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Third-Order Elastic Solution of the Stress Field Around a Wellbore

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Abstract

The stress distribution around a wellbore that is drilled in consolidated granular rock is solved numerically. The material is modeled as an isotropic third-order elastic material, and the material constants are taken from experimental data. The nonlinear problem is solved for transverse isotropic, and orthotropic stress boundary conditions. The tangential stress around the well face is different than the stress obtained from the from the linear elastic solution. This modifies the far-field stress that is interpreted of from hydraulic fracturing data. The stiffness components are considerably different from those of a related linear elastic material. Specifically, the resulting inhomogeneous stiffness suggests that surface, tube, and flexural waves, have two different wave velocities in two different polarizations that are determined by the far-field stress.

1. Introduction

Within a certain range of strain, consolidated granular materials may be characterized as nonlinear elastic solids. The nonlinearity can be easily observed by examining the effect of stress on the acoustical properties of the material (Walsh, 1965; Nur and Simmons, 1969). Ignoring damage evolution and failure that occur in higher strains and the hysteretic behavior due to intergranular friction, the material can be modeled as a nonlinear *hyperelastic* solid. A simple example of such a model is formulating the strain energy as a third-order polynomial of the strain invariants (Landau and Lifshitz, 1959, end of §26). This model is limited in the sense that the material is assumed to be isotropic with respect to the stress free state, and that the mechanical response of the material is described by only five material constants. Nevertheless, this model is appealing because it naturally exhibits stress dependent stiffness and stress induced anisotropy, and it allows a different mechanical response to positive and negative volume changes. In this work, this model is used to calculate the stress field around a wellbore.

Many well logging tools use acoustics (e.g., tube, surface, torsion, and flexural waves) to detect pore fluids and ore in the surrounding granular rock (White, 1965). By modeling the rock as an isotropic third-order elastic material the effects of the inhomogeneous stiffness and the stress induced anisotropy may be examined.

Analysis of the tangential stress around a wellbore in an isotropic third-order elastic (TOE) material yields different results than the same analysis in the related isotropic linear elastic (LE) material (i.e., both materials have the same stiffness tensor at the stress free state). This difference modifies the far-field stress that is interpreted of from hydraulic fracturing data (Hickman and Zoback; 1983).

The analysis in the present work is static and pore fluid effects are ignored.

2. Problem Formulation

The focus of this work is the stress field around a vertical wellbore of radius $r=a$ that is drilled in a nonlinear elastic material. A cylindrical coordinate system $\{r, q, z\}$ is chosen with origin at ground level such that z points down along the wellbore axis. The axial stress S_{zz} is determined by the over-burden and it is assumed that the axial displacement u_z is only a function of z (this assumption should be valid away of the well head). Furthermore, it is assumed that the radial displacement u_r and tangential displacement u_θ are independent of z. The radial tangential and axial displacements are therefore given by

$$u_r = u(r,\theta)\,, \tag{1a}$$

$$u_\theta = v(r,\theta) \ , \tag{1b}$$

$$u_z = \varepsilon_z z \ . \tag{1c}$$

According to the small strain approximation, the components of the strain tensor are then given by

$$E_{rr} = u' = \frac{\partial u}{\partial r} \ , \tag{2a}$$

$$E_{r\theta} = \frac{1}{2}\left(\frac{\partial u}{r\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right) \ , \tag{2b}$$

$$E_{\theta\theta} = \frac{u}{r} + \frac{\partial v}{r\partial\theta} \ , \tag{2c}$$

$$E_{zz} = \varepsilon_z \ , \tag{2d}$$

$$E_{rz} = E_{\theta z} = 0 \ , \tag{2e}$$

where $E_{ij} = E_{ji}$. The equilibrium equations for the problem are

$$\frac{\partial S_{rr}}{\partial r} + \frac{\partial S_{r\theta}}{r\partial\theta} + \frac{S_{rr} - S_{\theta\theta}}{r} = 0 \ , \tag{3a}$$

$$\frac{\partial S_{r\theta}}{\partial r} + \frac{\partial S_{\theta\theta}}{r\partial\theta} + 2\frac{S_{r\theta}}{r} = 0 \ , \tag{3b}$$

where S_{rr} ,$S_{r\theta}$ and $S_{\theta\theta}$ are the polar coordinate components of stress tensor.

