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*subject:* Failure Criteria and Temperature Dependent Elastic Constants in the 3-D Elastic Orthotropic Model

## 1 Introduction

Recent points of emphasis in the Library of Advanced Materials for Engineering (LAMÉ) [1] have been to enable flexibility in formulations via the adoption of a variety of modular frameworks [2–4]. While more established phenomenologies such as plasticity and viscoelasticity have been considered, elastically orthotropic models (*e.g.* `elastic_3D_orthotropic`) have not. For the elasticity component, not much can be modularized. However, a potential feature of interest would be the evaluation of failure criteria to consider the possibility of damage. Many such forms exist in the literature [5] providing a good basis for modularity.

An initial implementation of a `composite_failure` class is discussed in this effort. This class provides the basis for implementing various failure criterion for elastically orthotropic materials. Later extensions may be added to allow for evolution and coupling of the damage state, but this initial effort is focused on implementing decoupled formalisms. While many potential forms like Tsai-Hill or Tsai-Wu [5] exist, for this first attempt the multi-mode failure criterion of Xiao *et al.* [6] developed for fiber-reinforced composites is utilized as an exemplar. This model is selected for this effort, as a new elastically coupled damage formulation, `elastic_orthotropic_FRP_damage` [7] has recently been implemented using this damage model. The current desire for the added material failure related functionality to the `elastic_3D_orthotropic` model is only to store the driving force terms in the anisotropic damage criterion as state variables so that they can be easily visualized or used to control element deletion.

An additional effort to allow for temperature dependent elastic constants in `elastic_3D_orthotropic` was undertaken to match capabilities in other models. The added capability allows for each elastic constant to vary separately with user defined functions of temperature.

In this document, the theory and implementation of the `composite_failure` class, `multimode_exp_decay` and temperature dependent elasticity are presented and discussed. A user guide describing how to utilize the new models is also added. Finally, verification exercises establishing the capabilities of the model are discussed and presented.

# 2 Theory

## 2.1 Composite Failure Criterion

The `elastic_3D_orthotropic` model is a purely (hypo) elastic material model [1] which considers an orthotropic, linear elastic response. The elastic constants are specified in the material's coordinate system, with orthogonal axes denoted $A$, $B$ and $C$, which are arbitrary with respect to the global Cartesian coordinate system. For the developments made here, the material state is assumed to be uncoupled, that is, the failure criterion does not impact the constitutive relation at all.

The Xiao *et al.* damage criterion is motivated by woven composites. Therefore, the model assumes orthogonal principal axes that are oriented with the in-plane fill direction (the $A$ direction), the in-plane warp direction ($B$) and the out-of-plane direction ($C$). The damage criterion considers seven distinct failure mechanisms, resulting in a multi-surface representation. The seven mechanisms and their criteria are as follows:

1. Fiber tensile and shear failure in the $A$ direction:

$$f_1 = \left( \frac{E_{AA} \langle \tilde{\varepsilon}_{AA} \rangle_+}{S_{At}} \right)^2 + \left( \frac{2G_{CA}\tilde{\varepsilon}_{CA}}{S_{AFS}} \right)^2 \tag{1}$$

2. Fiber tensile and shear failure in the $B$ direction:

$$f_2 = \left( \frac{E_{BB} \langle \tilde{\varepsilon}_{BB} \rangle_+}{S_{Bt}} \right)^2 + \left( \frac{2G_{BC}\tilde{\varepsilon}_{BC}}{S_{BFS}} \right)^2 \tag{2}$$

3. Fiber compressive failure in the $A$ direction:

$$f_3 = \left( \frac{E_{AA} \langle \tilde{\varepsilon}'_{AA} \rangle_+}{S_{Ac}} \right)^2, \qquad \tilde{\varepsilon}'_{AA} = -\tilde{\varepsilon}_{AA} - \frac{E_{CC} \langle -\tilde{\varepsilon}_{CC} \rangle_+}{E_{AA}} \tag{3}$$

4. Fiber compressive failure in the $B$ direction:

$$f_4 = \left( \frac{E_{BB} \langle \tilde{\varepsilon}'_{BB} \rangle_+}{S_{Bc}} \right)^2, \qquad \tilde{\varepsilon}'_{BB} = -\tilde{\varepsilon}_{BB} - \frac{E_{CC} \langle -\tilde{\varepsilon}_{CC} \rangle_+}{E_{BB}} \tag{4}$$

5. Crush failure in the $C$ direction:

$$f_5 = \left( \frac{E_{CC} \langle \tilde{\varepsilon}_{CC} \rangle_-}{S_{Cc}} \right)^2 \tag{5}$$

6. Matrix shear failure in the $A - B$ plane:

$$f_6 = \left( \frac{2G_{AB}\tilde{\varepsilon}_{AB}}{S_{AB}} \right)^2 \tag{6}$$

7. Delamination failure:

$$f_7 = S^2 \left[ \left( \frac{E_{CC} \langle \tilde{\varepsilon}_{CC} \rangle_+}{S_{Ct}} \right)^2 + \left( \frac{2G_{BC}\tilde{\varepsilon}_{BC}}{S_{BC0} + S_{SR}} \right)^2 + \left( \frac{2G_{CA}\tilde{\varepsilon}_{CA}}{S_{CA0} + S_{SR}} \right)^2 \right] \tag{7}$$

   where, $S_{SR}$ is:

$$S_{SR} = -E_{CC} \langle \varepsilon_{33} \rangle_- \tan\phi. \tag{8}$$

The Macaulay bracket functions $\langle x \rangle_+$ and $\langle x \rangle_-$ are defined as:

$$\langle x \rangle_+ = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases} \tag{9}$$

$$\langle x \rangle_- = \begin{cases} 0 & x > 0 \\ x & x \leq 0 \end{cases} \quad . \tag{10}$$

