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Title: Lectures on Lie Group Analysis: Solving Differential Equations Using Symmetries

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Lectures on Lie Group Analysis: Solving Differential Equations Using Symmetries

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Chapter 1

Introduction

These notes are meant to be a supplemental reference for the beginner Lie Group Analyst. It is assumed that the reader has a basic concept of the fundamentals of Lie Group Theory (LGT), e.g. has seen the derivation of the infinitesimal generator and understands the mathematical meaning behind invariance. An excellent reference is Albright et al., "Symmetry Analysis of Differential Equations: A Primer," [1]. The reader is urged to read at least the first three chapters of that document to be able to follow the outset of Chapter 2 of this document. The reader should also have a general understanding of calculus, ordinary differential equations, and partial differential equations.

The purpose of these lectures is to provide advancing application to more difficult problems, starting with first-order ordinary differential equations (ODEs) in Chapter 2, then second-order ODEs in Chapter 3, followed by partial differential equations (PDEs) in Chapter 4, and finally an analysis of integro-differential equations in Chapter 5. Each example problem will demonstrate how to use LGT to find solutions to the wide variety of equations presented. We hope that the reader learns how to effectively move through the machinery of applying LGT to solving differential equations. Although all of the equations presented herein can be solved using more common "traditional" methods taught in applied math classes, LGT is powerful in that it permits one to find solutions to equations regardless of the type of equation. This is done by finding a Lie group, which allows one to find a new coordinate system that simplifies the equation. In simplifying the equation, either the number of dimensions is reduced or a solution can be found outright. If an exact closed-form solution does not exist for an equation, then that means that a Lie group does not exist. Finally, we note that these lectures are adapted from actual lectures given by the authors and therefore we avoid using too much language because most of what is written below is also to be written on a board. Thus, we let the math do the talking.

Chapter 2

Symmetry Analysis of First-Order ODEs

Consider the general first-order ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y). \tag{2.1}$$

Written as a surface equation:

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0 = \frac{\mathrm{d}y}{\mathrm{d}x} - f(x, y). \tag{2.2}$$

This defines a surface of three variables: x, y, and $\frac{\mathrm{d}y}{\mathrm{d}x}$. We call this surface a *differential function* to signify that it is not an algebraic function. We treat the derivative as an independent variable. In doing so, the derivative must remain invariant under a Lie group of transformations. Thus, the infinitesimal generator must be *prolonged* to accommodate this requirement.

As a reminder, the infinitesimal generator (IG) for an algebraic equation, AE, of three variables is:

$$V_{\mathrm{AE}} = \eta_1(x, y, z)\frac{\partial}{\partial x} + \eta_2(x, y, z)\frac{\partial}{\partial y} + \eta_3(x, y, z)\frac{\partial}{\partial z} \tag{2.3}$$

We need to derive a new form of the IG to account for invariance of the transformed derivatives.

2.1 Prolongation of the Infinitesimal Generator

The derivative is an independent variable, then

$$p = \frac{\mathrm{d}y}{\mathrm{d}x}. \tag{2.4}$$

The surface equation is:

$$F(x, y, p) = 0. \tag{2.5}$$

Our objective is to determine a set of infinitesimal transformations of the independent and (originally) dependent variables of the form:

$$\widetilde{x} = x + \epsilon\eta(x, y) + \mathcal{O}\left(\epsilon^2\right) \tag{2.6a}$$

$$\widetilde{y} = y + \epsilon\phi(x, y) + \mathcal{O}\left(\epsilon^2\right) \tag{2.6b}$$
$$\widetilde{p} = p + \epsilon\zeta(x, y, p) + \mathcal{O}\left(\epsilon^2\right) \tag{2.6c}$$

where $\mathcal{O}$ is the big-O notation for "of the order". We will use η to represent the coordinate functions of the original independent variables, ϕ for the original dependent variables, and ζ for the derivatives. Note that we have written the coordinate functions, η and ϕ, so that they are not functions of p. However, in general they can be, but we are going to restrict our analysis to *point symmetries*, i.e. symmetries that are not functions of derivatives. We will see that ζ is still a function of p for point symmetries; for higher-order symmetries, it would also depend on derivatives of p. Symmetries that depend on derivatives are called *contact symmetries*, and are typically much more complicated. These transformations can be written as finite transformations (aka mappings):

$$\widetilde{x} = \alpha(x, y; \epsilon) \tag{2.7a}$$
$$\widetilde{y} = \beta(x, y; \epsilon) \tag{2.7b}$$
$$\widetilde{p} = \gamma(x, y, p; \epsilon) \tag{2.7c}$$

which must form a Lie group of transformations called its *extension* or *prolongation.*

Because the coordinate function ζ leaves $p = \frac{\mathrm{d}y}{\mathrm{d}x}$ invariant, we must find the form of ζ that allows this. This is done by noting:

$$\widetilde{p} = \frac{\mathrm{d}\widetilde{y}}{\mathrm{d}\widetilde{x}} = \frac{\mathrm{d}\widetilde{y}/\,\mathrm{d}x}{\mathrm{d}\widetilde{x}/\,\mathrm{d}x} \tag{2.8}$$

The numerator of Eq. 2.8 can be found by taking the x-derivative of Eq. 2.6b:

$$\begin{aligned}\frac{\mathrm{d}\widetilde{y}}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}x} + \epsilon\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \mathcal{O}\left(\epsilon^2\right) \\ &= p + \epsilon\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}p\right) + \mathcal{O}\left(\epsilon^2\right).\end{aligned} \tag{2.9}$$

Similarly, the denominator is expanded by taking the x-derivative of Eq. 2.6a to find:

$$\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}x} = 1 + \epsilon\left(\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial y}p\right) + \mathcal{O}\left(\epsilon^2\right). \tag{2.10}$$

Equation 2.8 becomes:

$$\widetilde{p} = \frac{p + \epsilon\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}p\right) + \mathcal{O}\left(\epsilon^2\right)}{1 + \epsilon\left(\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial y}p\right) + \mathcal{O}\left(\epsilon^2\right)} \tag{2.11}$$

Recall our goal is to find ζ. Multiply Eq. 3.3c by the numerator and suppress $\mathcal{O}(\epsilon^2)$ terms to find:

$$\widetilde{p} + \widetilde{p}\epsilon\left(\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial y}p\right) = p + \epsilon\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}p\right). \tag{2.12}$$

Now substitute Eq. 2.6c into the second occurrence of $\widetilde{p}$ and suppress higher-order terms:

$$\widetilde{p} + (p + \epsilon\zeta)\epsilon\left(\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial y}p\right) = p + \epsilon\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}p\right). \tag{2.13}$$

Solve for $\widetilde{p}$ (and suppress $\mathcal{O}(\epsilon^2)$ terms):

$$\widetilde{p} = p + \epsilon\left(\frac{\partial\phi}{\partial x} + \left[\frac{\partial\phi}{\partial y} - \frac{\partial\eta}{\partial x}\right]p - \frac{\partial\eta}{\partial y}p^2\right). \tag{2.14}$$

Comparing Eq. 2.6c with Eq. 2.14, we may infer the functional form of ζ as:

$$\boxed{\zeta(x,y,p) = \frac{\partial \phi}{\partial x} + \left[\frac{\partial \phi}{\partial y} - \frac{\partial \eta}{\partial x}\right] p - \frac{\partial \eta}{\partial y} p^2} \tag{2.15}$$

As we can see, ζ is not necessarily a wholly new coordinate function, but rather an equation that defines the transformation $\widetilde{p}$ that leaves $\mathrm{d}y/\,\mathrm{d}x$ invariant under a Lie group of transformations. It is therefore implicitly a function of the other transformations η and ϕ.

The prolonged infinitesimal generator for a first-order ODE is then:

$$\mathrm{pr}V = \eta(x,y)\frac{\partial}{\partial x} + \phi(x,y)\frac{\partial}{\partial y} + \zeta(x,y,p)\frac{\partial}{\partial p}. \tag{2.16}$$

Note that the prolonged IG is a sum of the algebraic IG, $V = \eta\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y}$, with the added prolongation to account for transformations of the derivative that leave the original surface equation F invariant. Equations 2.15 and 2.16 are used to derive the symmetry determining equations (SDEs) of any first-order ODE. The solutions to the SDEs define the coordinate functions (η, ϕ) that form a Lie group of transformations.