The pressure in the wellbore can be measured directly and is assumed to be known

$$S_{rr\,(r=a)} = S_a \ . \tag{4}$$

The radial stress at the far field $r=b>>a$ and the principal directions ($\theta=0$, $\theta=\pi/2$) may be measured indirectly (Hickman and Zoback, 1983) a... are also assumed to be known

$$S_{rr(r=b>>a,\theta=0)} = S_x \ , \tag{5a}$$

$$S_{rr(r=b>>a,\theta=\pi/2)} = S_y \ . \tag{5b}$$

The rock in which the wellbore is drilled is modeled as an *isotropic, hyperelastic* material. Specifically, the strain energy ψ (per unit volume) of the material is assumed to be a third-order polynomial of strain and is given by

$$\psi = \frac{\lambda}{2} E_{mm}{}^2 + \mu E_{nk}E_{nk} + \frac{C}{3} E_{mm}{}^3 + BE_{nk}E_{nk}E_{mm} + \frac{A}{3} (E_{nk}E_{km}E_{mn}) \ , \tag{6}$$

where E_{ij} are the components of the Lagrangian strain tensor, the summation convention applies to repeated indices, and λ, μ, A, B, and C are material constants. The Symmetric Piola-Kirchhoff stress tensor for this material is given by

$$S_{ij} = \frac{\partial \psi}{\partial E_{ij}} = \lambda(E_{mm})\delta_{ij} + 2\mu E_{ij}$$

$$+ C E_{mm}{}^2\delta_{ij} + B[2E_{mm}E_{ij} + E_{nm}E_{nm}\delta_{ij}] + AE_{in}E_{nj} \ , \tag{7}$$

where δ_{ij} is the Kronecker delta and $S_{ij}=S_{ji}$. The five material constants λ, μ, A, B, and C of several rocks have been measured by Winkler and Liu (1996). Although the strain energy (6) is formulated in terms of finite strain, it is valid over a stress range that corresponds only to small strains for which the Symmetric Piola-Kirchhoff stress approaches the Cauchy stress. For the small strain considered, the equilibrium equations (3) may be formulated in terms of the Symmetric Piola-Kirchhoff stress, and the tangential stiffness tensor for this material is given by

$$K_{ijkl} = \frac{\partial S_{ij}}{\partial E_{kl}} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$+ 2C E_{mm}\delta_{ij}\delta_{kl} + B[E_{mm}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + 2E_{ij}\delta_{kl} + 2\delta_{ij}E_{kl}]$$

$$+ A/2(E_{ik}\delta_{jl} + E_{il}\delta_{jk} + \delta_{ik}E_{jl} + \delta_{il}E_{jk}) \ , \tag{8}$$

where $K_{ijkl}=K_{jikl}=K_{ijlk}=K_{klij}$. In the special case of $A=B=C=0$, the above equations describe a linear elastic material and λ and μ are the Lamé constants.

Substituting equation (2) into equation (7) yields

$$S_{rr} = \lambda\varepsilon + 2\mu E_{rr} + C\varepsilon^2 + B(2\varepsilon E_{rr} + \zeta) + A(E_{rr}{}^2 + E_{r\theta}{}^2) \ , \tag{9a}$$

$$S_{r\theta} = 2\mu E_{r\theta} + 2B\varepsilon E_{r\theta} + A(E_{rr} + E_{\theta\theta})E_{r\theta} \ , \tag{9b}$$

$$S_{\theta\theta} = \lambda\varepsilon + 2\mu E_{\theta\theta} + C\varepsilon^2 + B(2\varepsilon E_{\theta\theta} + \zeta) + A(E_{r\theta}{}^2 + E_{\theta\theta}{}^2) \ , \tag{9c}$$

$$S_{zz} = \lambda\varepsilon + 2\mu\varepsilon_z + C\varepsilon^2 + B(2\varepsilon\varepsilon_z + \zeta) + A\varepsilon_z{}^2 \ , \tag{9d}$$

where

$$\varepsilon = E_{rr} + E_{\theta\theta} + \varepsilon_z \ , \tag{10a}$$

$$\zeta = E_{rr}^2 + E_{\theta\theta}^2 + \varepsilon_z^2 + 2E_{r\theta}^2 \ . \tag{10b}$$

In the far-field ($r>>a$) the strain and stress are constant, and equations (9) may be solved for u, v, and ε_z. Then, for an arbitrary $r=b>>a$ the displacements u_b and v_b can be determined and boundary conditions (5) can be replaced by the conditions

$$u_{(r=b>>a)} = u_b \ , \tag{11a}$$

$$v_{(r=b>>a)} = v_b \ , \tag{11b}$$

which are simple to implement and improve the performance of the numerical scheme. Substituting the stress components (9) into the equilibrium equations (3) yields the field equation for the problem.

