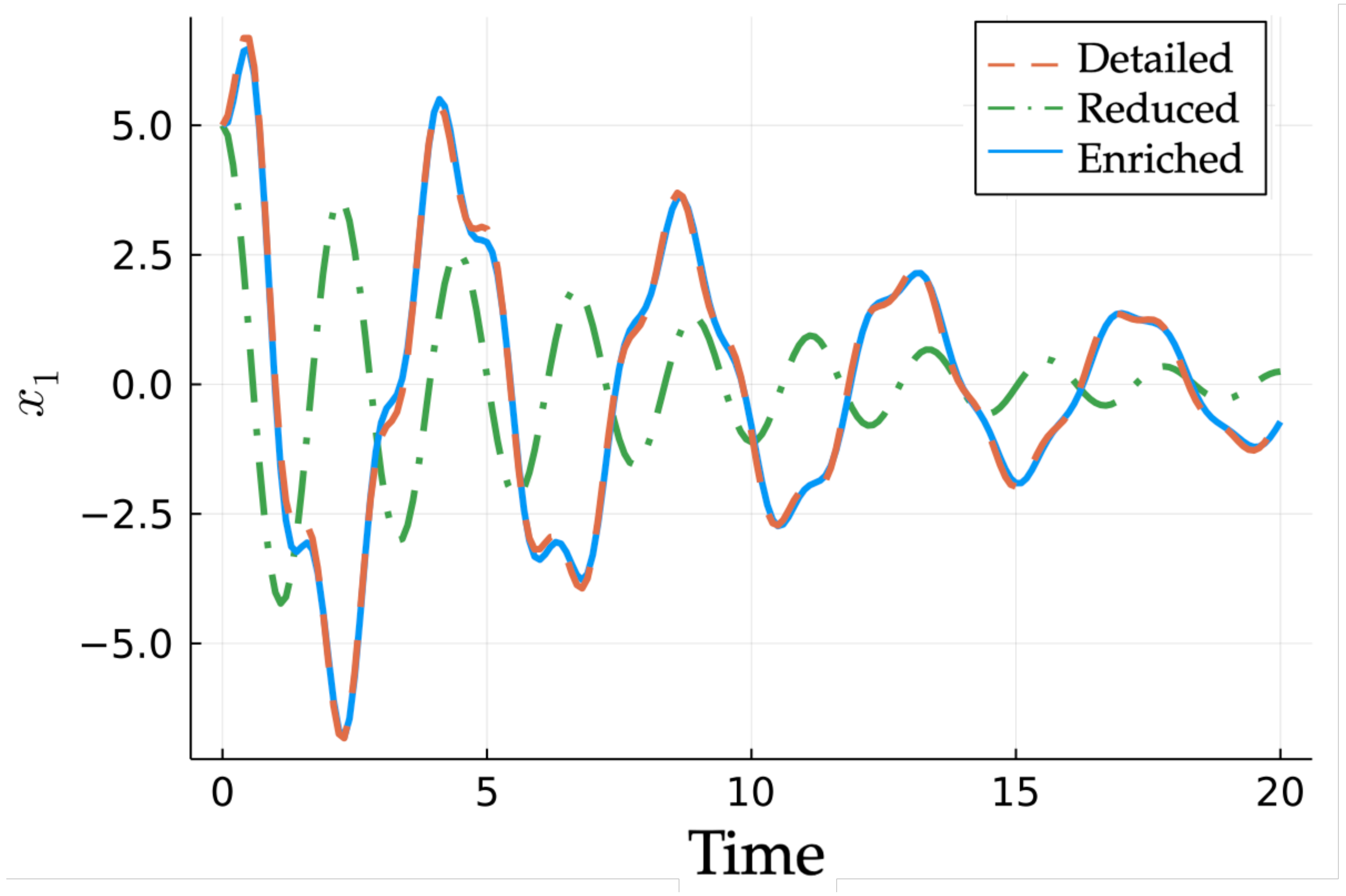
  $\theta_i \sim \mathcal{N}\left(\mu_i, \sigma_i^2\right)$, where $\mu_i \in \mathbb{R}$, $\sigma_i \in \mathbb{R}_{\geq 0}$, and $i = \{1,2,3,4\}$

- Hyperparameters: $\boldsymbol{\phi} = \left(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3, \boldsymbol{\phi}_4\right)$, where $\boldsymbol{\phi}_i = \left(\mu_i, \sigma_i\right)$

- We use hierarchical Bayesian calibration to sample a posterior on $\boldsymbol{\phi}$

The enriched model almost perfectly matches the trajectory of the **linear** detailed model

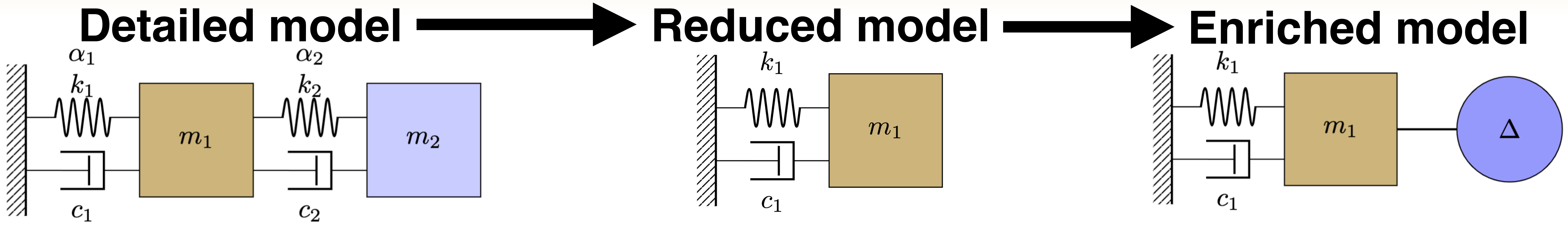


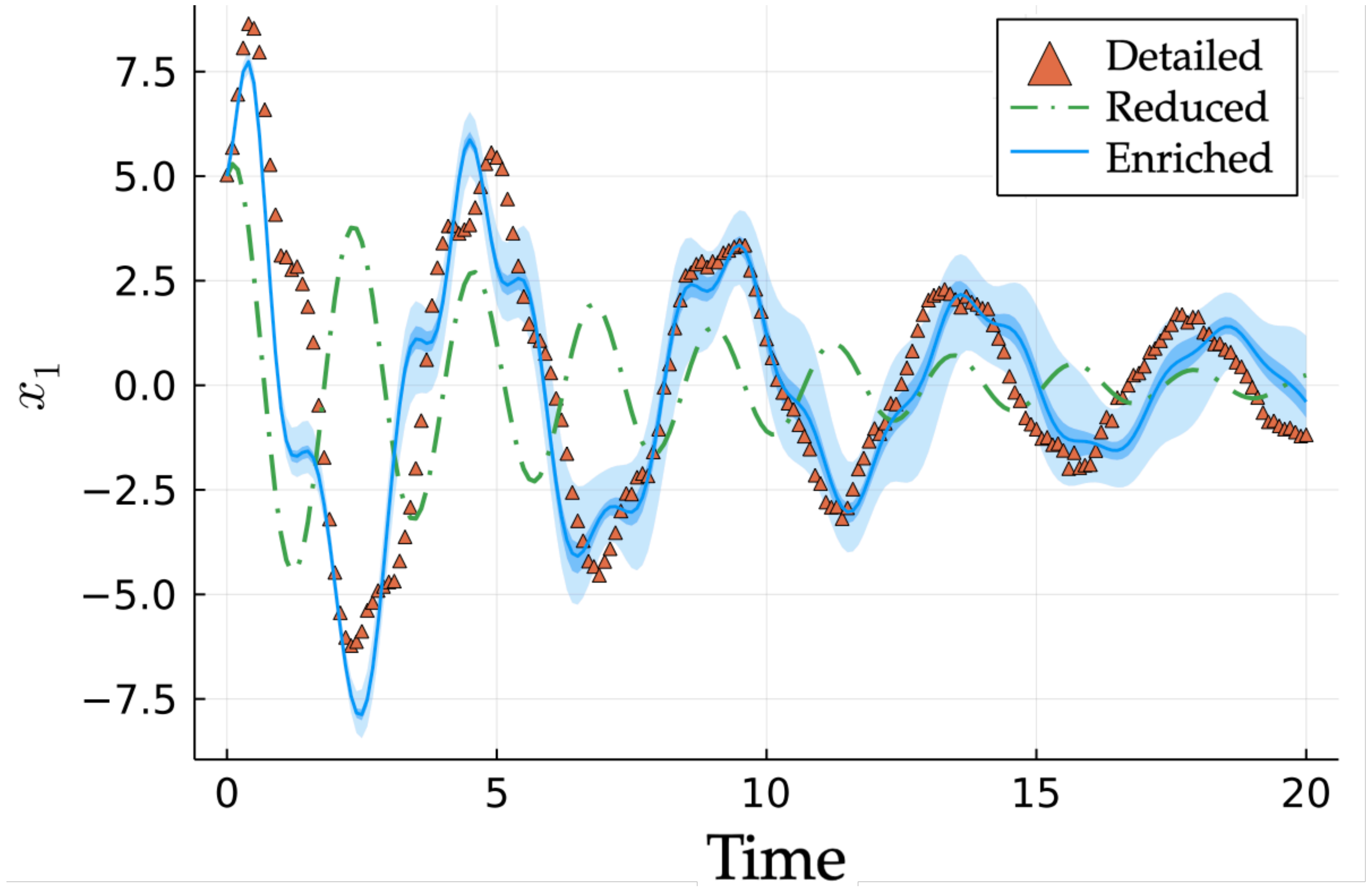


Enriched model is calibrated with mass one's displacement data.