In these equations, the strain tensor, $\tilde{\varepsilon}_{ij}$ (indexed by $A$, $B$, $C$) is written in the material's coordinate system and the $\tilde{\cdot}$ is used to emphasize this. Furthermore, the material dependent parameters are:

- $E_{AA}$ is the Young's modulus in the $A$ direction
- $E_{BB}$ is the Young's modulus in the $B$ direction
- $E_{CC}$ is the Young's modulus in the $C$ direction
- $G_{AB}$ is the Shear modulus in the $AB$ direction
- $G_{BC}$ is the Shear modulus in the $BC$ direction
- $G_{CA}$ is the Shear modulus in the $CA$ direction
- $S_{At}$ is the fiber tensile strength in the $A$ direction
- $S_{Bt}$ is the fiber tensile strength in the $B$ direction

- $S_{Ct}$ is the tensile strength in the $C$ direction
- $S_{AFS}$ is the shear strength of the fiber in the $A$ direction
- $S_{BFS}$ is the shear strength of the fiber in the $B$ direction
- $S_{Ac}$ is the fiber compression strength in the $A$ direction
- $S_{Bc}$ is the fiber compression strength in the $B$ direction
- $S_{Cc}$ is the fiber crush strength
- $S_{AB}$ is the in-plane shear strength in the $AB$ plane
- $S_{BC0}$ is the inter-laminar shear strength in the $BC$ direction
- $S_{AC0}$ is the inter-laminar shear strength in the $AC$ direction
- $S$ is the delamination scale factor
- $\phi$ is the internal friction angle.

$E_{AA}$, $E_{BB}$, $E_{CC}$, $G_{AB}$, $G_{BC}$, $G_{CA}$ are all elastic constants that are required for both the failure criterion and the mechanical constitutive relation. $S_{At}$, $S_{Bt}$, $S_{Ct}$, $S_{AFS}$, $S_{BFS}$, $S_{Ac}$, $S_{Bc}$, $S_{Cc}$, $S_{AB}$, $S_{BC0}$, $S_{AC0}$, $S$, and $\phi$ are specific to the failure criterion and as such are optional parameters unnecessary for pure elastic calculations.

### 2.2 Temperature Dependent Elastic Constants

Temperature dependent elasticity is considered by the multiplication of each elastic constant by an arbitrary function of temperature, $T$, as follows:

$$E_{AA} = f_{AA}(T)E^0_{AA} \tag{11}$$

$$E_{BB} = f_{BB}(T)E^0_{BB} \tag{12}$$

$$E_{CC} = f_{CC}(T)E^0_{CC} \tag{13}$$

$$\nu_{AB} = g_{AB}(T)\nu^0_{AB} \tag{14}$$

$$\nu_{BC} = g_{BC}(T)\nu^0_{BC} \tag{15}$$

$$\nu_{CA} = g_{CA}(T)\nu^0_{CA} \tag{16}$$

$$G_{AB} = h_{AB}(T)G^0_{AB} \tag{17}$$

$$G_{BC} = h_{BC}(T)G^0_{BC} \tag{18}$$

$$G_{CA} = h_{CA}(T)G^0_{CA} \tag{19}$$

Where the superscript "0" indicates a baseline elastic constant and the functions $f_{\alpha\beta}(T)$, $g_{\alpha\beta}(T)$ and $h_{\alpha\beta}(T)$ give the relative change in the baseline $\alpha\beta$ elastic constants. That is, $f_{\alpha\beta}(T)$, $g_{\alpha\beta}(T)$ and $h_{\alpha\beta}(T)$ are all unit-less scalar functions that should evaluate to be of order unity and simply scale the baseline elastic constant with temperature, where $\alpha$ and $\beta$ denote the principal directions. These functions are to be defined in the Sierra scope functions where either analytic or piece-wise functions can be used. For speed, piece-wise functions are recommended.

# 3 Implementation

## 3.1 Composite Failure Criterion

To facilitate the addition of multiple forms of failure criteria, a new `composite_failure` class was created. Each specific failure criterion is implemented as a member of the `composite_failure` class and the underlying constitutive model (`elastic_3D_orthotropic`) passes the relevant state variables to the new class. As an initial implementation, the damage criterion in Xiao *et al.* [6] is considered and implemented in the derived class `multimode_exp_decay`. The Xiao *et al.* [6] damage criterion is written in terms of strain, and thus, strains are passed into `multimode_exp_decay`. However, most failure criteria are written in terms of stresses (such as the Tsai-Wu [5] criterion); in such cases, stresses can be passed into `composite_failure` instead.

For a simulation involving an anisotropic material, two coordinate systems are relevant. The first is the global coordinate system of the finite element boundary value problem while the second is the local coordinate system of anisotropic media. Material characterization and parameterization is typically done in the local material coordinate system. Thus, for the `composite_failure` class, strains calculated in the global system, $\varepsilon_{ij}$, must be transformed for the local, $\tilde{\varepsilon}_{ij}$, prior to being passed in per:

$$\tilde{\varepsilon}_{ij} = Q_{ik}\varepsilon_{kl}Q_{jl}, \tag{20}$$

where $Q_{ij}$ is a rotation tensor which transforms from the global to the material coordinate system. This rotation tensor is already calculated in the `elastic_3D_orthotropic` model and its components are stored as state variables. Using the rotated strain, $\tilde{\varepsilon}_{ij}$, the equations (1) to (7) are evaluated and stored as state variables which are labeled `F1`, `F2`, `F3`, `F4`, `F5`, `F6`, and `F7`. On the other hand, most failure criteria do not have a multi-surface representation. In such cases, a single failure criterion should be evaluated and stored in a state variable called `F`. For the case of `multimode_exp_decay`, `F` stores the maximum value among `F1` - `F7`. Additionally, the equations used to update the failure criterion related state variables are explicitly defined. Thus, the model may be evaluated algebraically and not iteratively, minimizing the model cost.