3. Numerical Procedure

In the simple case of a linear material ($A=B=C=0$), equations (3) and (9d) with boundary conditions (4) and (5) may be solved analytically (Muskhelishvili, 1954, §56)

$$u = \frac{1}{2\mu}\left\{-S_a\frac{a^2}{r} - S_{zz}\frac{\lambda}{3\lambda+2\mu}r + \frac{S_x+S_y}{2}\left(\frac{a^2}{r} + \frac{\lambda+2\mu}{3\lambda+2\mu}r\right)\right.$$
$$\left. + \frac{S_x-S_y}{2}\left(-\frac{a^4}{r^3} + 2\,\frac{\lambda+2\mu}{\lambda+\mu}\frac{a^2}{r} + r\right)\cos 2\theta\right\} \ , \tag{12a}$$

$$v = -\frac{S_x-S_y}{2}\left(\frac{1}{2\mu}\frac{a^4}{r^3} + \frac{1}{\lambda+\mu}\frac{a^2}{r} + \frac{1}{2\mu}r\right)\sin 2\theta \ , \tag{12b}$$

$$\varepsilon_z = S_{zz}\frac{\lambda+\mu}{\mu(3\lambda+2\mu)} - \frac{S_x+S_y}{2}\,\frac{\lambda}{\mu(3\lambda+2\mu)} \ . \tag{12c}$$

In the more general case, the field equations of the problem are nonlinear and have no simple analytic solution and are therefore solved numerically using the Galerkin finite element method. The weighted residuals of the field equations are integrated over the domain

$$\int_{r=a}^{b}\int_{\theta=0}^{2\pi}\left(\frac{\partial S_{rr}}{\partial r}+\frac{\partial S_{r\theta}}{r\partial\theta}+\frac{S_{rr}-S_{\theta\theta}}{r}\right)w\,r\mathrm{d}\theta\mathrm{d}r = 0 \ , \tag{13a}$$

$$\int_{r=a}^{b}\int_{\theta=0}^{2\pi}\left(\frac{\partial S_{r\theta}}{\partial r}+\frac{\partial S_{\theta\theta}}{r\partial\theta}+2\,\frac{S_{r\theta}}{r}\right)w\,r\mathrm{d}\theta\mathrm{d}r = 0 \ . \tag{13b}$$

The weighing functions w are chosen to be the same as the interpolation functions that approximate the displacements u and v. Terms including derivatives of stress in the integrands of (13) may be integrated by parts to give

$$\int_{r=a}^{b}\int_{\theta=0}^{2\pi}\left(S_{rr}\frac{w}{\partial r}r+S_{r\theta}\frac{\partial w}{\partial\theta}+S_{\theta\theta}w\right)\mathrm{d}\theta\mathrm{d}r = \left(\int_{\theta=0}^{2\pi}S_{rr}\ w\,r\mathrm{d}\theta\right|_{r=a} \ , \tag{14a}$$

$$\int_{r=a}^{b}\int_{\theta=0}^{2\pi}\left(S_{r\theta}\frac{w}{\partial r}r+S_{\theta\theta}\frac{\partial w}{\partial\theta}-S_{r\theta}w\right)\mathrm{d}\theta\mathrm{d}r = 0 \ . \tag{14b}$$

In deriving these equations some boundary integrals were omitted due to the symmetry of the problem. Moreover, since the value of the variables u and v are known at the far-field [equations (11)], the weighing functions w are taken to be zero over the boundary $r=b>>a$. In the numerical scheme the integrands of equations (14) are linearized and solved iteratively by factoring first order terms of strain and evaluating the remaining terms using the solution obtained in the pervious iteration.

In order to evaluate the quality of the solution, the change in strain energy due to the drilling of wellbore is integrated over the problem domain

$$\Delta\psi = \int_{r=a}^{b}\int_{\theta=0}^{2\pi}(\psi(r,\theta,u,v,\varepsilon_z)-\psi(b,\theta,u_b,v_b b,\varepsilon_z))\,r\,\mathrm{d}\theta\mathrm{d}r \ . \tag{15}$$

It is postulated that this error has the following functional form

$$\|e\|_h = kh^c \ , \tag{16}$$

where k and c are constants and h is the element size. Using uniform meshes to solve the nonlinear axisymmetric problem presented in the next section, it was found that the numerical scheme converges quadraticly, i.e., $c\approx 2$ (c=2.38, 2.56, 2.40 for quadratic, cubic and quadruple elements, respectively). It was also verified that the solution is unchanged when the wellbore stress gradually varies (rather than in a single step) from ambient conditions $S_{rr\,(r=a)}=S_{rr\,(r=b)}$ to $S_{rr\,(r=a)}=S_a$.