2.2 The Symmetry Criterion

The symmetry criterion defines the symmetry determining equations (SDEs). A solution to the SDEs is a Lie group of infinitesimal transformations. The symmetry criterion is a statement that the surface equation remains invariant under a Lie group of transformations:

$$F(x,y,p) = F(\widetilde{x},\widetilde{y},\widetilde{p}) = 0. \tag{2.17}$$

This can be stated in the infinitesimal sense as follows:

$$\boxed{\mathrm{pr}VF(x,y,p)\Big|_{F=0} = 0.} \tag{2.18}$$

This is saying that, by applying the prolonged IG onto F and requiring $F = 0$, the entire operation must also be equivalently zero. Equation 2.18 is the general definition of the SDEs for a first-order ODE. In an expanded form, we have:

$$\mathrm{pr}VF(x,y,p)\Big|_{F=0} = 0 = \left[\eta(x,y)\frac{\partial F}{\partial x} + \phi(x,y)\frac{\partial F}{\partial y} + \zeta(x,y,p)\frac{\partial F}{\partial p}\right]\Bigg|_{F=0}. \tag{2.19}$$

A few comments:

- The SDEs, Eq. 2.18, form a system of first-order quasi-linear PDEs whose solutions are the coordinate functions η and ϕ.
- This may seem like a step in the wrong direction as we have arrived at a PDE from an ODE. The advantage to following this path is that the system of PDEs of this type can always be solved using the Method of Characteristics.

We will see how this is done in practice in the next section.

2.3 Example: A First-Order Nonlinear ODE

Consider the first-order nonlinear ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \mathrm{e}^{-x} y^2 - y - \mathrm{e}^{x} = 0, \tag{2.20}$$

or, as a surface equation

$$F(x, y, p) = 0 = p - \mathrm{e}^{-x} y^2 - y - \mathrm{e}^{x}\,. \tag{2.21}$$

We wish to find a solution to this ODE using LGT. First, we will find $\mathrm{pr}VF$, then $\mathrm{pr}VF|_{F=0}$ to find the SDEs. Then,

$$\begin{aligned}\mathrm{pr}VF &= \eta\frac{\partial F}{\partial x} + \phi\frac{\partial F}{\partial y} + \zeta\frac{\partial F}{\partial p} \\ &= \eta\left(\mathrm{e}^{-x} y^2 - \mathrm{e}^{x}\right) - \phi\left(2y\,\mathrm{e}^{-x} + 1\right) + \zeta \times 1 \\ &= \eta\left(\mathrm{e}^{-x} y^2 - \mathrm{e}^{x}\right) - \phi\left(2y\,\mathrm{e}^{-x} + 1\right) + \frac{\partial\phi}{\partial x} + \left[\frac{\partial\phi}{\partial y} - \frac{\partial\eta}{\partial x}\right] p - \frac{\partial\eta}{\partial y}p^2.\end{aligned} \tag{2.22}$$

Now we evaluate Eq. 2.22 at $F = 0$. This can be most easily enforced by selecting a form of p that makes $F = 0$. Thus, we choose p to be:

$$F = 0 \quad \text{when} \quad p = \mathrm{e}^{-x} y^2 + y + \mathrm{e}^{x}\,. \tag{2.23}$$

Inserting Eq. 2.23 into Eq. 2.22 gives us the SDEs. Thus, we find

$$\begin{aligned}\mathrm{pr}VF\Big|_{F=0} = 0 = {} & \eta\left(\mathrm{e}^{-x} y^2 - \mathrm{e}^{x}\right) - \phi\left(2y\,\mathrm{e}^{-x} + 1\right) + \frac{\partial\phi}{\partial x} + \left[\frac{\partial\phi}{\partial y} - \frac{\partial\eta}{\partial x}\right]\left(\mathrm{e}^{-x} y^2 + y + \mathrm{e}^{x}\right) \\ & - \frac{\partial\eta}{\partial y}\left(\mathrm{e}^{-x} y^2 + y + \mathrm{e}^{x}\right)^2.\end{aligned} \tag{2.24}$$

Equation 2.24 is the symmetry determining equation for Eq. 2.21. Without knowledge of boundary conditions, Eq. 2.24 has an infinite number of solutions. This is actually true for <u>all</u> first-order ODEs. For second-order ODEs, there are at most eight symmetries [2].

In light of this fact, we only need one Lie group to find a solution to this ODE. Therefore, we concede to make assumptions on the functional form of both η and ϕ. For example, let us chose η to be a constant and ϕ to be only a function of y, i.e.,

$$\eta(x, y) \to \text{constant} = \eta \tag{2.25a}$$

$$\phi(x, y) \to \phi(y). \tag{2.25b}$$

Then, $\frac{\partial\eta}{\partial x} = \frac{\partial\eta}{\partial y} = 0$ and $\frac{\partial\phi}{\partial x} = 0$ and Eq. 2.24 drastically simplifies to:

$$\mathrm{pr}VF\Big|_{F=0} = 0 = \eta\left(\mathrm{e}^{-x} y^2 - \mathrm{e}^{x}\right) - \phi\left(2y\,\mathrm{e}^{-x} + 1\right) + \frac{\partial\phi}{\partial y}\left(\mathrm{e}^{-x} y^2 + y + \mathrm{e}^{x}\right). \tag{2.26}$$

We can now multiply Eq. 2.26 by e^{x} and collect powers of e^{x} to find:

$$0 = \left(\frac{\mathrm{d}\phi}{\mathrm{d}y} - \eta\right)\mathrm{e}^{2x} + \left(y\frac{\mathrm{d}\phi}{\mathrm{d}y} - \phi\right)\mathrm{e}^{x} + y^2\frac{\mathrm{d}\phi}{\mathrm{d}y} + y^2\eta - 2y\phi. \tag{2.27}$$

Equation 2.27 is a polynomial is e^x whose coefficients are not functions of x. Therefore, if Eq. 2.27 is to be satisfied, then each of the coefficients must simultaneously equal zero for all values of x. Then, we may define three new SDEs with this information to find:

$$0 = \frac{d\phi}{dy} - \eta \tag{2.28a}$$

$$0 = y\frac{d\phi}{dy} - \phi \tag{2.28b}$$

$$0 = y^2\frac{d\phi}{dy} + y^2\eta - 2y\phi. \tag{2.28c}$$

Substituting Eq. 2.28a ($d\phi/dy = \eta$) into Eq. 2.28b gives:

$$\phi(y) = \eta y, \tag{2.29}$$

which also satisfies Eq. 2.28c. We can choose any value for η, then we have found a Lie group for Eq. 2.21:

$$\eta = 1 \tag{2.30a}$$

$$\phi = y. \tag{2.30b}$$

Plugging these into Eq. 2.27 confirms these are indeed a solution to the SDE under the assumptions given by Eq. 2.25.

Note that our IG is:

$$V = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{2.31}$$

We can use Eqs. 2.30 and 2.31 to form the Lagrange-Charpit equations, i.e. the characteristic system:

$$\frac{dx}{1} = \frac{dy}{y} = d\epsilon. \tag{2.32}$$

We use Eqs. 2.31 and 2.32 to find a new coordinate system that we can transform our original variables into that will make it easier to solve the original equation. To do this, we need to digress to discuss canonical coordinates, shown in the next section, from which we will pick up this problem afterwards.

