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Randomized Benchmarking of Spin Qudits

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Motivation

Qudits are not like “a bunch of qubits.” They have different physical realizations, modes of operation, and errors.

Primary example: spin qudits. Spin j → dimension $2j + 1$.

Hilbert space admits a natural action (irrep) of SU(2) = group of single-qubit unitaries. Suggests encoding a logical qubit. Can realize logical operations on spin codes.

[Gross ‘20]

Goal: develop a theoretical protocol for benchmarking native SU(2) rotations inside SU($2j + 1$) on a spin-j Hilbert space → FTQC with spin qudits.

Randomized Benchmarking

Idea: run many circuits of *m* randomly sampled gates $g_1, \ldots, g_m$ plus an inversion gate $g_{inv} = (g_m \cdots g_1)^\dagger$. Averaged survival probability $\langle end | g_{inv} g_m \cdots g_1 | start \rangle$ decays exponentially with *m*. Decay rate = average gate error rate.

Why? Benchmarking group $U(2^q) \supset G \ni g$ has superoperator representation

$$\mathcal{G}|\rho\rangle\rangle = |g\rho g^\dagger\rangle\rangle, \quad \mathcal{G} = \bigoplus_\lambda \phi_\lambda(g).$$

quality parameters depend only on gate set

coefficients depend only on SPAM

Noisy implementation: $\widetilde{\mathcal{G}} = \mathcal{E}\mathcal{G} \implies p_m = \frac{1}{|G|^m} \sum_{g_1, \ldots, g_m} \langle\langle \mathrm{end} | \widetilde{\mathcal{G}}_{\mathrm{inv}} \widetilde{\mathcal{G}}_m \cdots \widetilde{\mathcal{G}}_1 | \mathrm{start} \rangle\rangle = \sum_\lambda A_\lambda f_\lambda^m.$

projector onto irrep

Principle: RB only sensitive to twirled error channel $\frac{1}{|G|} \sum_{g \in G} \mathcal{G}^\dagger \mathcal{E} \mathcal{G} = \sum_\lambda f_\lambda \mathcal{P}_\lambda.$

Smaller benchmarking group → superoperator representation more reducible → more information, but harder to analyze.

Character Randomized Benchmarking

Goal: isolate quality parameters associated with each irrep. $$p_m = \sum_\lambda A_\lambda f_\lambda^m$$

[Helsen, Xue, Vandersypen, Wehner '18; Claes, Rieffel, Wang '20]

Idea: choose a "character group" $H \subset G$ whose superoperator representation has an irrep $\varphi_{\lambda'}$ inside the desired irrep φ_λ of G (e.g., $H = G$, $\lambda = \lambda'$). Then estimate average **weighted** survival probability

character of irrep $\varphi_{\lambda'}$

$$p_m = \frac{\dim \phi_{\lambda'}}{|H||G|^m} \sum_h \sum_{g_1,\dots,g_m} \chi_{\lambda'}(h)^* \langle\langle \mathrm{end}|\widetilde{\mathcal{G}}_{\mathrm{inv}}\widetilde{\mathcal{G}}_m \cdots \widetilde{\mathcal{G}}_2 \underbrace{\widetilde{\mathcal{G}_1 \mathcal{H}}|\mathrm{start}\rangle\rangle}_{} \propto f_\lambda^m$$

→ single exponential decay.

Principle: $$\frac{\dim \phi_{\lambda'}}{|H|} \sum_{g\in H} \chi_{\lambda'}(h)^* \mathcal{H} = \mathcal{P}_{\lambda'}.$$

Example: Pauli character RB.

SU(2) Operations

$$(J_x \pm iJ_y)|j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle$$
$$J_z|j,m\rangle = m|j,m\rangle$$

Superoperator representation of SU(2) is highly reducible: $j \otimes j^* = 0 \oplus 1 \oplus \cdots \oplus 2j.$

(pseudo)real

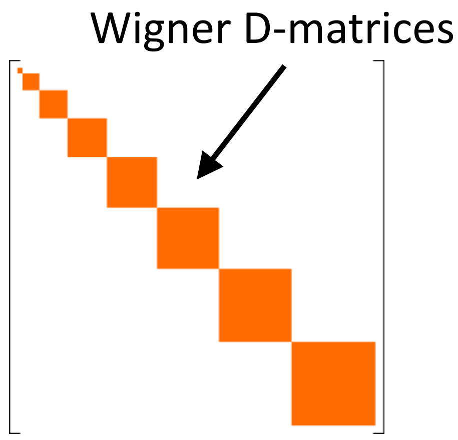
Any spin-*j* operator can be expanded in spherical tensors:

rank k = 0, 1, …, 2j
(degree k in J_x, J_y, J_z)

$$T_q^{(k)} = \sqrt{\frac{2k+1}{2j+1}} \sum_{m,m'=-j}^{j} C^{j,m}_{j,m';k,q} |j,m\rangle\langle j,m'|.$$

q = -k, …, k

→ Superoperator representation is block diagonal: $g \mapsto \mathcal{G}(g) =$



$$(2j+1)^2 = 1 + 3 + \cdots + (4j+1)$$

Any spin-*j* **super**operator can be expanded in Choi units:

$$\rho \mapsto T_q^{(k)} \rho (T_{q'}^{(k')})^\dagger.$$

We expect low-rank errors to dominate (cf. spin codes).