### 3.2 Temperature Dependent Elastic Constants

The temperature dependent elastic constants are implemented with the addition of 9 state variables, one for each elastic constant, which store the evaluations of the temperature dependent elastic constants. These state variables are updated at the beginning of each load step with the current temperature field and then passed into a routine which assembles the material's stiffness tensor and stores it as a state variable.

## 4 Model Specification

The syntax to use `multimode_exp_decay` and the temperature dependent elastic constants is given below.

```
BEGIN PARAMETERS FOR MODEL ELASTIC_3D_ORTHOTROPIC

YOUNGS MODULUS            = E
POISSONS RATIO            = ν
YOUNGS MODULUS AA         = E_AA
YOUNGS MODULUS BB         = E_BB
YOUNGS MODULUS CC         = E_CC
POISSONS RATIO AB         = ν_AB
POISSONS RATIO BC         = ν_BC
POISSONS RATIO CA         = ν_CA
SHEAR MODULUS AB          = G_AB
SHEAR MODULUS BC          = G_BC
SHEAR MODULUS CA          = G_CA

COMPOSITE FAILURE MODEL      = MULTIMODE_EXP_DECAY | NO_FAILURE (NO_FAILURE)
STRENGTH_TENSION_A           = S_At
STRENGTH_TENSION_B           = S_Bt
STRENGTH_TENSION_C           = S_Ct
STRENGTH_COMPRESSION_A       = S_Ac
STRENGTH_COMPRESSION_B       = S_Bc
```

```
STRENGTH_FIBER_SHEAR_A          = S_AFS
STRENGTH_FIBER_SHEAR_B          = S_BFS
STRENGTH_COMPRESSION_CRUSH_C    = S_Cc
STRENGTH_IN_PLANE_SHEAR         = S_AB
STRENGTH_SHEAR_BC               = S_BC0
STRENGTH_SHEAR_AC               = S_AC0
INTERNAL_FRICTION_ANGLE         = phi
DELAMINATION_SCALE_FACTOR       = S

YOUNGS_MODULUS_AA_FUNCTION      = f_AA(T)  (constant_one)
YOUNGS_MODULUS_BB_FUNCTION      = f_BB(T)  (constant_one)
YOUNGS_MODULUS_CC_FUNCTION      = f_CC(T)  (constant_one)
POISSONS_RATIO_AB_FUNCTION      = g_AB(T)  (constant_one)
POISSONS_RATIO_BC_FUNCTION      = g_BC(T)  (constant_one)
POISSONS_RATIO_CA_FUNCTION      = g_CA(T)  (constant_one)
SHEAR_MODULUS_AB_FUNCTION       = h_AB(T)  (constant_one)
SHEAR_MODULUS_BC_FUNCTION       = h_BC(T)  (constant_one)
SHEAR_MODULUS_CA_FUNCTION       = h_CA(T)  (constant_one)

END PARAMETERS FOR MODEL ELASTIC_3D_ORTHOTROPIC
```

The keywords related to the elastic constants ($E$, $\nu$, $E_{AA}$, $E_{BB}$, $E_{CC}$, $\nu_{AB}$, $\nu_{BC}$, $\nu_{CA}$, $G_{AB}$, $G_{BC}$, $G_{CA}$) are required input for `elastic_3d_orthotropic` while the keywords related to $S_{At}$, $S_{Bt}$, $S_{Ct}$, $S_{Ac}$, $S_{Bc}$, $S_{AFS}$, $S_{BFS}$, $S_{Cc}$, $S_{AB}$, $S_{BC0}$, $S_{AC0}$, $\phi$, $S$ are only required for the optional evaluation of the `multimode_exp_decay` failure criterion. The keywords related to $f_{AA}(T)$, $f_{BB}(T)$, $f_{CC}(T)$, $g_{AB}(T)$, $g_{BC}(T)$, $g_{CA}(T)$, $h_{AB}(T)$, $h_{BC}(T)$, $h_{CA}(T)$ are only required for the case of temperature dependent elastic constants.

## 5 Verification

The implementation of the failure criterion is verified with quasi-static uniaxial tensile and compressive strain tests as well as a simple shear test, all on a single element cube with an edge length of 1 mm. A single uniform gradient hex element is used, noting the homogeneous material state. For all of these cases, analytical solutions were derived for the values of $f_1$ - $f_7$. Furthermore, `elastic_3D_orthotropic` is a hypo-elastic model and the strain $\varepsilon_{ij}$ is calculated from time integration of the unrotated deformation rate so that the strain considered here can be interpreted like the logarithmic strain, which is the strain measure used to develop the analytical solutions. Representative elastic model parameters, used for all verification testing, are given below in Table 1 while the parameters related to `multimode_exp_decay` failure criteria are given in Table 2. These material parameters were chosen arbitrarily.

The implementation of temperature dependent elasticity is also verified with a quasi-static

| | | | | | |
|---|---|---|---|---|---|
| $E$ | 2 GPa | $\nu$ | 0.2 (-) | | |
| $E_{AA}$ | 1 GPa | $E_{BB}$ | 2 GPa | $E_{CC}$ | 3 GPa |
| $\nu_{AB}$ | 0.2 (-) | $\nu_{BC}$ | 0.1 (-) | $\nu_{CA}$ | 0.3 (-) |
| $G_{AB}$ | 2 GPa | $G_{BC}$ | 2 GPa | $G_{CA}$ | 3 GPa |

Table 1: Elastic constants used for verification testing

| | | | | | |
|---|---|---|---|---|---|
| $S_{At}$ | 100 MPa | $S_{Bt}$ | 200 MPa | $S_{Ct}$ | 100 MPa |
| $S_{Ac}$ | 200 MPa | $S_{Bc}$ | 300 MPa | $S_{Cc}$ | 100 MPa |
| $S_{AFS}$ | 50 MPa | $S_{BFS}$ | 50 MPa | | |
| $S_{AB}$ | 25 MPa | $S_{BC0}$ | 25 MPa | $S_{AC0}$ | 50 MPa |
| $\phi$ | 10 (-) | $S$ | 1.0 (-) | | |