2.4 Digression: Reduction of Order via Canonical Coordinates

A one-parameter group $\widetilde{x} = \alpha(x, y; \epsilon)$ and $\widetilde{y} = \beta(x, y; \epsilon)$ with the infinitesimal generator

$$V(x, y) = \eta(x, y)\frac{\partial}{\partial x} + \phi(x, y)\frac{\partial}{\partial y} \tag{2.33}$$

can be reduced by a change of variables

$$r = r(x, y) \tag{2.34a}$$

$$s = s(x, y) \tag{2.34b}$$

to the translation group

$$\widetilde{r} = r \tag{2.35a}$$

$$\widetilde{s} = s + \epsilon \tag{2.35b}$$

using the infinitesimal generator:

$$V(r,s) = \frac{\partial}{\partial s}. \tag{2.36}$$

This change of variables Eq. 2.34, when inserted into Eq. 2.33, gives:

$$\begin{aligned} V(r,s) &= \left(\eta\frac{\partial r}{\partial x} + \phi\frac{\partial r}{\partial y}\right)\frac{\partial}{\partial r} + \left(\eta\frac{\partial s}{\partial x} + \phi\frac{\partial s}{\partial y}\right)\frac{\partial}{\partial s} \\ &= (Vr)\frac{\partial}{\partial r} + (Vs)\frac{\partial}{\partial s}. \end{aligned} \tag{2.37}$$

Compare Eqs. 2.37 and 2.36 to find two equations that our canonical coordinates must satisfy:

$$Vr = 0 = \eta\frac{\partial r}{\partial x} + \phi\frac{\partial r}{\partial y} \tag{2.38a}$$

$$Vs = 1 = \eta\frac{\partial s}{\partial x} + \phi\frac{\partial s}{\partial y}. \tag{2.38b}$$

We can form the derivative $\mathrm{d}s/\,\mathrm{d}r$, which reduces $s(x,y) \to s(r)$, a function of a single variable r. The derivative is:

$$\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{\frac{\partial s}{\partial x} + p\frac{\partial s}{\partial y}}{\frac{\partial r}{\partial x} + p\frac{\partial r}{\partial y}} \tag{2.39}$$

where all we have done is treat the numerator and denominator independently and applied the total derivative to each variable. Upon transforming from (x,y)-space to (r,s)-space, the derivative Eq. 2.39 <u>will</u> be separable. It will be separable because we required (r,s) to satisfy the translation group with IG given by Eq. 2.36. Clearly, any translation group will therefore be separable in the respective transformed variables. Solving Eq. 2.39 via integration gives $s(r)$ and we have therefore reduced the number of variables from (x,y) to $(r,s(r))$. The process is as follows:

1. Find $r(x,y)$ by solving Eq. 2.38a. Alternatively, and what is typically done, is to solve the characteristic system, i.e., $\frac{\mathrm{d}x}{\eta} = \frac{\mathrm{d}y}{\phi}$ which may need to be rearranged to isolate x and y on the appropriate side. Solving $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\phi}{\eta}$, we can determine r to be the constant of integration.
2. Solve Eq. 2.38b for $s(x,y)$.
3. Solve Eq. 2.39 for $s(r)$.
4. Revert back to (x,y)-space to obtain a reduced equation, a similarity variable, or an outright solution to the original equation.

Back to our example problem!

2.5 Solution of the Example ODE

As a reminder, we have the characteristic system and IG:

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{y} = \mathrm{d}\epsilon \tag{2.40a}$$

$$V = \eta\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{2.40b}$$

Following the steps outlined at the end of Sec. 2.4, we find:

1. Find $r(x, y)$: Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\phi}{\eta} = y$.

   $$\int \frac{\mathrm{d}y}{y} = \int \mathrm{d}x = \ln(y) = x + c$$

   Thus we have

   $$y = r\,\mathrm{e}^{x} \tag{2.41a}$$
   $$r = y\,\mathrm{e}^{-x} \tag{2.41b}$$

   It is easy to show the above r satisfies $Vr = 0$.

2. Find $s(x, y)$ by solving $Vs = 1$:

   It can be shown that the general solution to

   $$Vs = 1 = \frac{\partial s}{\partial x} + y\frac{\partial s}{\partial y}$$

   is:

   $$s(x, y) = x + c_1 y\,\mathrm{e}^{-x} \tag{2.42}$$

   where c_1 is a constant. The new coordinate system is represented as:

   $$(r, s) = \left(y\,\mathrm{e}^{-x}, x + c_1 y\,\mathrm{e}^{-x}\right)$$

   which is an interesting form, to say the least.

3. Form the derivative $\mathrm{d}s/\,\mathrm{d}r$:

   The necessary partials to evaluate Eq. 2.39 are:

   $$\begin{aligned} \frac{\partial r}{\partial x} &= -y\,\mathrm{e}^{-x}, & \frac{\partial r}{\partial y} &= \mathrm{e}^{-x} \\ \frac{\partial s}{\partial x} &= 1 - c_1 y\,\mathrm{e}^{-x}, & \frac{\partial s}{\partial y} &= c_1\,\mathrm{e}^{-x} \end{aligned} \tag{2.43}$$

   Then we find:

   $$\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{1 - c_1 y\,\mathrm{e}^{-x} + pc_1\,\mathrm{e}^{-x}}{-y\,\mathrm{e}^{-x} + p\,\mathrm{e}^{-x}} \tag{2.44}$$

   Next, we recall $p = \mathrm{e}^{-x}\, y^2 + y + \mathrm{e}^{x}$, which can be written in terms of r as:

   $$p = (r^2 + r + 1)\,\mathrm{e}^{x}, \tag{2.45}$$

   Plugging this expression and the expression for $r = y\,\mathrm{e}^{-x}$, we find:

   $$\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{c_1 r^2 + 1 + c_1}{r^2 + 1}. \tag{2.46}$$

   This ODE is separable, again by design due to the canonical coordinates we found, thus, we find:

   $$\int \mathrm{d}s = \int \mathrm{d}r \frac{c_1 r^2 + 1 + c_1}{r^2 + 1} = s(r) = c_1 r + \tan^{-1}(r) + c_2. \tag{2.47}$$

4. Revert back to original coordinate system:

   Using Eqs. 2.41b and 2.42, we can convert the obtained solution, Eq. 2.47, back to the original coordinate system to find:

$$s(r) = c_1 r + \tan^{-1}(r) + c_2$$
$$x + c_1 y\,\mathrm{e}^{-x} = c_1 y\,\mathrm{e}^{-x} + \tan^{-1}\left(y\,\mathrm{e}^{-x}\right) + c_2.$$

   Upon cancelling and isolating y, we find our desired solution to Eq. 2.20:

$$\boxed{y(x) = \mathrm{e}^{x}\tan\left(x + c_2\right)} \tag{2.48}$$

This solution is admitted by the Lie group $(\eta, \phi) = (1, y)$. There could be other solutions admitted by other Lie groups (recall we made an assumption on the functional form of η and ϕ). Note that, although the canonical coordinates permit separability, there is no guarantee that the integral is invertible, i.e. we might not be able to solve for $y(x)$ every time. Another point to make is that we did not make any assumptions on the functional form of the canonical coordinates; thus, we found the general solution pertaining to this Lie group.

2.6 Global Transformations and Invariance Verification

Aside from finding solutions to ODEs, we can also more easily see what the Lie group physically means by mapping from the infinitesimal transformation (η, ϕ) to the *global transformation* $(\widetilde{x}, \widetilde{y})$. We can map from one to the other via the system of equations:

$$\frac{\partial \widetilde{x}}{\partial \epsilon} = \eta(\widetilde{x}, \widetilde{y}), \qquad \widetilde{x}(x, y; \epsilon = 0) = x \tag{2.49a}$$
$$\frac{\partial \widetilde{y}}{\partial \epsilon} = \phi(\widetilde{x}, \widetilde{y}), \qquad \widetilde{y}(x, y; \epsilon = 0) = y. \tag{2.49b}$$

The terminal conditions are an expression of the identity transformation, one requirement for a continuous transformation group to be classified as a Lie group. Equations 2.49a and 2.49b are readily solvable[1]. Starting with $\widetilde{x}$:

$$\frac{\partial \widetilde{x}}{\partial \epsilon} = \eta = 1 \quad \rightarrow \quad \int \mathrm{d}\widetilde{x} = \int \mathrm{d}\epsilon = \widetilde{x} = \epsilon + c \tag{2.50}$$

Using the terminal condition, $\widetilde{x}(x; \epsilon = 0) = x$, we find $c = x$ and thus $\widetilde{x}(x; \epsilon) = x + \epsilon$. Similarly,

$$\frac{\partial \widetilde{y}}{\partial \epsilon} = \phi(\widetilde{y}) = \widetilde{y} \quad \rightarrow \quad \int \frac{\mathrm{d}\widetilde{y}}{\widetilde{y}} = \int \mathrm{d}\epsilon = \ln(\widetilde{y}) = \epsilon + c \quad \rightarrow \quad \widetilde{y} = c\,\mathrm{e}^{\epsilon} \tag{2.51}$$

[1]Note that this is not always true! The ensuing integrals are not always invertible and therefore the global transformations might need to be left in an integral form. Such global transformations should still leave the original surface equation invariant.