[Gross '20; Omanakuttan, Gross '23; Omanakuttan, Buchemmavari, Gross, Deutsch, Marvian '24]

$$\left(g_{k'}^{(k)} = \frac{1}{(2j+1)^2} \sum_{q=-k}^{k} \sum_{q'=-k'}^{k'} (-1)^{q+q'} \sum_{m=-j}^{j} C_{k,q;j,m}^{j,m+q} C_{k',q';j,m+q}^{j,m+q+q'} C_{k,-q;j,m+q+q'}^{j,m+q'} C_{k',-q';j,m+q'}^{j,m} \right)$$

SU(2) Randomized Benchmarking

Novelties: • Infinite (but compact) group. • Only viable character group is SU(2) itself. • Infinite character group increases sampling complexity. BUT finitely many group elements suffice to build a projector. Suggests generalization of character RB.

SU(2)-twirled error channel can be expanded in twirled Choi units or irrep projectors:

$$\langle \mathcal{G}^\dagger \mathcal{E} \mathcal{G} \rangle_{SU(2)} = \sum_{k=0}^{2j} p_k \mathcal{G}^{(k)} = \sum_{k=0}^{2j} \gamma_k \mathcal{P}_k \qquad \mathcal{G}^{(k)} : \rho \mapsto \frac{1}{2k+1} \sum_{q=-k}^{k} T_q^{(k)} \rho (T_q^{(k)})^\dagger$$

probability of weight-k SU(2) error

RB quality parameters

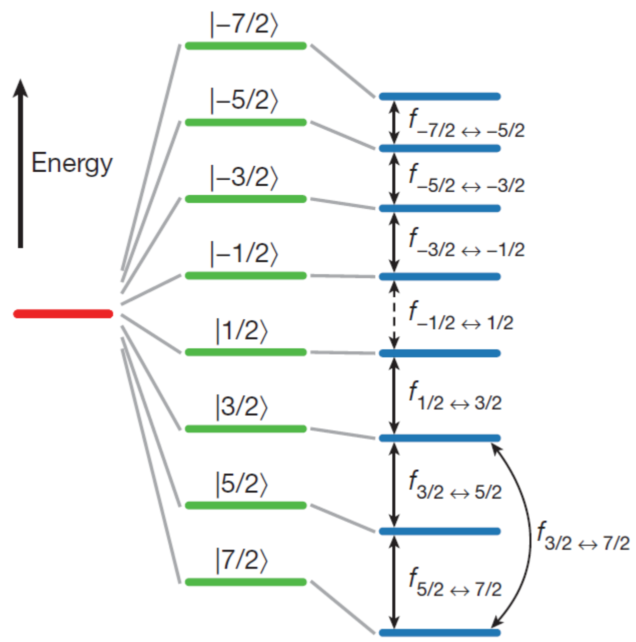
(uniformly random weight-k errors)

Coefficients related by: $\gamma_{k'} = \sum_{k=0}^{2j} g_{k'}^{(k)} p_k, \quad \mathcal{G}^{(k)} : T_{q'}^{(k')} \mapsto g_{k'}^{(k)} T_{q'}^{(k')}$

→ Generalized SU(2) character RB can distinguish rates of low-*k* vs. high-*k* errors.

Example

Spin-7/2 nuclear spin of antimony (^{123}Sb). Irreps k = 0, 1, …, 7.



[Asaad et al. ‘20]

$$\vec{\gamma} = M\vec{p}, \qquad M = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{59}{63} & \frac{17}{21} & \frac{13}{21} & \frac{23}{63} & \frac{1}{21} & -\frac{1}{3} & -\frac{7}{9} \\ 1 & \frac{17}{21} & \frac{7}{15} & \frac{1}{21} & -\frac{1}{3} & -\frac{11}{21} & -\frac{1}{3} & \frac{7}{15} \\ 1 & \frac{13}{21} & \frac{1}{21} & -\frac{31}{77} & -\frac{101}{231} & \frac{1}{77} & \frac{17}{33} & -\frac{7}{33} \\ 1 & \frac{23}{63} & -\frac{1}{3} & -\frac{101}{231} & \frac{1}{9} & \frac{103}{231} & -\frac{1}{3} & \frac{7}{99} \\ 1 & \frac{1}{21} & -\frac{11}{21} & \frac{1}{77} & \frac{103}{231} & -\frac{33}{91} & \frac{53}{429} & -\frac{7}{429} \\ 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{17}{33} & -\frac{1}{3} & \frac{53}{429} & -\frac{1}{39} & \frac{1}{429} \\ 1 & -\frac{7}{9} & \frac{7}{15} & -\frac{7}{33} & \frac{7}{99} & -\frac{7}{429} & \frac{1}{429} & -\frac{1}{6435} \end{bmatrix}$$

Summary

- Spin qudits provide an elegant platform for QEC and FTQC.

- We have developed a protocol for benchmarking spin qudits using the representation theory of SU(2).

- This suggests interesting generalizations of character RB without a group structure (applicable beyond spin qudits).

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