[3] R. BANDY AND R. MORRISON, *Quantifying model form uncertainty in spring-mass-damper systems,* Model Validation and Uncertainty Quantification, Volume 3, Conference Proceedings of the Society for Experimental Mechanics Series, (2024).

Discrepancies emerge between the enriched model and the **nonlinear** detailed model





[3] R. BANDY AND R. MORRISON, *Quantifying model form uncertainty in spring-mass-damper systems,* Model Validation and Uncertainty Quantification, Volume 3, Conference Proceedings of the Society for Experimental Mechanics Series, (2024).

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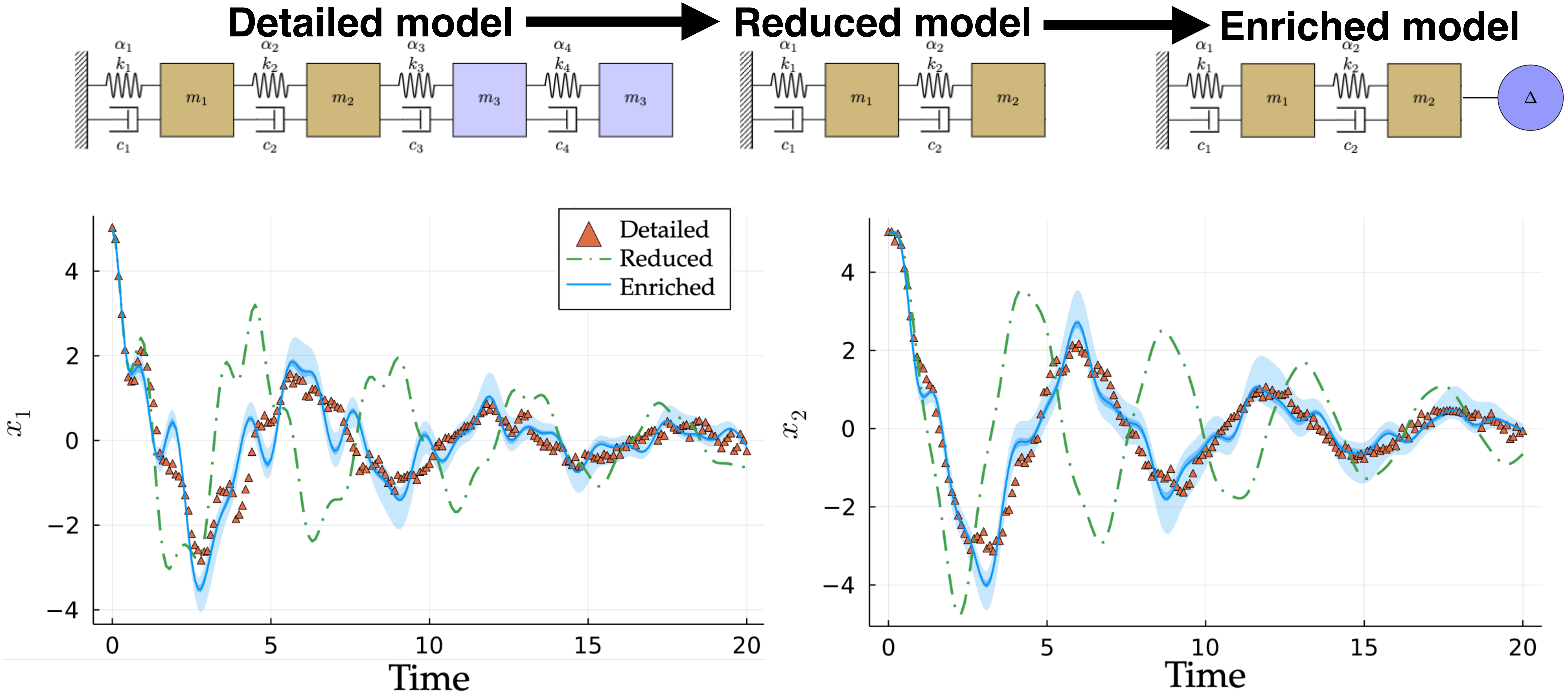
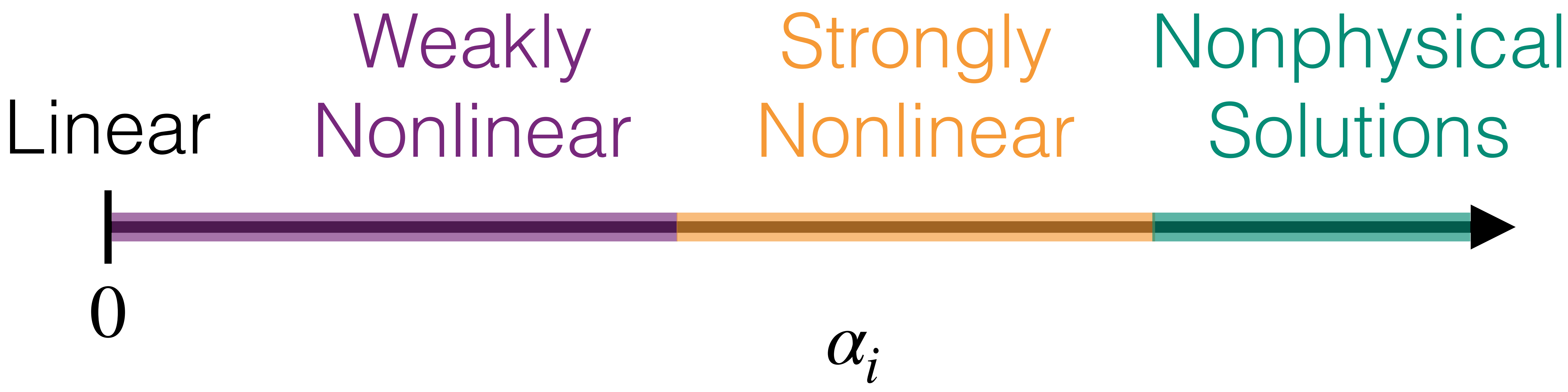


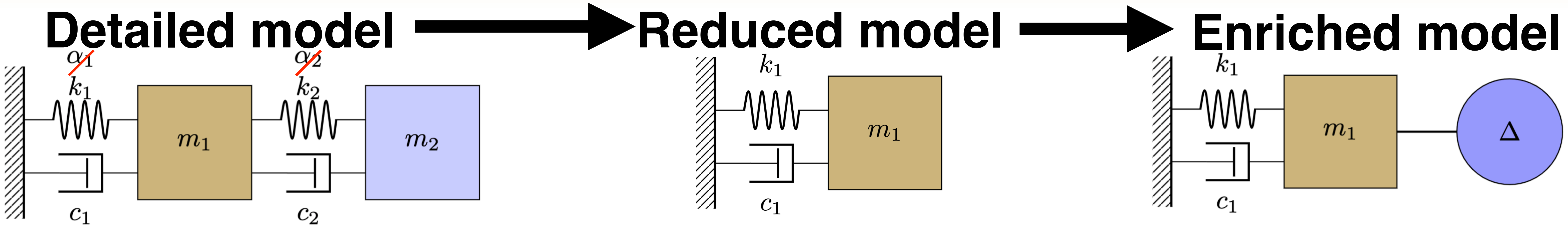
[3] R. BANDY AND R. MORRISON, *Quantifying model form uncertainty in spring-mass-damper systems,* Model Validation and Uncertainty Quantification, Volume 3, Conference Proceedings of the Society for Experimental Mechanics Series, (2024).