4. Results

The results of two simulations are presented here. In both cases the material constants of Massillon sandstone are used (Winkler and Liu, 1996)

$$\lambda = 1.9 \text{ GPa} , \tag{17a}$$

$$\mu = 6.3 \text{ GPa} , \tag{17b}$$

$$A = -17530 \text{ GPa} , \tag{17c}$$

$$B = -5670 \text{ GPa} , \tag{17d}$$

$$C = -2230 \text{ GPa} . \tag{17e}$$

The first problem is axisymmetric and is defined by the stress boundary conditions

$$S_a = 0 \text{ MPa} , \tag{18a}$$

$$S_x = -4 \text{ MPa} , \tag{18b}$$

$$S_y = -4 \text{ MPa} , \tag{18c}$$

$$S_{zz} = -6 \text{ MPa} . \tag{18d}$$

In this case, equation (3b) is trivial and only equation (3a) is solved. The stress distribution around the well bore is described in Fig. 1, where the solutions for the linear elastic (LE) material ($A=B=C=0$) and the third-order elastic (TOE) material are compared. The maximal difference in tangential stress between the two solutions is about 10%. A posteriori examination of the solution of the nonlinear problem verifies that the maximal absolute value of all strain components is 2.1×10^{-4}, and that the maximal absolute value of the volumetric strain is 7.7×10^{-4}. It is therefore concluded that the small strain assumption is justified.

Due to the symmetry of the problem, the tangential stiffness tensor has the same symmetries as an orthotropic material with symmetry axes parallel to r, θ, and z. The nine independent components of the tangential stiffness tensor are described in Fig. 2, as functions of normalized radius. As can be seen, the stiffness of the stressed TOE material is inhomogeneous and the stiffness components are considerably higher than the corresponding values of the LE material (i.e. $K_{rrrr}=K_{\theta\theta\theta\theta}=K_{zzzz}=\lambda+2\mu$, $K_{rr\theta\theta}=K_{rrzz}=K_{\theta\theta zz}=\lambda$, $K_{r\theta r\theta}=K_{rzrz}=K_{\theta z\theta z}=\mu$).

The second problem is not axisymmetric, and is defined by the stress boundary conditions

$$S_a = 0 \text{ MPa} \,, \tag{19a}$$

$$S_x = -6 \text{ MPa} \,, \tag{19b}$$

$$S_y = -2 \text{ MPa} \,, \tag{19c}$$

$$S_{zz} = -6 \text{ MPa} \,. \tag{19d}$$

The stress distribution around the wellbore is described in Fig. 3, and the stress distributions in three radial directions (θ=0, θ=π/4, and θ=π/2), are described in Fig. 4. As can be seen, the stress distribution of the TOE material resembles that of the LE material with maximal differences close to the well face. It can be seen from Figs. 1 and 3, that the difference in tangential stress between the TOE and the LE materials is greater in the non axisymmetric problem. The stressed TOE material is orthotropic with one symmetry axis in the vertical direction, but the other two symmetry axes are not necessarily parallel to the radial and tangential directions - except on the well face. Some components of the stiffness tensor are plotted in Fig. 5. The variation of these components around the well face suggests that surface, tube, and flexural waves, have two different wave velocities in two different polarizations that are determined by the far-field stress. This is in agreement with the findings of Barton and Zoback (1988) that related the polarization of borehole guided waves to the in situ stress orientation. As in the previous problem, a posteriori examination of the solution of the nonlinear problem verifies that the maximal absolute value of all strain components is 3.3×10^{-4} and that the maximal absolute value of the volumetric strain is 5.5×10^{-4}, and therefore the small strain assumption is justified.