Table 2: Common failure parameters for verification testing of `multimode_exp_decay`.

uniaxial tensile strain test on the same element. Again, the material coordinate system is rotated 90 degrees about the $\hat{e}_3$ axis from the global coordinate system so that the $B$ axis is aligned with the global $\hat{e}_1$ axis. In this case, $f_{AA} = f_{BB} = f_{CC}$, $g_{AB} = g_{BC} = g_{CA}$ and $h_{AB} = h_{BC} = h_{CA}$. The specific forms of these functions used in the verification testing are:

$$f_{\alpha\beta} = \frac{\exp(-1.35)}{\exp(-0.005T)}, \tag{21}$$

$$g_{\alpha\beta} = \frac{T}{270}, \tag{22}$$

$$h_{\alpha\beta} = \frac{\exp(-2.16)}{\exp(-0.008T)}. \tag{23}$$

### 5.1 Uniaxial Tensile Strain - `multimode_exp_decay`

For the case of uniaxial tensile strain, a constant velocity magnitude, $v$, is applied in the global $\hat{e}_1$ direction. Meanwhile, the material coordinate system is rotated 90 degrees about the $\hat{e}_3$ direction in the global coordinate system so that the $B$ axis is aligned with the global $\hat{e}_1$ direction. The displacement as a function of time, $t$, is of the form:

$$u_i = vtX_i\delta_{i1} \qquad \text{no sum on } i. \tag{24}$$

The $BB$ component of the logarithmic strain in the material coordinate system is:

$$\tilde{\varepsilon}_{BB} = \varepsilon_{XX} = \ln\left(1 + \frac{v}{L}t\right), \tag{25}$$

while all other strain components are 0 and $L$ is the side length of the cube. Equation (25) is then inserted into (2) to give:

$$f_2 = \left[\frac{E_{BB}\ln\left(1 + \frac{v}{L}t\right)}{S_{Bt}}\right]^2, \tag{26}$$

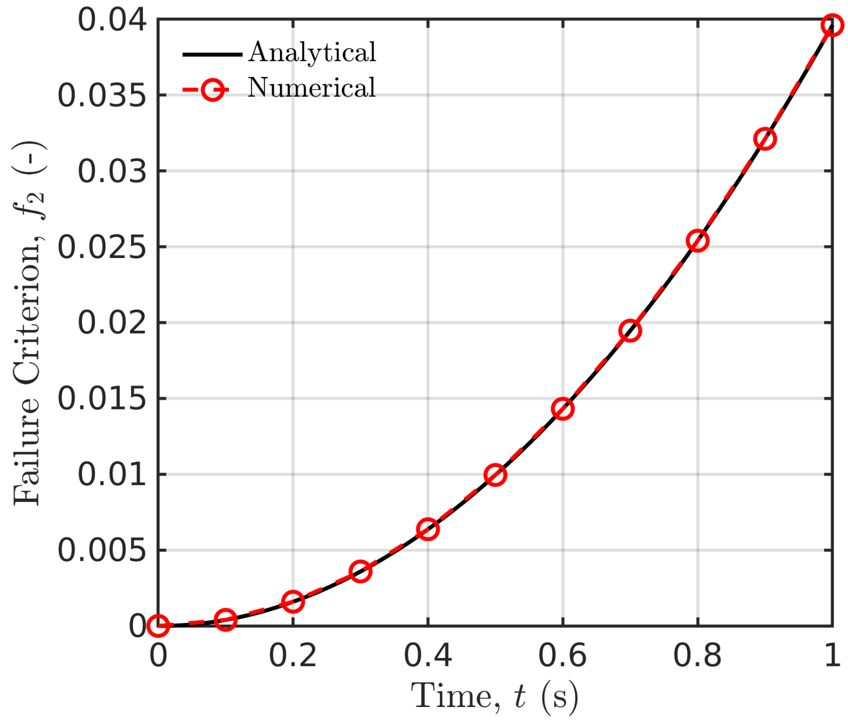


Figure 1: Comparison of analytical and numerical solution for the evolution of $f_2$

while all the other failure criteria, $f_i$, evaluate to 0 and `F=F2`. Results for `multimode_exp_decay` are plotted against the analytical result (26) with $v$ set to $0.01 \frac{\text{mm}}{\text{s}}$ in Figure 1. While not presented, the other failure state variables were compared to the analytical values and found to be in agreement.

### 5.2 Uniaxial Compressive Strain - `multimode_exp_decay`

For the case of uniaxial compressive strain, a constant velocity magnitude, $v$, is applied in the global $-\hat{e}_1$ direction. Again, the material coordinate system is rotated 90 degrees about the $\hat{e}_3$ direction in the global coordinate system so that the $B$ axis is aligned with the global $\hat{e}_1$ direction. The displacement as a function of time, $t$, is of the form:

$$u_i = -vtX_i\delta_{i1} \qquad \text{no sum on } i. \tag{27}$$

The $BB$ component of the logarithmic strain in the material coordinate system is:

$$\tilde{\varepsilon}_{BB} = \varepsilon_{XX} = \ln\Big(1 - \frac{v}{L}t\Big), \tag{28}$$

while all other strain components are zero. Equation (28) is then inserted into (4) to give:

$$f_4 = \left[\frac{E_{BB}\ln\big(1 - \frac{v}{L}t\big)}{S_{Bc}}\right]^2, \tag{29}$$