Using the terminal condition, $\widetilde{y}(y;\epsilon=0)=y$, we find $c=y$ and thus $\widetilde{y}=y\,\mathrm{e}^{\epsilon}$ [2]. Thus, our Lie group of global transformations pertaining the the infinitesimal Lie group $(\eta,\phi)=(1,y)$ is:

$$\boxed{\begin{aligned}\widetilde{x}(x;\epsilon) &= x+\epsilon \\ \widetilde{y}(y;\epsilon) &= y\,\mathrm{e}^{\epsilon}\end{aligned}} \qquad \begin{aligned}&(2.52\text{a})\\&(2.52\text{b})\end{aligned}$$

We can, and should, use this form of the Lie group to confirm that the symmetry criterion, Eq. 2.17, is satisfied. Sometimes it is actually better to do this before looking for a solution via the canonical coordinates. This is because one might mis-derive the symmetry determining equations (e.g., missing a negative sign) and thus a solution to the SDEs are not necessarily an actual Lie group of the original equation. This may save you time, if you run into this all-too-common mistake.

We proceed by verifying the symmetry criterion, i.e. $F(x,y,p)=F(\widetilde{x},\widetilde{y},\widetilde{p})=0$, is satisfied. We first invert the Lie group (i.e., solve for $x(\widetilde{x})$ and $y(\widetilde{y})$), form the derivative $p(\widetilde{p})=\frac{\mathrm{d}y(\widetilde{y})}{\mathrm{d}x(\widetilde{x})}$, plug those expressions into $F(x(\widetilde{x}),y(\widetilde{y}),p(\widetilde{p}))$ and ensure it is the same as the original surface equation. Note that this means the equation should look exactly the same, just that all the x,y,p have tildes above them.

The inverted Lie group is:

$$x(\widetilde{x})=\widetilde{x}-\epsilon \tag{2.53a}$$

$$y(\widetilde{y})=\widetilde{y}\,\mathrm{e}^{-\epsilon}\,. \tag{2.53b}$$

Then the derivative becomes:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}(\widetilde{y}\,\mathrm{e}^{-\epsilon})}{\mathrm{d}\widetilde{x}}\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}x}=\mathrm{e}^{-\epsilon}\,\frac{\mathrm{d}\widetilde{y}}{\mathrm{d}\widetilde{x}}.$$

Other useful transformations include:

$$\begin{aligned}\mathrm{e}^{x}&=\mathrm{e}^{\widetilde{x}-\epsilon}=\mathrm{e}^{\widetilde{x}}\,\mathrm{e}^{-\epsilon}\\ \mathrm{e}^{-x}\,y^2&=\mathrm{e}^{-\widetilde{x}+\epsilon}\,\widetilde{y}^2\,\mathrm{e}^{-2\epsilon}=\mathrm{e}^{-\epsilon}\,\mathrm{e}^{-\widetilde{x}}\,\widetilde{y}^2.\end{aligned}$$

Recalling the original equation and inserting the above expressions, we find:

$$\begin{aligned}0&=\frac{\mathrm{d}y}{\mathrm{d}x}-\mathrm{e}^{-x}\,y^2-y-\mathrm{e}^{x}\\ &=\mathrm{e}^{-\epsilon}\,\frac{\mathrm{d}\widetilde{y}}{\mathrm{d}\widetilde{x}}-\mathrm{e}^{-\epsilon}\,\mathrm{e}^{-\widetilde{x}}\,\widetilde{y}^2-\mathrm{e}^{-\epsilon}\,\widetilde{y}-\mathrm{e}^{-\epsilon}\,\mathrm{e}^{\widetilde{x}}\\ &=\frac{\mathrm{d}\widetilde{y}}{\mathrm{d}\widetilde{x}}-\mathrm{e}^{-\widetilde{x}}\,\widetilde{y}^2-\widetilde{y}-\mathrm{e}^{\widetilde{x}}\end{aligned}$$

Thus, because the first and third lines of the above are equivalent, the equation is indeed left invariant under the Lie group of transformations. This would not be true if, say, one term had an extra $\mathrm{e}^{-\epsilon}$ attached to it. Such an exercise is vital to ensuring 1. that you derived the SDEs correctly and 2. that you do indeed have a Lie group of transformations.

[2] We can also incorporate the terminal conditions directly into the integration limits as:

$$\int_{\widetilde{x}(\epsilon=0)}^{\widetilde{x}}\mathrm{d}\widetilde{x}'=\int_{x}^{\widetilde{x}}\mathrm{d}\widetilde{x}'=\int_0^{\epsilon}\mathrm{d}\epsilon'=\widetilde{x}-x=\epsilon\to\widetilde{x}(x;\epsilon)=x+\epsilon$$

and similarly for $\widetilde{y}$:

$$\int_y^{\widetilde{y}}\frac{\mathrm{d}\widetilde{y}'}{\widetilde{y}'}=\int_0^{\epsilon}\mathrm{d}\epsilon'=\ln\left(\widetilde{y}/y\right)=\epsilon\to\widetilde{y}(y;\epsilon)=y\,\mathrm{e}^{\epsilon}$$

2.7 Exercises

1. Using LGT, solve the ODE:

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + xy - x^2 \tag{2.55}$$

2. Using LGT, solve the ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + a\lambda + a(\lambda - a)\cot^2(\lambda x) \tag{2.56}$$

   where a and λ are real constants.

3. Using LGT, solve the ODE:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2 - a^2 x \ln^{2k}(\beta x) + ak\ln^{k-1}(\beta x) \tag{2.57}$$

   where a and β are real constants and $k = 1, 2, \ldots$.

4. Consider the general Bernoulli ODE,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y = g(x)y^a, \quad a \neq 0, 1. \tag{2.58}$$

   (a) Find the Lie group that permits the general solution:

$$[y(x)]^{1-a} = C\,\mathrm{e}^{-F(x)} + (1-a)\,\mathrm{e}^{-F(x)} \int^x \mathrm{d}x'\,\mathrm{e}^{-F(x')}\,g(x'), \quad \text{where } C \text{ is a constant and } F(x) = (1-a)\int^x \mathrm{d}x' f(x'). \tag{2.59}$$

   (b) Comment on the form of this solution compared to the general solution of the inhomogeneous separable ODE: $\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y = S(x)$ (i.e., solve this ODE with the integrating factor method and compare solutions).

   (c) Now derive the symmetry determining equations for the inhomogeneous separable ODE: $\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y = S(x)$. (Hint: can you manipulate the determining equations for Eq. 2.59 instead of re-deriving the determining equations?)

   (d) Do the solutions share a Lie group, aside from the value of a? Start by simplifying the Lie group from part (a) and see if that gives the desired solution.

   (e) Comment on this Lie group and its global transformations.

5. <u>Extra</u>: Consider the generalized homogeneous first-order ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} f(x^n y^m) \tag{2.60}$$

   Can you find a Lie group that allows for this equation to be separable? (Hint: the substitution $z = x^n y^m$ leads to the separable equation $x\frac{\mathrm{d}z}{\mathrm{d}x} = nz + mzf(z)$. Thus there must be a Lie group corresponding to the canonical coordinate transformation that permits this reduction to a separable equation).