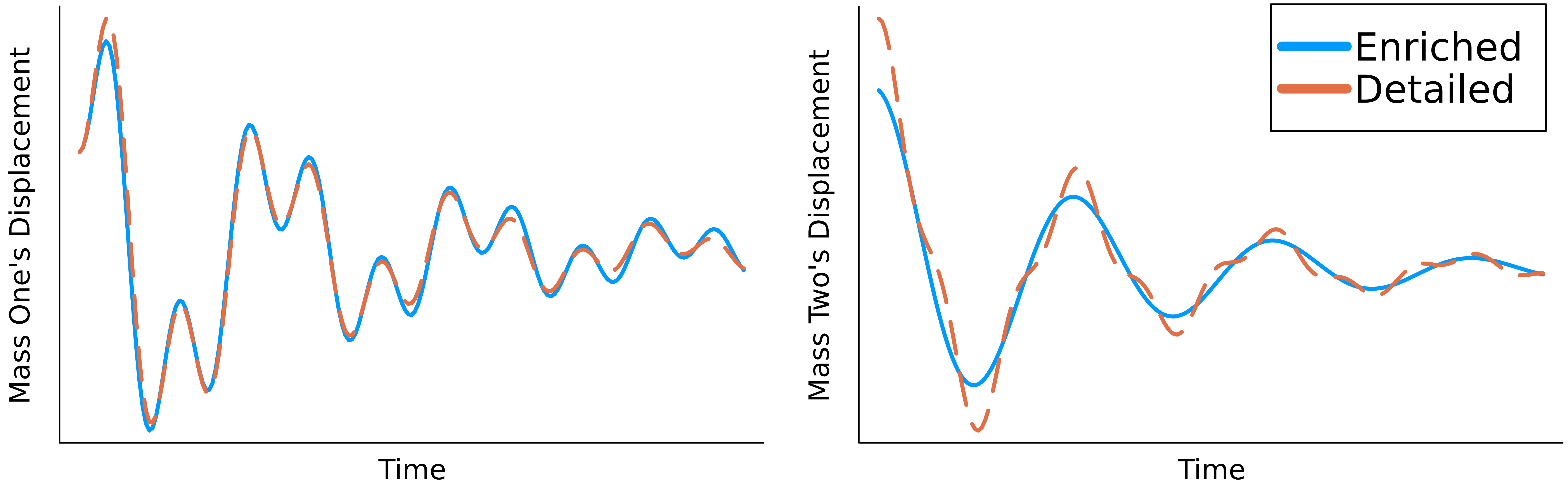
How does the degree of nonlinearity impact model discrepancy?

$$f_{s,i}(t) = \left[\underbrace{-k_i(x_i(t) - x_{i-1}(t)) + k_{i+1}(x_{i+1}(t) - x_i(t))}_{\text{Hooke's law}} \right] \left(1 + \underbrace{\alpha_i(x_{i+1}(t) - x_{i-1}(t))}_{\text{nonlinear term}} \right)$$



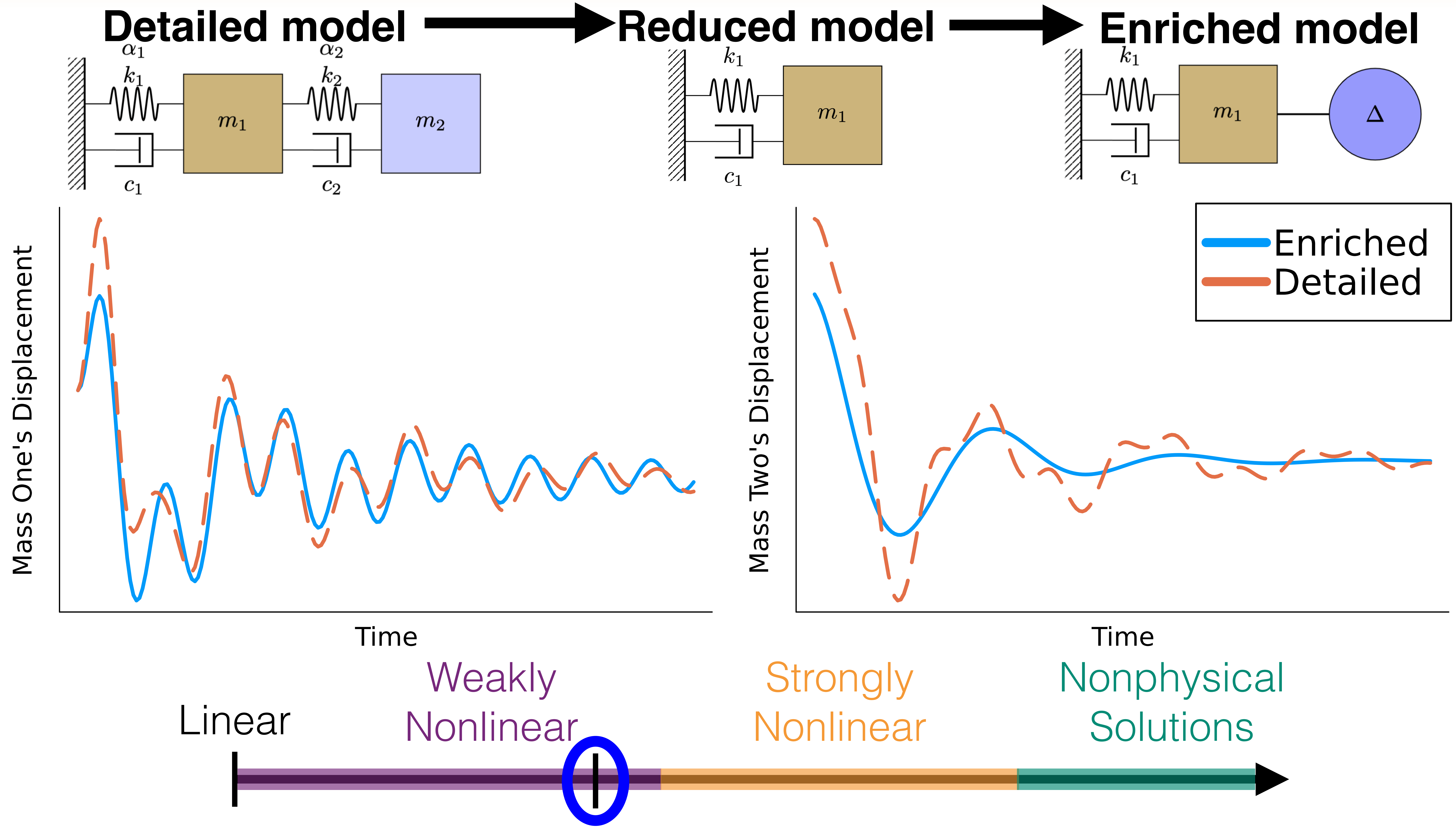
The enriched model closely matches the **linear** detailed trajectories