5. Discussion

The stress distribution of the third-order elastic material resembles that of the linear elastic material but the radial displacement of the TOE material at the well face is about half the radial displacement of the LE material. A greater difference in the stress distribution of the two materials would have been observed if the problems had included displacement rather than stress boundary conditions. In situ, the material is best modeled as a semi-infinite medium with a stress-free boundary at ground level. The reference state of the in situ material is never known, and therefore the strains can never be measured directly. In contrast, the stress boundary conditions can be measured indirectly and therefore, in this work only stress boundary conditions are considered. Still, the difference in stiffness components between the TOE and LE materials is considerable. The inhomogeneous stiffness of the axisymmetric nonlinear problem, suggests that waves traveling between two different points along the wellbore may travel in curved trajectories. Consequently, the measured wave velocities may depend on the distance between the seismic source and receiver. The stress and stiffness variations in the nonaxisymmetric problem are more complex. Here, the results suggest that surface, tube, and flexural waves, have two different wave velocities in two different polarizations that are determined by the far-field stress.

The tangential stress at the well face for the two materials is shown to be different. This will affect the interpretation of far-field stresses from hydraulic fracturing data that is based on the Kirsch solution for a linear elastic material (Hickman and Zoback; 1983).

The specific wave forms and velocities related to wellbore waves in TOE materials are currently under investigation.

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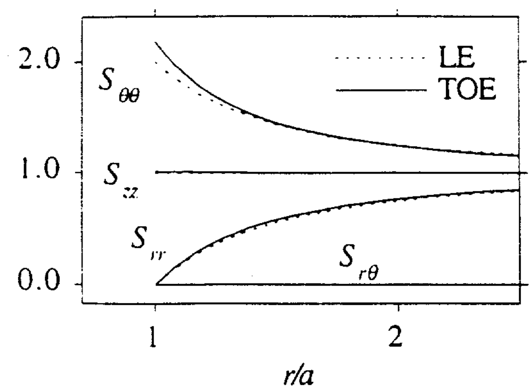

Fig. 1. The axisymmetric stress distribution in the linear elastic (LE) and third-order elastic (TOE) materials. S_{rr}, $S_{\theta\theta}$ and $S_{r\theta}$ are normalized by -4 MPa and S_{zz} is normalized by -6 MPa.

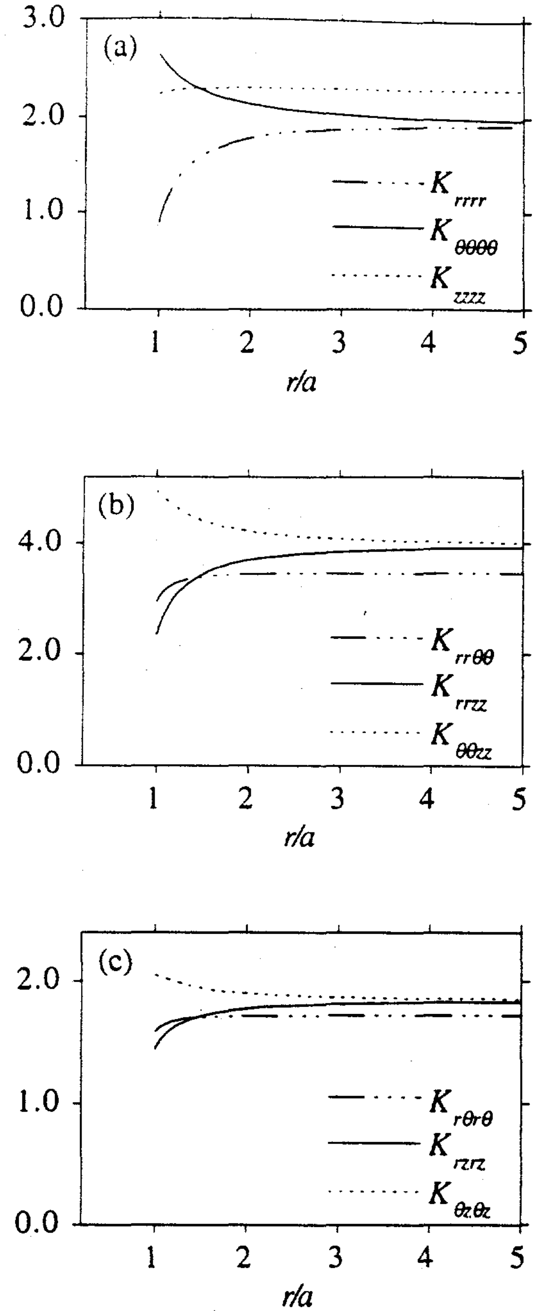

Fig. 2. The nine independent components of the tangential stiffness normalized by $\lambda+2\mu$ in (a), λ in (b) and by μ in (c).