Chapter 3

Symmetry Analysis of Second-Order ODEs

3.1 Generalized Prolongation Formula for ODEs

If the ODE we wish to solve is of order-(n), then we must prolong our infinitesimal generator n times to account for leaving all n derivatives invariant under the Lie group of transformations. Given the K-order ODE:

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \dots, \frac{\mathrm{d}^K y}{\mathrm{d}x^K}\right) = 0, \tag{3.1}$$

we denote the k-th derivative as:

$$p^{(k)} = \frac{\mathrm{d}^k y}{\mathrm{d}x^k}. \tag{3.2}$$

The transformations written as their convergent Lie series are:

$$\widetilde{x} = x + \epsilon\eta(x, y) + \mathcal{O}\left(\epsilon^2\right) = \alpha(x, y; \epsilon) \tag{3.3a}$$

$$\widetilde{y} = y + \epsilon\phi(x, y) + \mathcal{O}\left(\epsilon^2\right) = \beta(x, y; \epsilon) \tag{3.3b}$$

$$\widetilde{p}^{(k)} = p^{(k)} + \epsilon\zeta^{(k)}(x, y, p^{(1)}, \dots, p^{(k)}) + \mathcal{O}\left(\epsilon^2\right) = \gamma^{(k)}(x, y, p^{(1)}, \dots, p^{(k)}; \epsilon) \tag{3.3c}$$

for $k = 1, 2, \dots, K$.

The k-th prolongation of the infinitesimal generator is given by:

$$\mathrm{pr}^{(k)}V = \eta(x, y)\frac{\partial}{\partial x} + \phi(x, y)\frac{\partial}{\partial y} + \sum_{k=1}^{K} \zeta^{(k)}(x, y, p^{(1)}, \dots, p^{(k)})\frac{\partial}{\partial p^{(k)}}. \tag{3.4}$$

As before in Sec. 2.1, we want to derive an expression for $\zeta^{(k)}$. This is done by noting the total derivative:

$$\mathrm{D} = \frac{\partial}{\partial x} + p^{(1)}\frac{\partial}{\partial y} + p^{(2)}\frac{\partial}{\partial p^{(1)}} + \cdots + p^{(k+1)}\frac{\partial}{\partial p^{(k)}} \tag{3.5}$$

then we can define the k-th transformed derivative as:

$$\begin{aligned} \widetilde{p}^{(k)} &= \frac{\mathrm{D}\widetilde{p}^{(k-1)}}{\mathrm{D}\widetilde{x}} \\ &= \frac{\mathrm{D}\gamma^{(k-1)}(x, y, p^{(1)}, \dots, p^{(k-1)}; \epsilon)}{\mathrm{D}\alpha(x, y; \epsilon)} \end{aligned} \tag{3.6}$$

We use Eq. 3.6 to find expressions for $\zeta^{(k)}$. Thus, we treat the numerator and denominator of Eq. 3.6 independently and use the Lie series given by Eqs. 3.3a and 3.3c to find:

$$\begin{aligned}
\widetilde{p}^{(k)} &= \frac{\mathrm{D}\widetilde{p}^{(k-1)}}{\mathrm{D}\widetilde{x}} \\
&= \frac{\mathrm{D}\left[p^{(k-1)} + \epsilon\zeta^{(k-1)} + \mathcal{O}(\epsilon^2)\right]}{\mathrm{D}\left[x + \epsilon\eta + \mathcal{O}(\epsilon^2)\right]} \\
&= \left(\frac{p^{(k)} + \epsilon\,\mathrm{D}\zeta^{(k-1)}}{1 + \epsilon\,\mathrm{D}\eta}\right) + \mathcal{O}(\epsilon^2)
\end{aligned}$$

By now multiplying the last line by the denominator, we find:

$$\begin{aligned}
\widetilde{p}^{(k)}(1 + \epsilon\,\mathrm{D}\eta) &= p^{(k)} + \epsilon\,\mathrm{D}\zeta^{(k-1)} + (1 + \epsilon\,\mathrm{D}\eta)\mathcal{O}(\epsilon^2) \\
\widetilde{p}^{(k)} + \widetilde{p}^{(k)}\epsilon\,\mathrm{D}\eta &= p^{(k)} + \epsilon\,\mathrm{D}\zeta^{(k-1)} + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Now inserting Eq. 3.3c into the second occurrence of $\widetilde{p}^{(k)}$:

$$\widetilde{p}^{(k)} + (p^{(k)} + \epsilon\zeta^{(k)} + \mathcal{O}(\epsilon^2))\epsilon\,\mathrm{D}\eta = p^{(k)} + \epsilon\,\mathrm{D}\zeta^{(k-1)} + \mathcal{O}(\epsilon^2)$$

Now solving for $\widetilde{p}^{(k)}$, we find:

$$\widetilde{p}^{(k)} = p^{(k)} + \epsilon\left[\mathrm{D}\zeta^{(k-1)} - p^{(k)}\,\mathrm{D}\eta\right] + \mathcal{O}(\epsilon^2) \tag{3.7}$$

Finally, if we compare Eq. 3.3c with Eq. 3.7, we may infer the definition of the coordinate function for the k-th derivative as:

$$\boxed{\zeta^{(k)} = \mathrm{D}\zeta^{(k-1)} - p^{(k)}\,\mathrm{D}\eta, \quad k \geq 1} \tag{3.8}$$

By induction, we may also write:

$$\zeta^{(k)} = \mathrm{D}^k\phi - \sum_{j=1}^{k} \frac{k!}{(k-j)!j!} p^{(k-j+1)}\,\mathrm{D}^j\eta \tag{3.9}$$

where $D^2 f = D(Df)$, $D^3 f = D(D(Df))$, and so on for $D^k f$. As an example, we derive the first and second prolongation formula for the coordinate functions.

3.1.1 Derivation of the prolonged Infinitesimals

Example: derive the first prolonged infinitesimal, $\zeta^{(1)}$, using Eq. 3.9.

$$\begin{aligned}
\zeta^{(1)} &= \mathrm{D}\phi(x,y) - p^{(1)}\,\mathrm{D}\eta(x,y) \\
&= \frac{\partial\phi}{\partial x} + p^{(1)}\frac{\partial\phi}{\partial y} - p^{(1)}\left[\frac{\partial\eta}{\partial x} + p^{(1)}\frac{\partial\eta}{\partial y}\right] \\
&= \frac{\partial\phi}{\partial x} + \left(\frac{\partial\phi}{\partial y} - \frac{\partial\eta}{\partial x}\right)p^{(1)} - \frac{\partial\eta}{\partial y}\left(p^{(1)}\right)^2
\end{aligned} \tag{3.10}$$

Clearly, the last line agrees exactly with Eq. 2.15.

Example: derive the second prolonged infinitesimal, $\zeta^{(2)}$, using Eq. 3.8. We choose to use $p^{(1)} = y'$ and $p^{(2)} = y''$ as well as using subscript notation for partials, e.g. $\frac{\partial \eta}{\partial x} = \eta_x$, as that will make the notation less confusing. Then,

$$\begin{aligned} \zeta^{(2)} &= \mathrm{D}\zeta^{(1)} - p^{(2)}\,\mathrm{D}\eta \\ &= \mathrm{D}\zeta^{(1)} - y''\left[\eta_x + \eta_y y'\right] \\ &= \mathrm{D}\zeta^{(1)} - \eta_x y'' - \eta_y y' y''. \end{aligned} \tag{3.11}$$

Note that for these functions,

$$\mathrm{D} = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'}. \tag{3.12}$$

We can use Eq. 3.10 to evaluate $\mathrm{D}\zeta^{(1)}$:

$$\begin{aligned} \mathrm{D}\zeta^{(1)} &= \mathrm{D}\phi_x + \mathrm{D}[\phi_y y'] - \mathrm{D}[\eta_x y'] - \mathrm{D}[\eta_y (y')^2] \\ &= \mathrm{D}\phi_x + y'\,\mathrm{D}\phi_y + \phi_y\,\mathrm{D}[y'] - y'\,\mathrm{D}\eta_x - \eta_x\,\mathrm{D}[y'] - (y')^2\,\mathrm{D}\eta_y - \eta_y\,\mathrm{D}[(y')^2] \end{aligned} \tag{3.13}$$