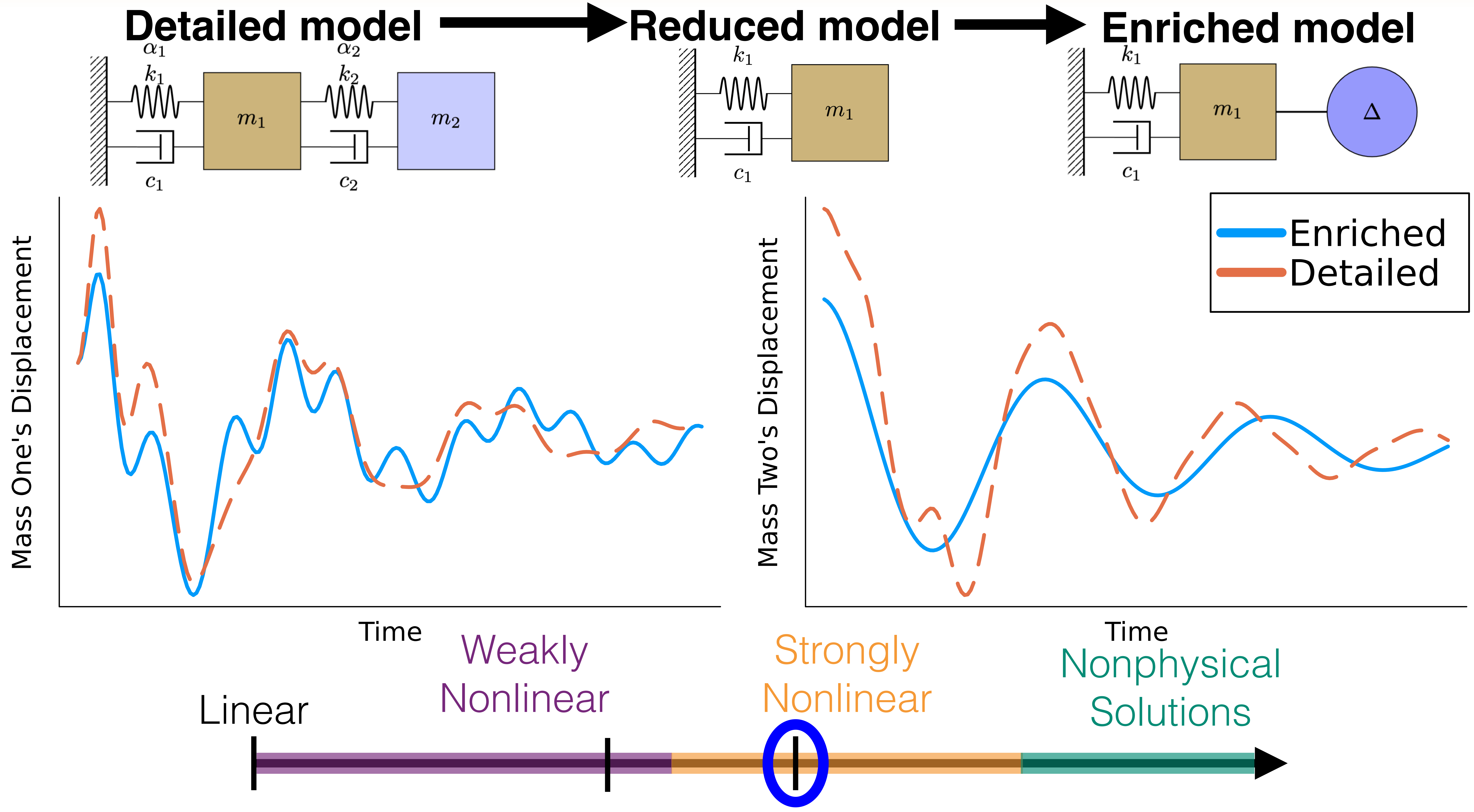


Enriched model predictions are using MAP estimates of model parameters.

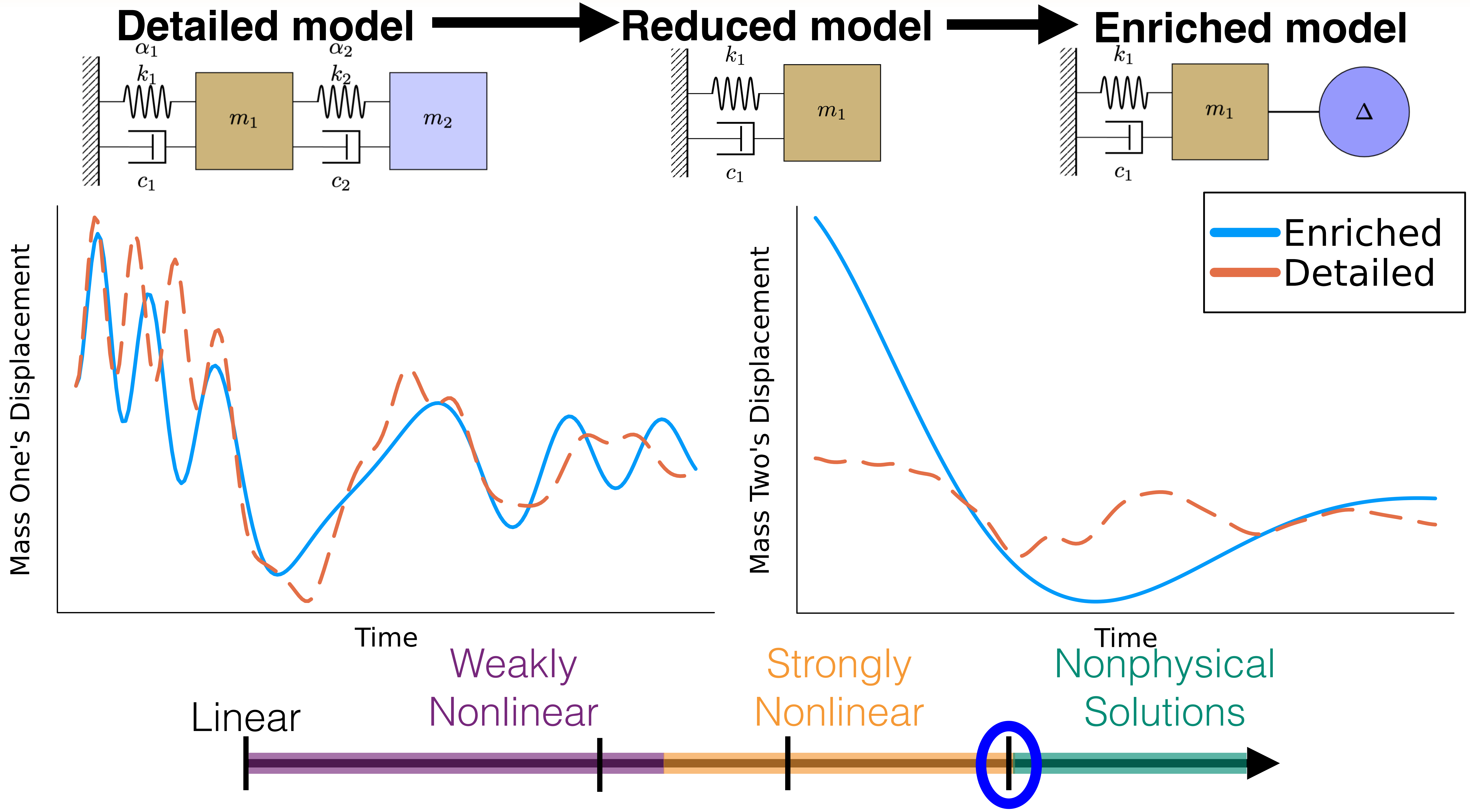
The enriched model overestimates the dampening of mass two for the **weakly nonlinear** model



The enriched model overestimates the dampening and frequency of mass two for the **stronger nonlinear** model