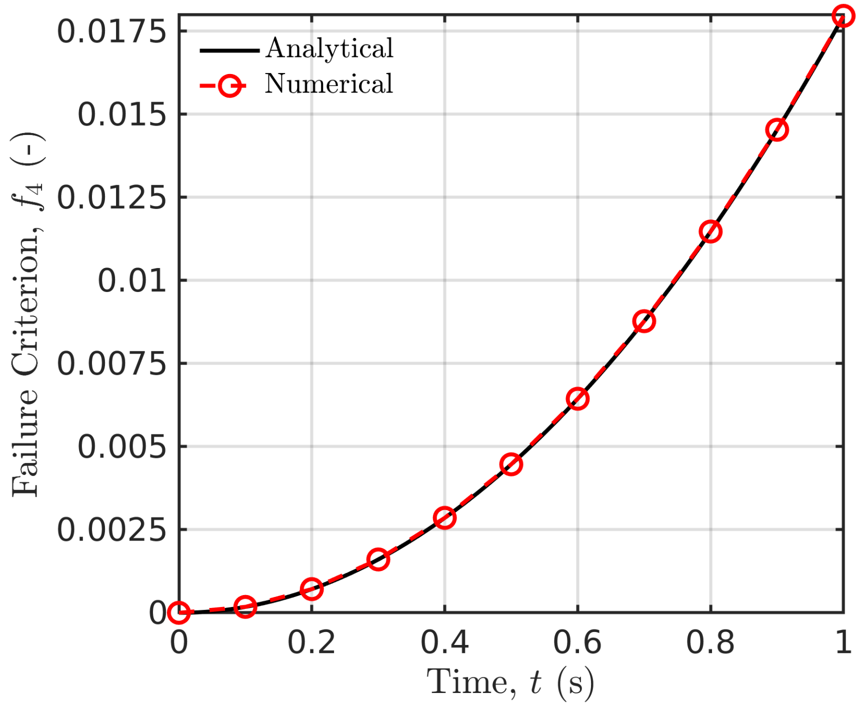


Figure 2: Comparison of analytical and numerical solution for the evolution of $f_4$

while all the other failure criteria, $f_i$, evaluate to 0 and `F=F4`. Results for `multimode_exp_decay` are plotted against the analytical result (29) with $v$ set to $0.01\frac{\text{mm}}{\text{s}}$ in Figure 2. While not presented, the other failure components were confirmed to match their expected values.

### 5.3 Simple Shear - `multimode_exp_decay`

For the case of simple shear, the following displacement is applied:

$$u_i = vtX_2\delta_{i1}. \tag{30}$$

The imposed displacement yields the following deformation gradient:

$$F_{ij} = \delta_{ij} + \gamma\delta_{i1}\delta_{j2}, \tag{31}$$

where $\gamma = \frac{v}{L}t$. To calculate the logarithmic strain, the right Cauchy-Green tensor, $C_{ij}$, may be written in tensorial form as,

$$C_{ij} = F_{ki}F_{kj} \tag{32}$$

and in matrix form,

$$\mathbf{C} = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{33}$$

Next, an eigen decomposition of $C_{ij}$ is required. The principal orientation of $C_{ij}$ can be calculated as,

$$\theta = \frac{1}{2}\arctan(\frac{2C_{12}}{C_{22}-C_{11}}) = \frac{1}{2}\arctan(\frac{2}{\gamma}), \tag{34}$$

which forms a decomposition rotation matrix, $Q_{ij}$, of,

$$\mathbf{Q} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\frac{1}{2}\arctan(\frac{2}{\gamma})) & -\sin(\frac{1}{2}\arctan(\frac{2}{\gamma})) & 0 \\ \sin(\frac{1}{2}\arctan(\frac{2}{\gamma})) & \cos(\frac{1}{2}\arctan(\frac{2}{\gamma})) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{35}$$

The principal form of the tensor, $C'_{ij}$, may be calculated by applying the orthogonal transformation,

$$C'_{ml} = Q_{mi} C_{ij} Q_{lj} \tag{36}$$

which results in:

$$\mathbf{C}' = \begin{bmatrix} \sin(\theta)[(1+\gamma^2)\sin(\theta) - 2\gamma\sin(\theta)] + \cos^2(\theta) & 0 & 0 \\ 0 & \cos(\theta)[(1+\gamma^2)\cos(\theta) + 2\gamma\sin(\theta)] + \sin^2(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{37}$$

Substituting in the value of $\theta$, the expression can be simplified using the product sum identities and inverse trigonometric identities:

$$C'_1 = 1 + \frac{\gamma^2}{2} - \frac{4+\gamma^2}{2\sqrt{1+4/\gamma^2}}, \tag{38}$$

$$C'_2 = 1 + \frac{\gamma^2}{2} + \frac{4+\gamma^2}{2\sqrt{1+4/\gamma^2}}, \tag{39}$$

and the principal logarithmic strains are,

$$\varepsilon'_i = \frac{1}{2}\ln\left(C'_i\right) \tag{40}$$

with,

$$\varepsilon'_1 = \frac{1}{2}\ln\left(1 + \frac{\gamma^2}{2} - \frac{4+\gamma^2}{2\sqrt{1+4/\gamma^2}}\right), \tag{41}$$

$$\varepsilon'_2 = \frac{1}{2}\ln\left(1 + \frac{\gamma^2}{2} + \frac{4+\gamma^2}{2\sqrt{1+4/\gamma^2}}\right), \tag{42}$$

$$\varepsilon'_3 = 0. \tag{43}$$

The global strain may be found by transforming the principal strains via,

$$\varepsilon_{ij} = Q_{ki}\varepsilon'_{kl}Q_{lj}, \tag{44}$$

with the non-zero components being,

$$\varepsilon_{11} = \frac{1}{4}\left(1 + \frac{1}{\sqrt{1+4/\gamma^2}}\right)\ln\left(1 + \frac{\gamma^2}{2} - \frac{4+\gamma^2}{2\sqrt{1+4/\gamma^2}}\right)$$
$$+\frac{1}{4}\left(1 - \frac{1}{\sqrt{1+4/\gamma^2}}\right)\ln\left(1 + \frac{\gamma^2}{2} + \frac{4+\gamma^2}{2\sqrt{1+4/\gamma^2}}\right), \quad (45)$$