The necessary total derivatives are:

$$\begin{aligned} \mathrm{D}\phi_x &= \phi_{xx} + y'\phi_{xy} \qquad & \mathrm{D}\phi_y &= \phi_{xy} + y'\phi_{yy} \\ \mathrm{D}\eta_x &= \eta_{xx} + y'\eta_{xy} \qquad & \mathrm{D}\eta_y &= \eta_{xy} + y'\eta_{yy} \\ \mathrm{D}[y'] &= y'' \qquad & \mathrm{D}[(y')^2] &= 2y'y'' \end{aligned}$$

Assembling all of these derivatives, we find:

$$\begin{aligned} \mathrm{D}\zeta^{(1)} = {} & \phi_{xx} + y'\phi_{xy} + y'[\phi_{xy} + y'\phi_{yy}] + \phi_y y'' - y'[\eta_{xx} + y'\eta_{xy}] - \eta_x y'' \\ & - (y')^2[\eta_{xy} + y'\eta_{yy}] - 2\eta_y y' y'' \\ = {} & \phi_{xx} + [2\phi_{xy} - \eta_{xx}]y' + [\phi_{yy} - 2\eta_{xy}](y')^2 - \eta_{yy}(y')^3 \\ & + [\phi_y - \eta_x]y'' - 2\eta_y y' y'' \end{aligned} \tag{3.14}$$

Finally, we find

$$\begin{aligned} \zeta^{(2)} = {} & \phi_{xx} + [2\phi_{xy} - \eta_{xx}]y' + [\phi_{yy} - 2\eta_{xy}](y')^2 - \eta_{yy}(y')^3 \\ & + [\phi_y - 2\eta_x]y'' - 3\eta_y y' y'' \end{aligned} \tag{3.15}$$

or written in the original notation

$$\boxed{\begin{aligned} \zeta^{(2)} = {} & \frac{\partial^2 \phi}{\partial x^2} + \left[2\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \eta}{\partial x^2}\right] p^{(1)} + \left[\frac{\partial^2 \phi}{\partial y^2} - 2\frac{\partial^2 \eta}{\partial x \partial y}\right]\left(p^{(1)}\right)^2 - \frac{\partial^2 \eta}{\partial y^2}\left(p^{(1)}\right)^3 \\ & + \left[\frac{\partial \phi}{\partial y} - 2\frac{\partial \eta}{\partial x}\right] p^{(2)} - 3\frac{\partial \eta}{\partial y} p^{(1)} p^{(2)} \end{aligned}} \tag{3.16}$$

3.2 Formulae for Second-Order Canonical Coordinate Reductions

Along with needing the second prolongation for second-order ODEs, we will also need additional higher-order derivatives for the canonical coordinates, namely $\mathrm{d}^2 s/\,\mathrm{d}r^2$. Without derivation, we will simply state the formulas for the derivatives and how to invert back to the original space once the newly-obtained separable equation, $\mathrm{d}^2 s/\,\mathrm{d}r^2$, is solved via integration.

Recall the first derivative is:

$$\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{\frac{\partial s}{\partial x} + y' \frac{\partial s}{\partial y}}{\frac{\partial r}{\partial x} + y' \frac{\partial r}{\partial y}} \tag{3.17}$$

and the second-order derivative is:

$$\frac{\mathrm{d}^2 s}{\mathrm{d}r^2} = y'' f_1 \left(r, s, \frac{\mathrm{d}s}{\mathrm{d}r} \right) + g_1 \left(r, s, \frac{\mathrm{d}s}{\mathrm{d}r} \right) \tag{3.18}$$

where we have defined:

$$f_1 = \frac{1}{\delta^3} \left(\frac{\partial s}{\partial y} \frac{\partial r}{\partial x} - \frac{\partial s}{\partial x} \frac{\partial r}{\partial y} \right) \tag{3.19a}$$

$$\begin{aligned} g_1 = \frac{1}{\delta^3} \Bigg[& \left(\frac{\partial r}{\partial y} \frac{\partial^2 s}{\partial y^2} - \frac{\partial s}{\partial y} \frac{\partial^2 r}{\partial y^2} \right) (y')^3 + \left(2 \frac{\partial r}{\partial y} \frac{\partial^2 s}{\partial x \partial y} + \frac{\partial r}{\partial x} \frac{\partial^2 s}{\partial y^2} - 2 \frac{\partial s}{\partial y} \frac{\partial^2 r}{\partial x \partial y} - \frac{\partial s}{\partial x} \frac{\partial^2 r}{\partial y^2} \right) (y')^2 \\ & + \left(2 \frac{\partial r}{\partial x} \frac{\partial^2 s}{\partial x \partial y} + \frac{\partial r}{\partial y} \frac{\partial^2 s}{\partial x^2} - 2 \frac{\partial s}{\partial x} \frac{\partial^2 r}{\partial x \partial y} - \frac{\partial s}{\partial y} \frac{\partial^2 r}{\partial x^2} \right) y' + \frac{\partial r}{\partial x} \frac{\partial^2 s}{\partial x^2} - \frac{\partial s}{\partial x} \frac{\partial^2 r}{\partial x^2} \Bigg] \end{aligned} \tag{3.19b}$$

$$\delta = \frac{\partial r}{\partial x} + y' \frac{\partial r}{\partial y}. \tag{3.19c}$$

We will also need to write y, y', and y'' in terms of s and r and plug those into the above formulae. The inversions for the first and second derivatives in the original space are:

$$y' = \frac{\frac{\partial s}{\partial x} - \frac{\partial r}{\partial x} \frac{\mathrm{d}s}{\mathrm{d}r}}{\frac{\partial r}{\partial y} \frac{\mathrm{d}s}{\mathrm{d}r} - \frac{\partial s}{\partial y}} \tag{3.20a}$$

$$y'' = \frac{1}{f_1} \frac{\mathrm{d}^2 s}{\mathrm{d}r^2} - \frac{g_1}{f_1}. \tag{3.20b}$$

Once we have formed the derivatives of $s(r)$, we reduce the order of the equations as:

$$v(r) = \frac{\mathrm{d}s}{\mathrm{d}r} \tag{3.21a}$$

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{\mathrm{d}^2 s}{\mathrm{d}r^2}. \tag{3.21b}$$

We can then integrate (and hopefully solve) Eq. 3.21b as it will be separable, and finally revert back to the original variable space to obtain the desired solution.

We are now prepared to apply LGT to second-order ODEs, shown in the next section.

3.3 Example: A Second-Order Nonlinear ODE

In this section, we show how to solve a second-order nonlinear ODE. We want to show an equation that has actual physics involved to give the reader an idea of how to contextualize LGT with a not-so-abstract example. There will be multiple times in the process where we stop and consider the physical meaning of a result or step in the process, which help to guide our decisions.

3.3.1 The Physics Model

Consider the "point-reactor power equation," which is a well-known equation in the nuclear engineering community:

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\rho(t) - \beta}{\ell} n(t), \tag{3.22}$$

where the following are defined:

- $n(t)$ is the power output of the nuclear reactor due to neutron-induced fission reactions in the uranium fuel, measured in Watts,
- $\rho(t)$ is the *reactivity*, which is a measure of how the reactor responds to changes in its' fuel; e.g., moving control rods in/out of the reactor or the change in the isotopic composition of the uranium fuel. We see that a positive reactivity increases the power production rate while a negative reactivity reduces the power production rate.
- β is the total delayed neutron fraction, which accounts for neutrons that are emitted slightly later due to radioactive decay of isotopes produced in the fission of uranium atoms. We see that β always reduces the power output rate of a reactor (this is actually a good thing as it allows us to control the reactor more effectively).
- ℓ is the neutron generation time and indicates how quickly fission chains are evolving. A typical neutron generation time is on the order of a 100 nanoseconds to few nanoseconds.

Next, suppose the reactivity is a function of the reactor temperature T, but the reactor temperature is a function of the reactor power, i.e. $T(n)$. This is known as "reactor feedback". We can also allow for mechanical changes in the reactor due to motion of the control rods; think of pulling a control rod out as a linear function in time, or perhaps the reactor operators move the control rods up/down motion as a sine wave. In general, we can assume the mechanical motion of the control rods is represented as a polynomial in time. Then the reactivity is expressed as:

$$\rho(t) = \rho_o - \alpha T(n) + \sum_{j=1}^{J} a_j t^j \tag{3.23}$$

where ρ_o is the initial reactivity of the reactor, α is the temperature coefficient of reactivity and is known, and the a_j are known polynomial coefficients. The most common feedback model is written in the form:

$$T(n) = \kappa\omega^{1-\gamma} \int_0^t \mathrm{d}\tau [n(\tau)]^\gamma \tag{3.24}$$

where κ is the inverse heat capacity of the reactor, ω is a compensation coefficient, and γ is a known exponent. Note that there are more complicated forms of the integrand, e.g. arbitrary-order derivatives of n, but we avoid those for this lecture series.