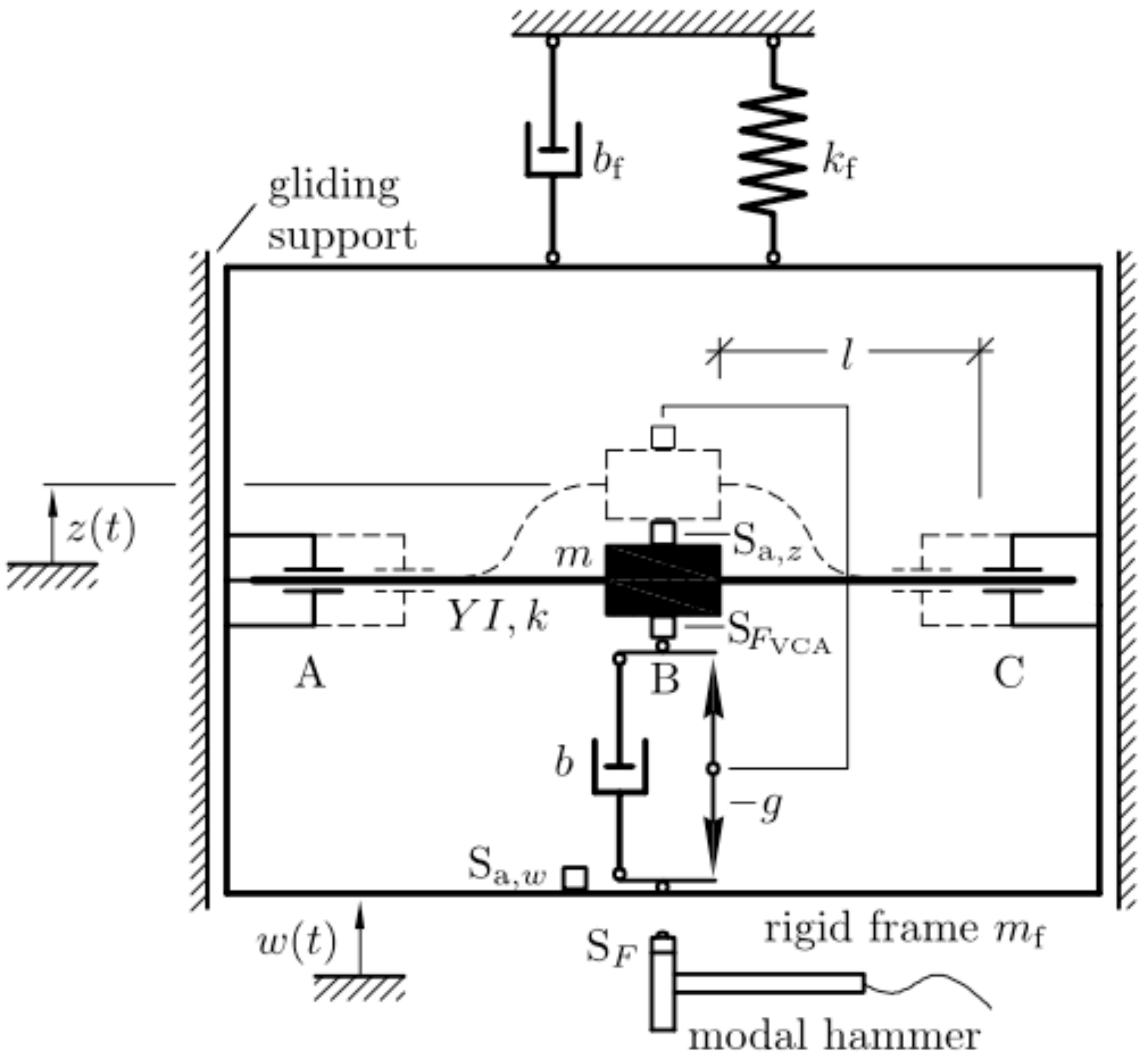


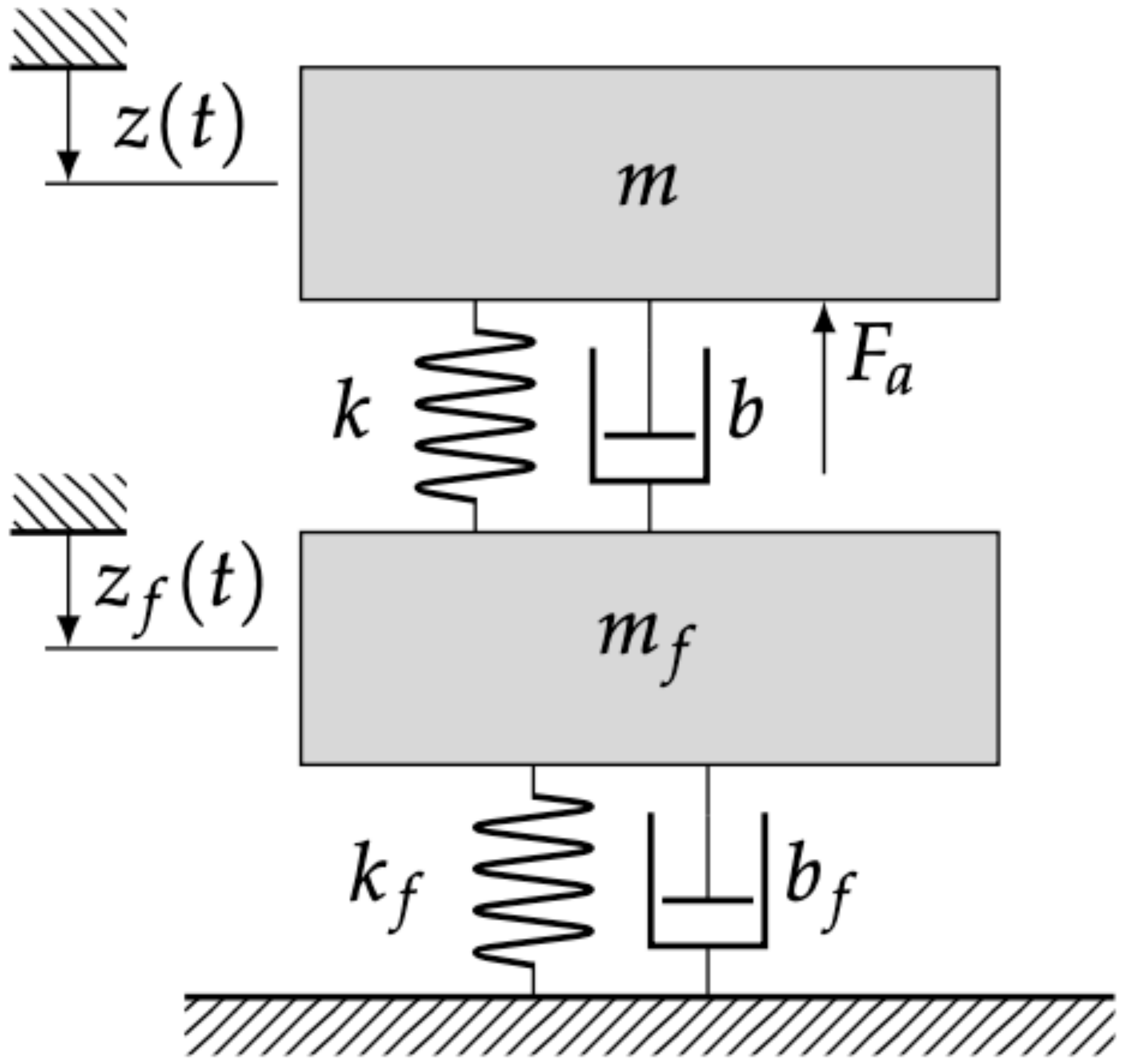
The enriched model overestimates the initial amplitude of mass two for the **strongest nonlinear** model



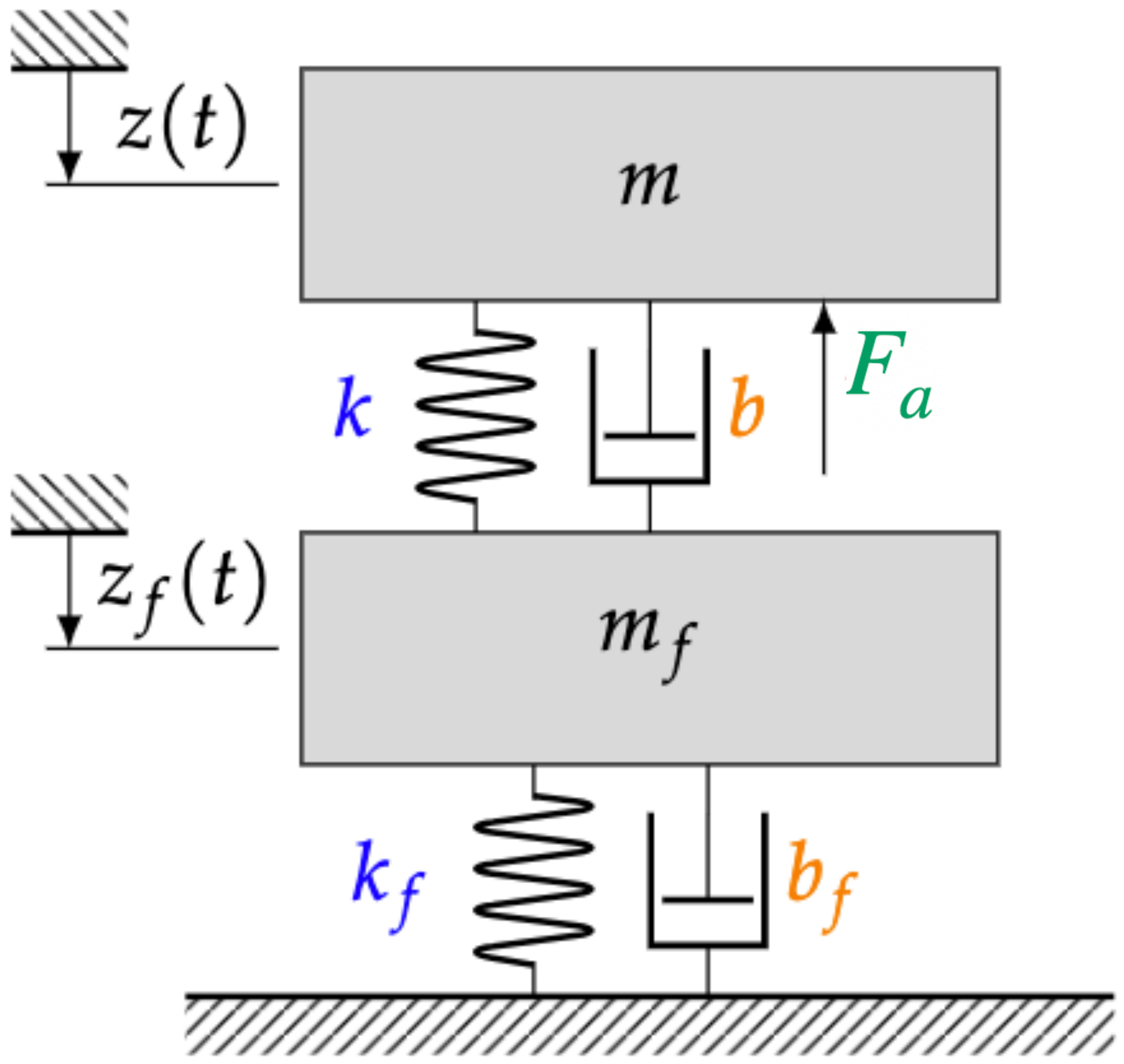
Case 2

Experiment → **Low-fidelity model**





The low-fidelity model is a two-mass oscillator with ***linear*** forces



$$g_0 \dot{z}(t)$$

$$\ddot{z}(t) = \frac{1}{m}\left\{-b\left(\dot{z}(t) - \dot{z}_f(t)\right) - F_a\left(\dot{z}(t)\right) - k\left(z(t) - z_f(t)\right)\right\}$$