$$\varepsilon_{22} = \frac{1}{4}\left(1 + \frac{1}{\sqrt{1+4/\gamma^2}}\right)\ln\left(1 + \frac{\gamma^2}{2} + \frac{4+\gamma^2}{2\sqrt{1+4/\gamma^2}}\right)$$
$$+\frac{1}{4}\left(1 - \frac{1}{\sqrt{1+4/\gamma^2}}\right)\ln\left(1 + \frac{\gamma^2}{2} - \frac{4+\gamma^2}{2\sqrt{1+4/\gamma^2}}\right), \quad (46)$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{1}{\gamma\sqrt{1+4/\gamma^2}}\right)\ln\left(1 + \frac{\gamma^2}{2} + \frac{4+\gamma^2}{2\sqrt{1+4/\gamma^2}}\right)$$
$$-\frac{1}{2}\left(\frac{1}{\gamma\sqrt{1+4/\gamma^2}}\right)\ln\left(1 + \frac{\gamma^2}{2} - \frac{4+\gamma^2}{2\sqrt{1+4/\gamma^2}}\right). \quad (47)$$

Although the strains are undefined for $\gamma = 0$, examination of the principal right Cauchy Green tensor (38), (39) shows the limit of $C_i'$ as $\gamma \to 0$ to be 1 for all components. Therefore, in the limit of $\gamma \to 0$ the strains will all be 0.

The log strain can then be rotated to align with the material axes. For the 90 degree rotation about the $\hat{e}_3$ direction, the transformation is:

$$\boldsymbol{Q}^m = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (48)$$

and,

$$\tilde{\varepsilon}_{ij} = Q^m_{ik}\varepsilon_{kl}Q^m_{jl}. \quad (49)$$

The non-zero, components of the material strain are thus,

$$\tilde{\varepsilon}_{AA} = \varepsilon_{22}, \quad (50)$$

$$\tilde{\varepsilon}_{BB} = \varepsilon_{11}, \quad (51)$$

$$\tilde{\varepsilon}_{AB} = -\varepsilon_{12}. \quad (52)$$

Then the failure criterion is evaluated with $\tilde{\varepsilon}_{ij}$. The nonzero components of the failure criterion are $f_1$, $f_4$, and $f_6$ which may be written,

$$f_1 = \left(\frac{E_{AA}\varepsilon_{22}}{S_{At}}\right)^2, \quad (53)$$

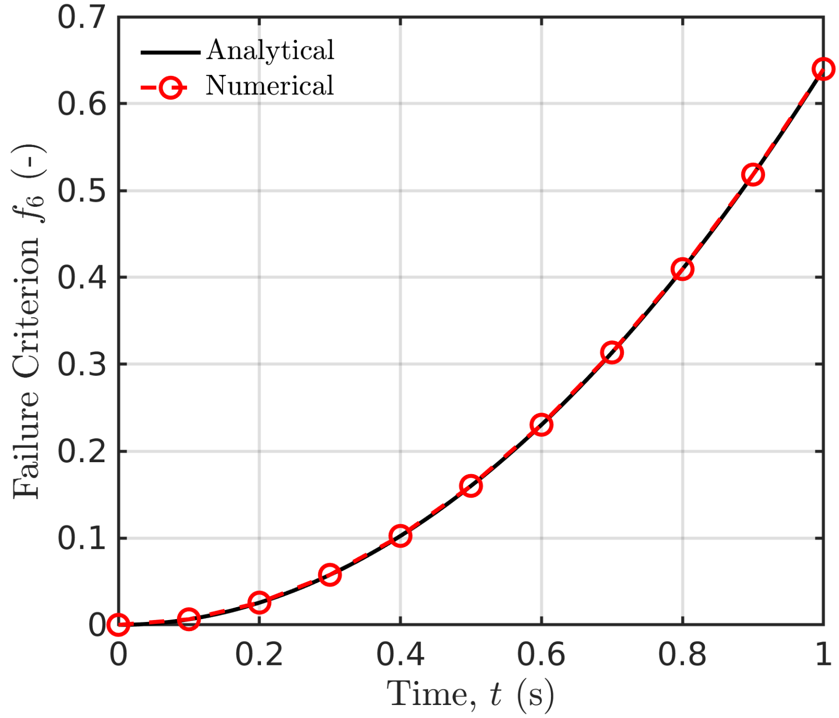


Figure 3: Comparison of analytical and numerical solution for the evolution of $f_6$

$$f_4 = \left(\frac{E_{BB}\varepsilon_{11}}{S_{Bc}}\right)^2, \tag{54}$$

$$f_6 = \left(\frac{2G_{AB}\varepsilon_{12}}{S_{AB}}\right)^2. \tag{55}$$

Results for `multimode_exp_decay` are plotted against the analytical result (53) - (55), with $v$ set to $0.01\frac{\text{mm}}{\text{s}}$, in Figure 3 and Figure 4. For the material parameters chosen here, $f_6$ will be the greatest among the failure criteria. Thus, `F=F6` in this example.