We want to avoid dealing with integro-differential equations, which we would get by directly inserting Eq. 3.23 into Eq. 3.22. Thus, we will take the time-derivative of Eq. 3.23,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\alpha\kappa\omega^{1-\gamma}[n(t)]^\gamma + \sum_{j=1}^{J} j a_j t^{j-1} \tag{3.25}$$

which can then be inserted into the time-derivative of Eq. 3.22 to yield the equation we will apply LGT to:

$$\boxed{n(t)\frac{\mathrm{d}^2 n}{\mathrm{d}t^2} = -A[n(t)]^{\gamma+2} + \frac{[n(t)]^2}{\ell}\sum_{j=1}^{J} j a_j t^{j-1} + \left(\frac{\mathrm{d}n}{\mathrm{d}t}\right)^2} \tag{3.26}$$

where $A = \alpha\kappa\omega^{1-\gamma}/\ell$ is a known coefficient.

3.3.2 Derivation of the Symmetry Determining Equations

Equation 3.26 can be written as a surface equation as:

$$F(t, n, n', n'') = 0 = nn'' + An^{\gamma+2} - \frac{n^2}{\ell}\sum_{j=1}^{J} j a_j t^{j-1} + (n')^2 . \tag{3.27}$$

The symmetry determining equations are given by:

$$\mathrm{pr}^{(2)}VF\Big|_{F=0} = 0 = \left[\eta(t,n)\frac{\partial F}{\partial t} + \phi(t,n)\frac{\partial F}{\partial n} + \zeta^{(1)}\frac{\partial F}{\partial n'} + \zeta^{(2)}\frac{\partial F}{\partial n''}\right]\Bigg|_{F=0} \tag{3.28}$$

The partials to consider are:

$$\frac{\partial F}{\partial t} = -\frac{n^2}{\ell}\sum_j j(j-1)a_j t^{j-2} \tag{3.29a}$$

$$\frac{\partial F}{\partial n} = n'' + (\gamma+2)An^{\gamma+1} - \frac{2n}{\ell}\sum_j j a_j t^{j-1} \tag{3.29b}$$

$$\frac{\partial F}{\partial n'} = -2n' \tag{3.29c}$$

$$\frac{\partial F}{\partial n''} = n \tag{3.29d}$$

For additional simplification, we define:

$$S_k = \frac{1}{\ell}\sum_{j=1}^{J} j(j-1)\cdots(j-k+1)a_j t^{j-k} \tag{3.30}$$

The once- and twice-prolonged coordinate functions are:

$$\zeta^{(1)} = \phi_t + (\phi_n - \eta_t)n' - \eta_n(n')^2 \tag{3.31a}$$

$$\zeta^{(2)} = \phi_{tt} + (2\phi_{tn} - \eta_{tt})n' + (\phi_{nn} - 2\eta_{tn})(n')^2 - \eta_{nn}(n')^3 + (\phi_n - 2\eta_t)n'' - 3\eta_n n' n'', \tag{3.31b}$$

where we have switched from explicit partial derivative notation to subscript notation for brevity.

Inserting Eqs. 3.29 and 3.31 into Eq. 3.28 and enforcing invariance (i.e., forcing $F = 0$) by setting

$$n'' = -An^{\gamma+1} + nS_1 + \frac{1}{n}(n')^2, \tag{3.32}$$

provides the overarching determining equation, after some algebraic manipulations:

$$
\begin{aligned}
\operatorname{pr}^{(2)} VF\Big|_{F=0} = 0 = &-(n')^3 n\Big[\eta_{nn} n + \eta_n\Big] \\
&+ (n')^2\Big[\big(\phi_{nn} - 2\eta_{tn}\big)n^2 - \phi_n n + \phi\Big] \\
&+ n'\Big[3\eta_n\big(An^\gamma - S_1\big)n^2 + \big(2\phi_{tn} - \eta_{tt}\big)n - 2\phi_t\Big] \\
&- \eta S_2 n^2 + \Big[\phi_{tt} + \gamma\phi A n^\gamma + \big(2\eta_t n - \phi_n n + \phi\big)\big(An^\gamma - S_1\big)\Big]n,
\end{aligned}
\tag{3.33}
$$

The quantity n' can take on arbitrary values limited only by the range of n' covered by the family of solutions of Eq. 3.27. Therefore, the only way the invariance condition given by Eq. 3.33 can be satisfied is if each coefficient in n' is individually equal to zero (this decomposition is strictly permitted because we are searching for *point transformations*, which are transformations that are not functions of the derivatives of the solution n' and n''). The infinitesimals must then satisfy the following system of determining equations:

$$0 = n\frac{\partial^2 \eta}{\partial n^2} + \frac{\partial \eta}{\partial n} \tag{3.34a}$$

$$0 = \left(\frac{\partial^2 \phi}{\partial n^2} - 2\frac{\partial^2 \eta}{\partial t \partial n}\right)n^2 - \frac{\partial \phi}{\partial n}n + \phi \tag{3.34b}$$

$$0 = 3\Big(An^\gamma - S_1\Big)\frac{\partial \eta}{\partial n}n^2 + \left(2\frac{\partial^2 \phi}{\partial t \partial n} - \frac{\partial^2 \eta}{\partial t^2}\right)n - 2\frac{\partial \phi}{\partial t} \tag{3.34c}$$

$$0 = -\eta S_2 n + \frac{\partial^2 \phi}{\partial t^2} + \gamma\phi A n^\gamma + \left(2\frac{\partial \eta}{\partial t}n - \frac{\partial \phi}{\partial n}n + \phi\right)\big(An^\gamma - S_1\big). \tag{3.34d}$$

We proceed by first solving Eq. 3.34a as it as an equation that is a function only of $\eta(t, n)$. This equation is a homogeneous linear parabolic partial differential equation (PDE) with a variable coefficient which has the solution:

$$\boxed{\eta(t, n) = \ln(n)c_1(t) + c_2(t),} \tag{3.35}$$

where $c_1(t)$ and $c_2(t)$ are unknown functions of time. Next, we solve Eq. 3.34b with Eq. 3.35 for $\phi(t, n)$, which is an inhomogeneous linear parabolic PDE with the solution:

$$\boxed{\phi(t, n) = n\ln^2(n)\frac{\mathrm{d}c_1}{\mathrm{d}t} + nc_3(t) + n\ln(n)c_4(t).} \tag{3.36}$$

where $c_3(t)$ and $c_4(t)$ are unknown functions of time. We note that the above general expressions for η and ϕ do not depend on γ nor S_k and we will therefore only see alterations in the time-dependent coefficients for differing reactivity models described later.