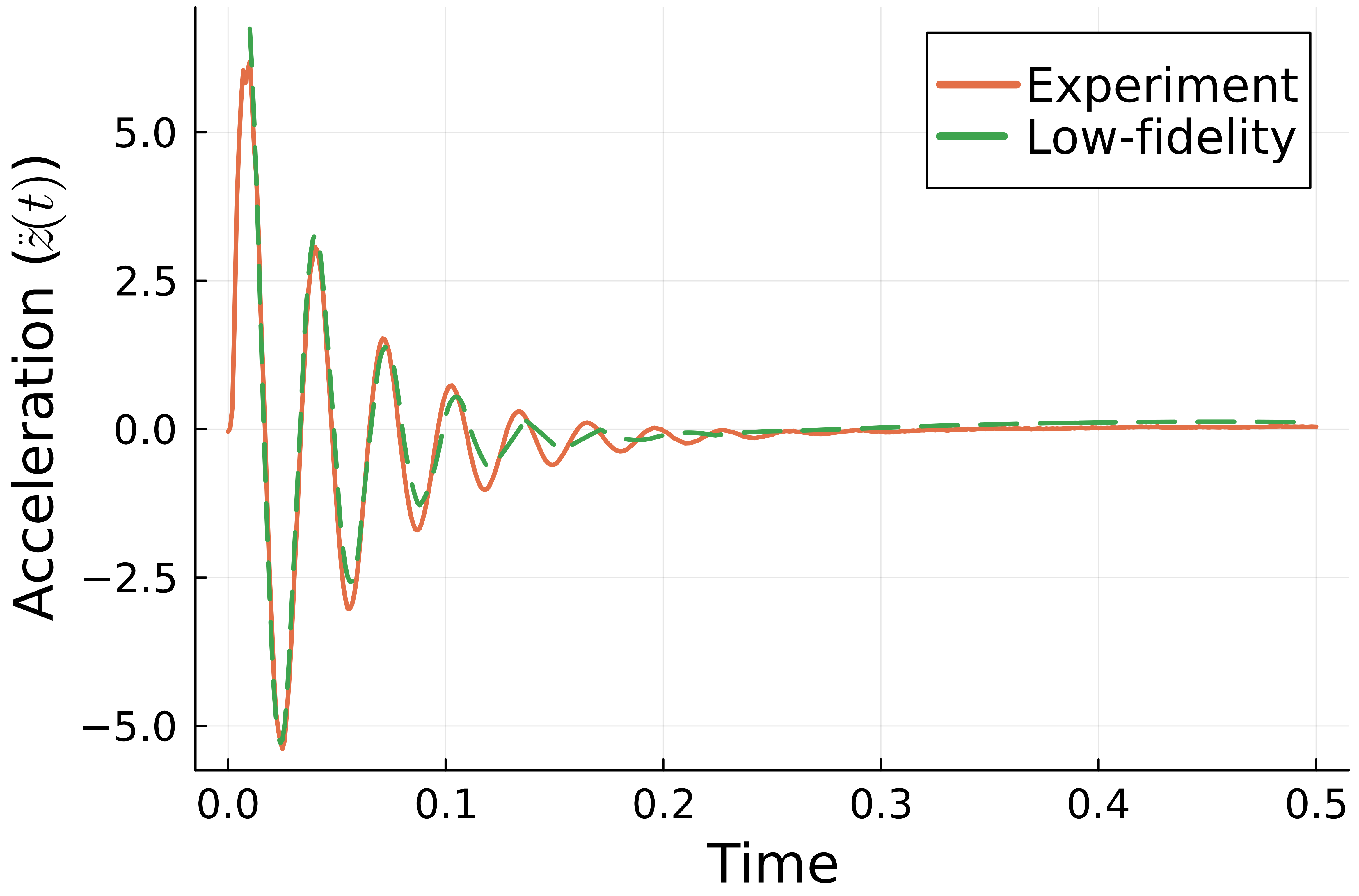
$$\ddot{z}_f(t) = \frac{1}{m_f}\left\{b\left(\dot{z}(t) - \dot{z}_f(t)\right) + k\left(z(t) - z_f(t)\right) - b_f \dot{z}_f - k_f z_f\right\}$$