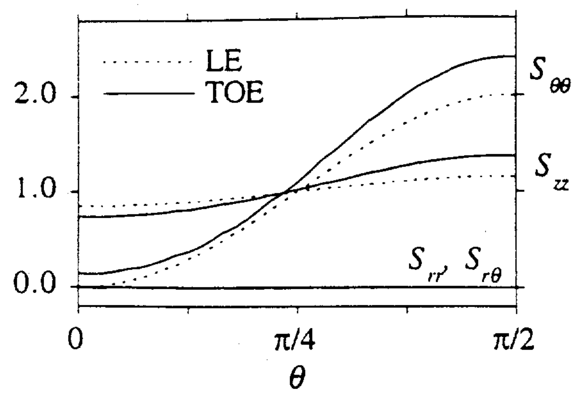

Fig. 3. Stress distribution at the wellbore. The nonzero components are normalized by their value at $\theta=\pi/4$ for LE material ($S_{\theta\theta}=-8$ MPa, $S_{zz}=-6$ MPa).

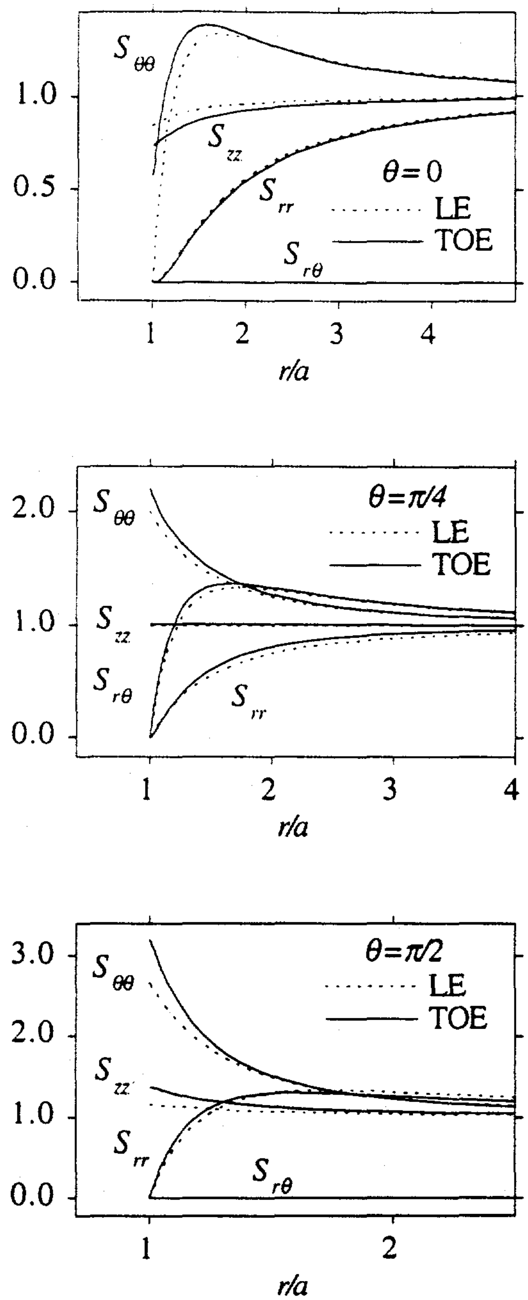

Fig. 4. Stress distribution in three different radial directions. The nonzero components are normalized by their far-field values.

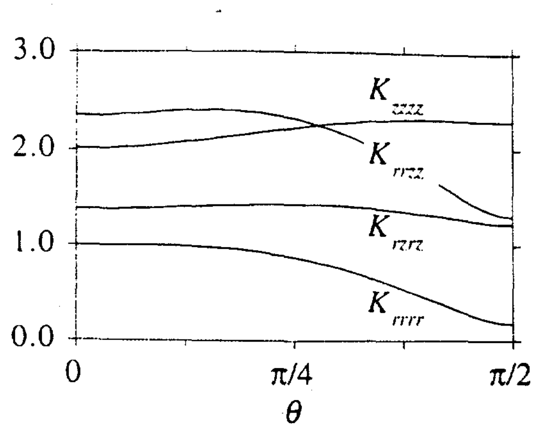

Fig. 5. Variation of stiffness components around the well face normalized by their respective values in a LE material (K_{rrrr} and K_{zzzz} are normalized by $\lambda+2\mu$, K_{rrzz} by λ, and K_{rzrz} is normalized by μ).

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