### 5.4 Uniaxial Tensile Strain - Temperature Dependent Elasticity

The same displacement as in the uniaxial tensile strain verification problem for `multimode_exp_decay` is used in this verification test, which results in the same strain field (see Section 5.1). The temperature is set to a constant 500 K and the accuracy of the solution is bench-marked with the non-zero stress components $\sigma_{11}$, $\sigma_{22}$ and $\sigma_{33}$. The expressions for these stress components are:

$$\sigma_{11} = D_{22} \ln\left(1 + \frac{v}{L}t\right), \tag{56}$$

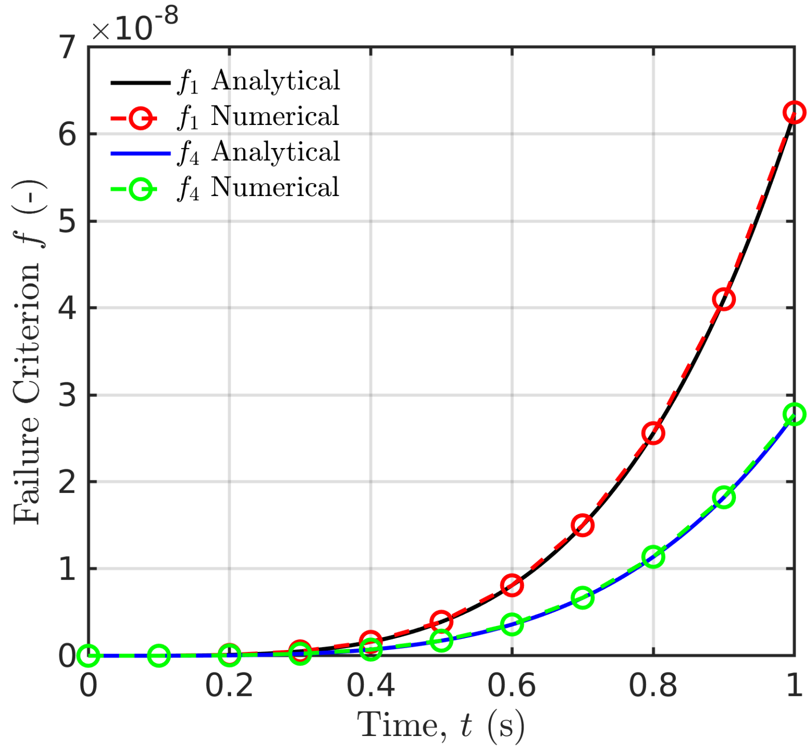


Figure 4: Comparison of analytical and numerical solution for the evolution of $f_1$ and $f_4$

$$\sigma_{22} = D_{12} \ln\Big(1 + \frac{v}{L}t\Big), \tag{57}$$

$$\sigma_{33} = D_{23} \ln\Big(1 + \frac{v}{L}t\Big). \tag{58}$$

Where $D_{22}$, $D_{12}$ and $D_{23}$ determined from the following equations:

$$D_{22} = \frac{1 - \nu_{CA}\nu_{AC}}{\Delta} E_{BB}, \tag{59}$$

$$D_{12} = \frac{\nu_{BA} + \nu_{CA}\nu_{BC}}{\Delta} E_{AA}, \tag{60}$$

$$D_{23} = \frac{\nu_{CB} + \nu_{AB}\nu_{CA}}{\Delta} E_{BB}, \tag{61}$$

$$\Delta = 1 - \nu_{AB}\nu_{BA} - \nu_{BC}\nu_{CB} - \nu_{CA}\nu_{AC} - 2\nu_{AB}\nu_{BC}\nu_{CA}, \tag{62}$$

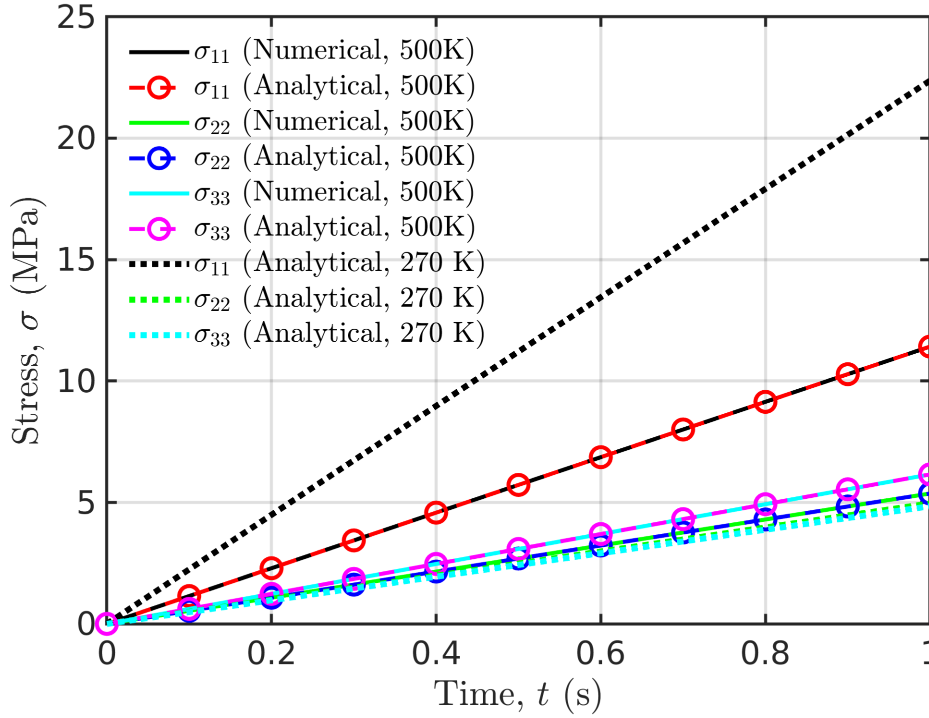


Figure 5: Comparison of analytical and numerical solution for stress components

$$\nu_{BA} = \nu_{AB}\frac{E_{BB}}{E_{AA}}, \tag{63}$$

$$\nu_{CB} = \nu_{BC}\frac{E_{CC}}{E_{BB}}, \tag{64}$$

$$\nu_{AC} = \nu_{CA}\frac{E_{AA}}{E_{CC}}, \tag{65}$$

$$\nu_{BA} = \nu_{AB}\frac{E_{BB}}{E_{AA}}. \tag{66}$$

Results for the temperature dependent elasticity verification problem are shown in Figure 5, with analytical results for the 270K case for reference.