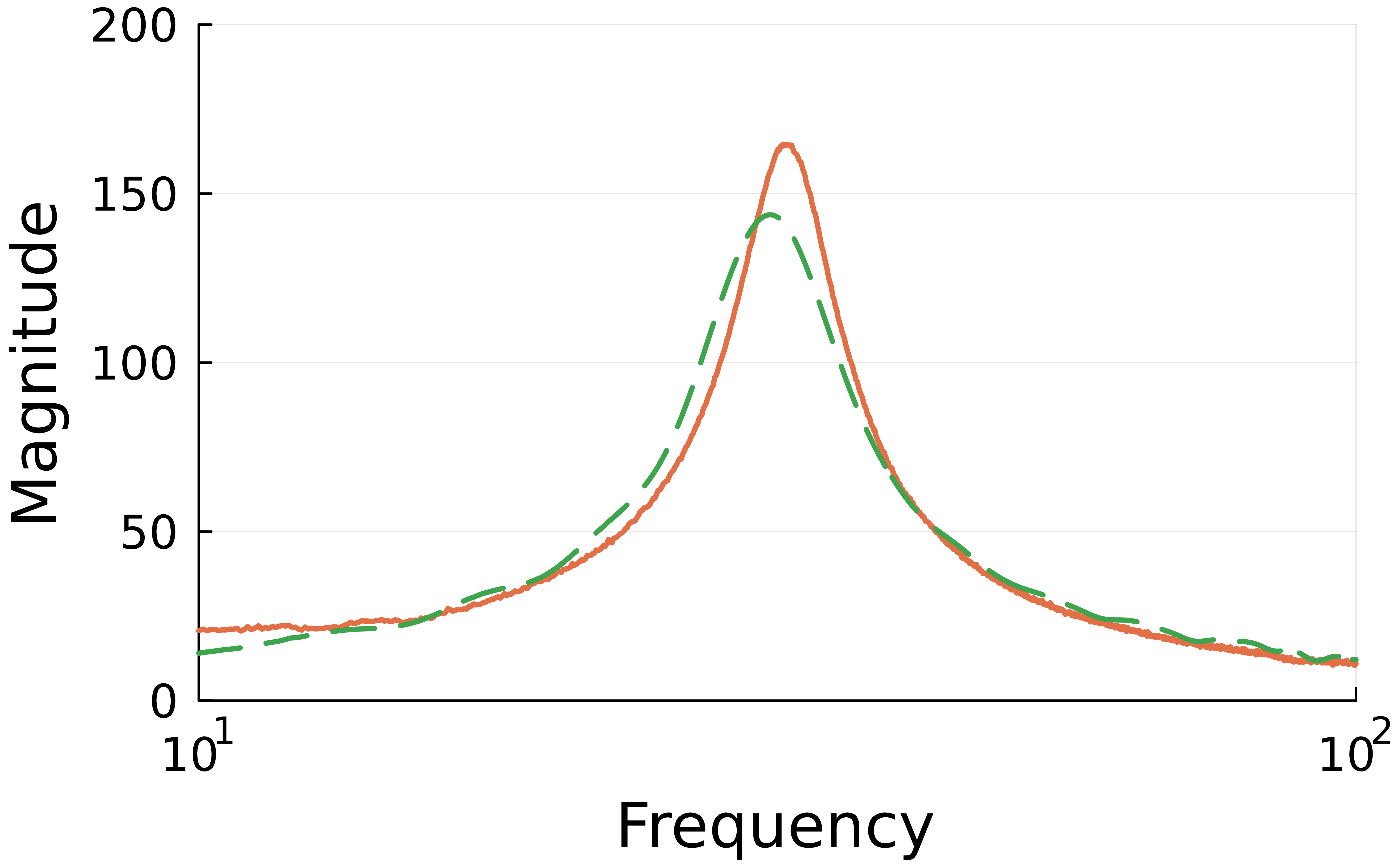
Spring force

Passive damping force

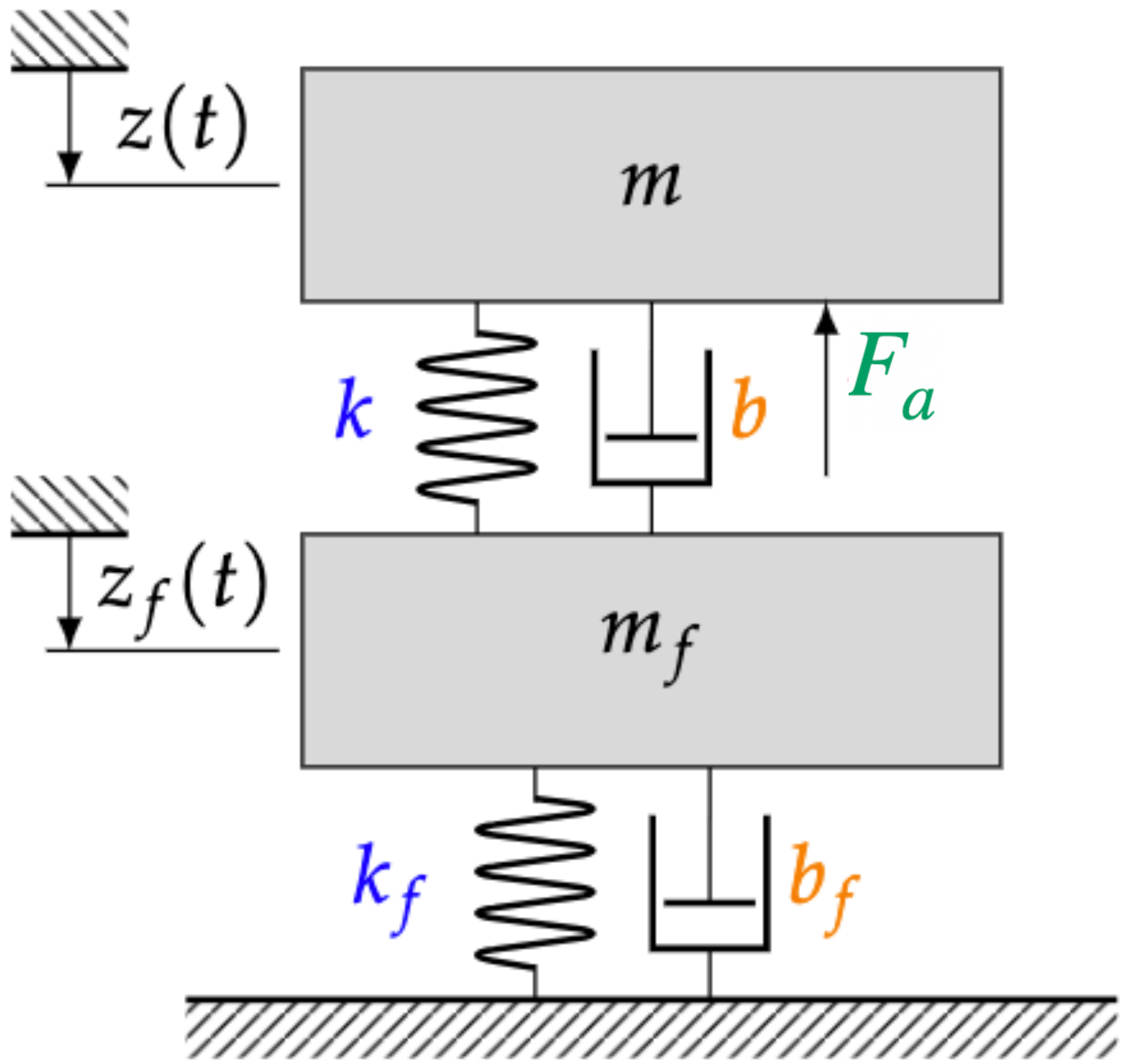
Active damping force

Assumed ***linear*** forces cause error in the low-fidelity model





An enrichment operator is embedded into the low-fidelity model



$$g_0\dot{z}(t) + g_1|\dot{z}(t)|\dot{z}(t) + g_2\dot{z}(t)^3$$

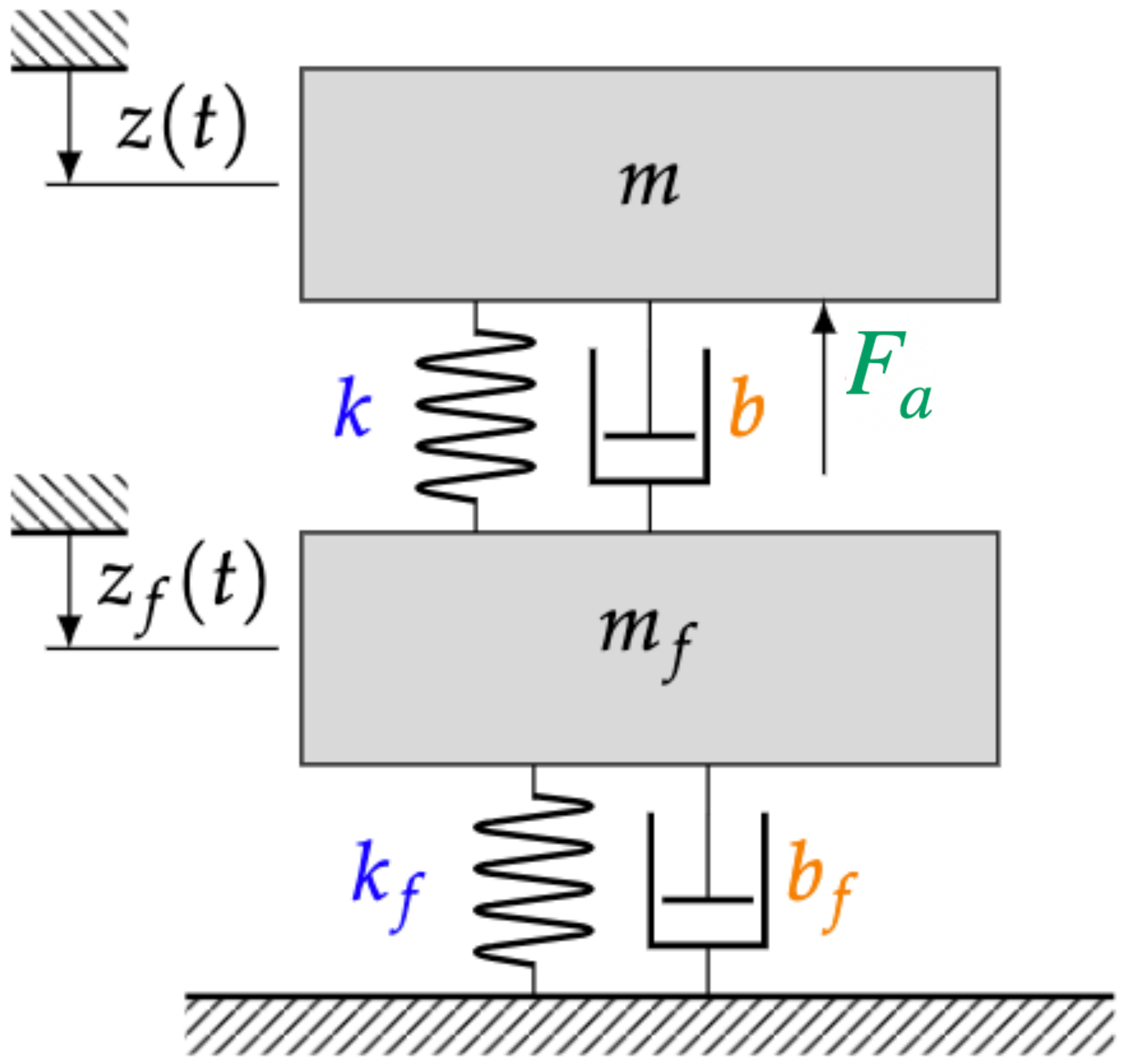
Enrichment operator

$$\ddot{z}(t) = \frac{1}{m}\left\{-b\left(\dot{z}(t) - \dot{z}_f(t)\right) - F_a\left(\dot{z}(t)\right) - k\left(z(t) - z_f(t)\right)\right\}$$

$$\ddot{z}_f(t) = \frac{1}{m_f}\left\{b\left(\dot{z}(t) - \dot{z}_f(t)\right) + k\left(z(t) - z_f(t)\right) - b_f\dot{z}_f - k_f z_f\right\}$$

[4] D. ROETTEGEN, B.R. PACINI, AND R. MAYES. *Techniques for nonlinear identification and maximizing modal response,* Nonlinear Structures and Systems, Volume 1, Conference Proceedings of the Society for Experimental Mechanics Series, (2024).

The enriched model is a two-mass oscillator with *nonlinear* active damping



$$g_0\dot{z}(t) + \underbrace{g_1|\dot{z}(t)|\dot{z}(t) + g_2\dot{z}(t)^3}_{\text{Enrichment operator}}$$

$$\ddot{z}(t) = \frac{1}{m}\left\{-b\left(\dot{z}(t) - \dot{z}_f(t)\right) - F_a\left(\dot{z}(t)\right) - k\left(z(t) - z_f(t)\right)\right\}$$

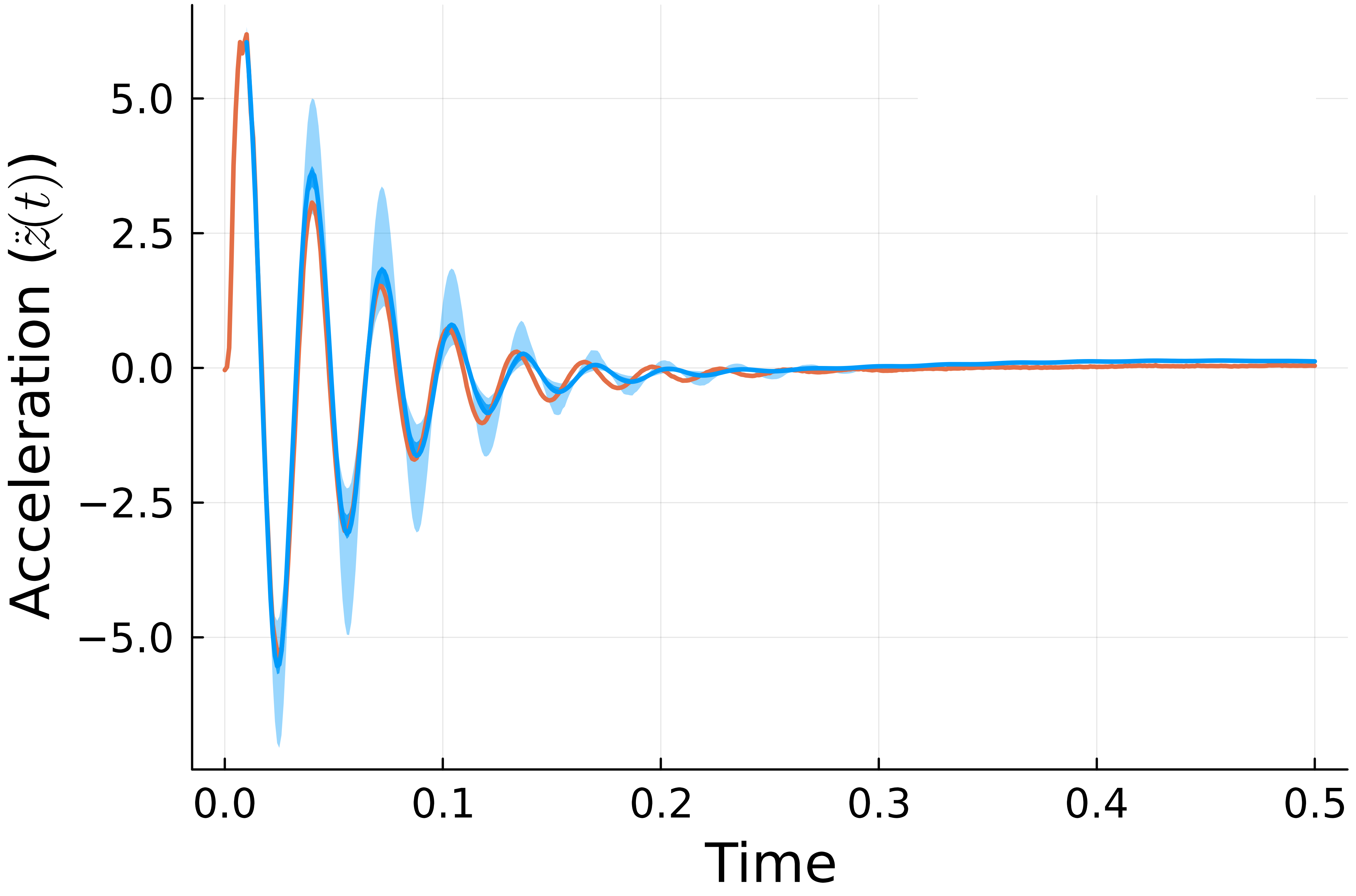
- Model parameters: $\boldsymbol{\theta} = (g_1, g_2)$

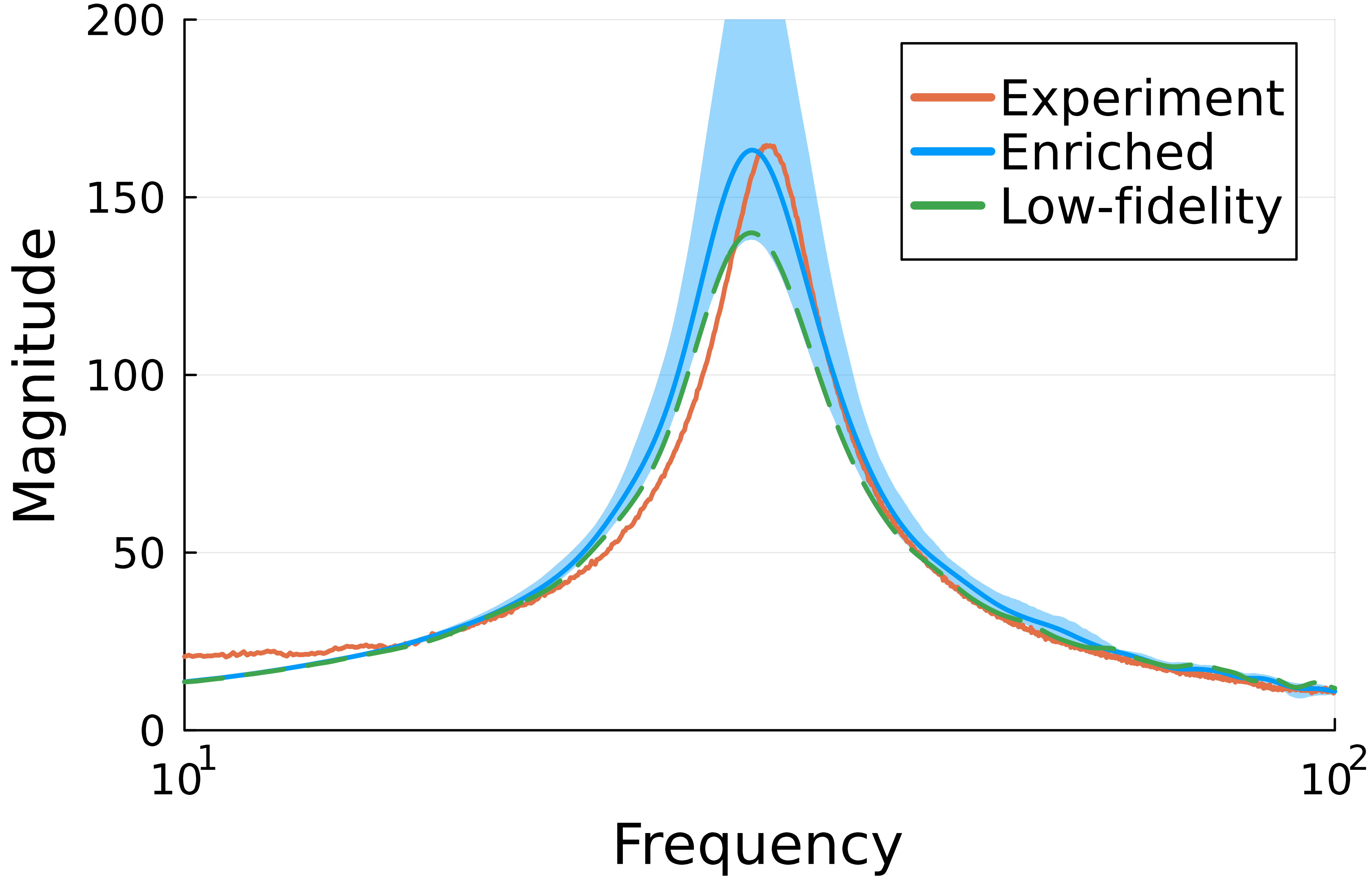
  $g_i \sim \mathcal{N}\left(\mu_i, \sigma_i^2\right)$, where $\mu_i \in \mathbb{R}$, $\sigma_i \in \mathbb{R}_{\geq 0}$, and $i = \{1,2\}$

- Hyperparameters: $\boldsymbol{\phi} = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$, where $\boldsymbol{\phi}_i = (\mu_i, \sigma_i)$

- We use hierarchical Bayesian calibration to sample a posterior on $\boldsymbol{\phi}$

The enriched model covers most experimental observations





[5] R. BANDY, T. PORTONE, AND R. MORRISON, *Stochastic model correction for the adaptive vibration isolation round-robin challenge, Model Validation and Uncertainty Quantification*, Volume 3, Conference Proceedings of the Society for Experimental Mechanics Series, (to be released).

Conclusions

- Mass-spring-damper models illustrate model-form error that can arise in many structural dynamics applications.
- Expert knowledge about a potential source of model-form error informs the enrichment operators.
- Enriched models decrease discrepancies and retains interpretability.

Questions

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