### 5.5 Unit Testing for `multimode_exp_decay`

In addition to the regression tests, unit tests were done to verify the correctness of the implementation. These tests of the failure criteria consider a single material point and are directly given a strain tensor as input. The following cases were tested against analytical solutions:

1. The first case considers uniaxial tensile strain in the $\hat{e}_1$ direction with the material coordinate system aligned with the global coordinate system. In this case, the only non-zero failure criterion will be $f_1$ which evaluates to:

$$f_1 = \left(\frac{E_{AA}\varepsilon_{11}}{S_{At}}\right)^2. \tag{67}$$

2. The next case considers pure shear in the $\hat{e}_1\hat{e}_3$ direction with the material coordinate system aligned with the global coordinate system. In the interest of only testing the $f_1$ criterion for this case, $S$ is set to 0.0 so that $f_7$ evaluates to 0. $f_1$ then evaluates to:

$$f_1 = \left(2\frac{G_{CA}\varepsilon_{13}}{S_{AFS}}\right)^2. \tag{68}$$

3. This case considers uniaxial tensile strain in the $\hat{e}_2$ direction with the material coordinate system material coordinate system aligned with the global coordinate system. In this case, the only non-zero failure criterion will be $f_2$ which evaluates to:

$$f_2 = \left(\frac{E_{BB}\varepsilon_{22}}{S_{Bt}}\right)^2. \tag{69}$$

4. This case considers pure shear in the $\hat{e}_2\hat{e}_3$ direction with the material coordinate system aligned with the global coordinate system. In the interest of only testing the $f_2$ criterion for this case, $S$ is set to 0.0 so that $f_7$ evaluates to 0. $f_2$ then evaluates to:

$$f_2 = \left(2\frac{G_{BC}\varepsilon_{23}}{S_{BFS}}\right)^2. \tag{70}$$

5. This case considers uniaxial compressive strain in the $\hat{e}_1$ direction with the material coordinate system aligned with the global coordinate system. In this case, the only non-zero failure criterion will be $f_3$ which evaluates to:

$$f_3 = \left(\frac{E_{AA}\varepsilon_{11}}{S_{Ac}}\right)^2. \tag{71}$$

6. This case considers uniaxial compressive strain in the $\hat{e}_2$ direction with the material coordinate system aligned with the global coordinate system. In this case, the only non-zero failure criterion will be $f_4$ which evaluates to:

$$f_4 = \left(\frac{E_{BB}\varepsilon_{22}}{S_{Bc}}\right)^2. \tag{72}$$

7. This case considers uniaxial compressive strain in the $\hat{e}_3$ direction with the material coordinate system aligned with the global coordinate system. In this case, the only non-zero failure criterion will be $f_5$ which evaluates to:

$$f_5 = \left( \frac{E_{CC}\varepsilon_{33}}{S_{Cc}} \right)^2 . \tag{73}$$

8. This case considers pure shear in the $\hat{e}_1\hat{e}_2$ direction with the material coordinate system aligned with the global coordinate system. In this case, the only non-zero failure criterion will be $f_6$ which evaluates to:

$$f_6 = \left( 2\frac{G_{AB}\varepsilon_{12}}{S_{AB}} \right)^2 . \tag{74}$$

9. This case considers uniaxial tensile strain in the $\hat{e}_3$ direction with the material coordinate system aligned with the global coordinate system. In this case, the only non-zero failure criterion will be $f_7$ which evaluates to:

$$f_7 = \left( S\frac{E_{CC}\varepsilon_{33}}{S_{Ct}} \right)^2 . \tag{75}$$

10. This case considers pure shear in the $\hat{e}_2\hat{e}_3$ direction with the material coordinate system aligned with the global coordinate system. In this case, both $f_2$ and $f_7$ will be non-zero and evaluate to:

$$f_2 = \left( 2\frac{G_{BC}\varepsilon_{23}}{S_{BFS}} \right)^2 , \tag{76}$$

$$f_7 = \left( 2S\frac{G_{BC}\varepsilon_{23}}{S_{BC0}} \right)^2 . \tag{77}$$

11. This case considers pure shear in the $\hat{e}_1\hat{e}_3$ direction with the material coordinate system aligned with the global coordinate system. In this case, both $f_1$ and $f_7$ will be non-zero and evaluate to:

$$f_1 = \left( 2\frac{G_{CA}\varepsilon_{13}}{S_{AFS}} \right)^2 , \tag{78}$$

$$f_7 = \left( 2S\frac{G_{BC}\varepsilon_{13}}{S_{AC0}} \right)^2 . \tag{79}$$

12. This case considers combined axial compressive strain in the $\hat{e}_1$ and $\hat{e}_1\hat{e}_3$ directions with $\varepsilon_{33}$ = -$\varepsilon_{13}$ and $\varepsilon_{13} > 0$. In this case, $f_1$, $f_5$ and $f_7$ will be non-zero and evaluate to:

$$f_1 = \left(2\frac{G_{CA}\varepsilon_{13}}{S_{AFS}}\right)^2, \quad (80)$$

$$f_5 = \left(\frac{E_{CC}\varepsilon_{33}}{S_{Cc}}\right)^2, \quad (81)$$

$$f_7 = \left(2S\frac{G_{CA}\varepsilon_{13}}{S_{AC0} - E_{CC}\varepsilon_{33}\tan(\phi)}\right)^2. \quad (82)$$

In each of these unit tests, the analytical solution matches the output from the implemented code within a tolerance of $1 \cdot 10^{-8}$.

## 6 Summary and Conclusion

Recently, the `multimode_exp_decay` composite failure criterion was added to the `elastic_3d_orthotropic` model in LAMÉ. Details regarding its equations, implementation, and verification have been discussed in this work. The new capability is intended to provide a physically motivated criterion for controlling element deletion in orthotropic materials or for simple visualization of where damage could occur. To further these capabilities, a softening model could be added to more closely replicate the behavior of real materials and to alleviate mesh dependence. Additionally, a capability to allow for each elastic constant to vary separately with user defined functions of temperature was added and its implementation and verification was presented.

## 7 Acknowledgements

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