At this point, the system is underdetermined as we have two remaining determining equations and four unknown time-dependent coefficients. By inserting the solutions of η and ϕ, given by Eqs. 3.35 and 3.36, into the remaining determining equations, Eqs. 3.34c and 3.34d, we devise two constraining equations for the time-dependent coefficients:

$$0 = 3\ln(n)\frac{\mathrm{d}^2 c_1}{\mathrm{d}t^2} - \frac{\mathrm{d}^2 c_2}{\mathrm{d}t^2} + 2\frac{\mathrm{d}c_4}{\mathrm{d}t} + 3(An^\gamma - S_1)c_1 \tag{3.37a}$$

$$0 = \ln^2(n)\frac{\mathrm{d}^3 c_1}{\mathrm{d}t^3} + \frac{\mathrm{d}^2 c_3}{\mathrm{d}t^2} + \ln(n)\frac{\mathrm{d}^2 c_4}{\mathrm{d}t^2} - 2S_1\frac{\mathrm{d}c_2}{\mathrm{d}t} - \ln(n)S_2 c_1 - S_2 c_2 + S_1 c_4$$
$$+ An^\gamma\left[\gamma\ln^2(n)\frac{\mathrm{d}c_1}{\mathrm{d}t} + 2\frac{\mathrm{d}c_2}{\mathrm{d}t} + \gamma c_3 + (\gamma\ln(n) - 1)c_4\right]. \tag{3.37b}$$

In order for Eqs. 3.37 to be satisfied, we note that all n-dependence must be removed as these are ODEs for time. Concerning Eq. 3.37a, there does not appear to be simple rearranging to remove the n dependence except to set

$$\boxed{c_1 \equiv 0} \tag{3.38}$$

this is effectively a statement that η is a purely time-dependent function for all reactivity models encapsulated by the rate function Eq. 3.25. With this enforcement, our determining equations for the three remaining time-dependent coefficients are:

$$0 = -\frac{\mathrm{d}^2 c_2}{\mathrm{d}t^2} + 2\frac{\mathrm{d}c_4}{\mathrm{d}t} \tag{3.39a}$$

$$0 = \frac{\mathrm{d}^2 c_3}{\mathrm{d}t^2} + \ln(n)\frac{\mathrm{d}^2 c_4}{\mathrm{d}t^2} - 2S_1\frac{\mathrm{d}c_2}{\mathrm{d}t} - S_2 c_2 + S_1 c_4 + An^\gamma\left[2\frac{\mathrm{d}c_2}{\mathrm{d}t} + \gamma c_3 + (\gamma\ln(n) - 1)c_4\right]. \tag{3.39b}$$

We are leaving c_4 in the above because there are scenarios where we may remove the n-dependence without setting c_4 to zero (e.g., say $A = 0$, then we need only say $c_4'' = 0$ which indicates c_4 is at-most a linear function in time). Thus, our infinitesimal Lie group is:

$$\eta(t) = c_2(t) \tag{3.40a}$$

$$\phi(t, n) = nc_3(t) + n\ln(n)c_4(t) \tag{3.40b}$$

where, again, we note that we have found the explicit functional dependence the Lie group has on n, but because we have arbitrary-order polynomial-in-t, this is the most precise form of the Lie group we can have while maintaining generality of the reactivity model. In the next section, we consider a well-known reactivity model to then solve the system, Eq. 3.39.

3.3.3 A Specific Reactivity Model

The Nordheim-Fuchs (NF) Model assumes the reactivity rate is a linear function of the reactor power and thus $\gamma = 1$ and we set $a_j = 0$ for all j in Eq. 3.25. However, we will retain γ as an arbitrary value in an attempt to capture a family of permitted analytical solutions. The corresponding reactivity rate expression is $\mathrm{d}\rho/\mathrm{d}t = -\alpha\kappa\omega^{1-\gamma}n^\gamma$ and the power equation, Eq. 3.26, reduces to

$$n(t)\frac{\mathrm{d}^2 n(t)}{\mathrm{d}t^2} = -A[n(t)]^{\gamma+2} + \left(\frac{\mathrm{d}n(t)}{\mathrm{d}t}\right)^2, \tag{3.41}$$

where $A = \alpha\kappa\omega^{1-\gamma}/\ell$.

Since $a_j = 0$ for all j, then $S_k = 0$ and the coefficient system, Eq. 3.39, becomes:

$$0 = -\frac{\mathrm{d}^2 c_2}{\mathrm{d}t^2} + 2\frac{\mathrm{d}c_4}{\mathrm{d}t} \tag{3.42a}$$

$$0 = \frac{\mathrm{d}^2 c_3}{\mathrm{d}t^2} + \ln(n)\frac{\mathrm{d}^2 c_4}{\mathrm{d}t^2} + An^\gamma\left[2\frac{\mathrm{d}c_2}{\mathrm{d}t} + \gamma c_3 + (\gamma\ln(n) - 1)c_4\right]. \tag{3.42b}$$

We see that we are now required to set

$$c_4 \equiv 0 \tag{3.43}$$

to further untangle the n-dependence from Eq. 3.42b. Doing so, Eqs. 3.42 simplifies to:

$$0 = \frac{\mathrm{d}^2 c_2}{\mathrm{d}t^2} \tag{3.44a}$$

$$0 = \frac{\mathrm{d}^2 c_3}{\mathrm{d}t^2} + An^{\gamma}\left[2\frac{\mathrm{d}c_2}{\mathrm{d}t} + \gamma c_3\right]. \tag{3.44b}$$

Equation 3.44a informs us that c_2 is, at most, a linear function in time:

$$c_2(t) = m_2 t + b_2, \tag{3.45}$$

where m_2 is the slope of c_2 and b_2 is the arbitrary initial condition. Equation 3.44b is a polynomial in n whose coefficients are not functions of n. Thus, in order for Eq. 3.44b to be satisfied for all arbitrary values of n, its coefficients must individually equal zero. This reveals two more equations:

$$\frac{\mathrm{d}^2 c_3}{\mathrm{d}t^2} = 0 \tag{3.46a}$$

$$2\frac{\mathrm{d}c_2}{\mathrm{d}t} + \gamma c_3 = 0. \tag{3.46b}$$

Equation 3.46a informs us that c_3 is also a linear function in time:

$$c_3(t) = m_3 t + b_3, \tag{3.47}$$

where m_3 is the slope of c_3 and b_3 is the arbitrary initial condition. Finally, we use Eq. 3.46b to find $2m_2 + \gamma m_3 t + \gamma b_3 = 0$, which can only be satisfied if $m_3 = 0$ and we find $m_2 = -\gamma b_3/2$. Therefore c_3 is not a function of time and ϕ is purely a function of n. To summarize, the coordinate functions of the NF model are:

$$\boxed{\eta(t) = -\frac{\gamma b_3}{2}t + b_2} \tag{3.48a}$$

$$\boxed{\phi(n) = b_3 n,} \tag{3.48b}$$

which is a two-parameter symmetry group. In order to obtain analytical solutions through the canonical coordinates method, we must consider the one-parameter symmetry groups separately by setting either $b_2 = 0$ or $b_3 = 0$.

3.3.4 Solutions Admitted by the b_2-group

We first derive the canonical coordinates for the infinitesimal generator corresponding to the one-parameter b_2 symmetry group:

$$\eta(t) = b_2 \tag{3.49a}$$

$$\phi(n) = 0, \tag{3.49b}$$

whose infinitesimal generator is:

$$V = b_2 \frac{\partial}{\partial t}. \tag{3.50}$$

The canonical coordinates must satisfy the PDEs:

$$Vr = 0 = b_2 \frac{\partial r}{\partial t} \tag{3.51a}$$

$$Vs = 1 = b_2 \frac{\partial s}{\partial t}. \tag{3.51b}$$

We first obtain r from the relation: $\frac{\mathrm{d}t}{b_2} = \frac{\mathrm{d}n}{0} \to \frac{\mathrm{d}n}{\mathrm{d}t} = 0$ which, upon integrating and solving for the constant, results in:

$$r(n) = n. \tag{3.52}$$

The next coordinate, s, is found by solving Eq. 3.51b to find:

$$s(t) = \frac{t}{b_2} + k, \tag{3.53}$$

where k is an arbitrary constant. The relevant partial derivatives are: $r_t = 0$, $r_n = 1$, $s_t = 1/b_2$, and $s_n = 0$. From Eq. 3.17, we find the constraint equation:

$$\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{1}{b_2 n'} \tag{3.54}$$

where we have evoked $y' = n'$. Next, we introduce v as a new variable to effectively reduce the order of the original equation:

$$v(r) = \frac{\mathrm{d}s}{\mathrm{d}r}. \tag{3.55}$$

The next step is to determine the derivative: $\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{\mathrm{d}^2 s}{\mathrm{d}r^2}$ corresponding to Eq. 3.18. For the canonical coordinates of this symmetry group, all second-order partial derivatives are zero, and thus $g_1 = 0$ and we find

$$\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{b_2} \frac{n''}{(n')^3}. \tag{3.56}$$

We will now use Eqs. 3.52, 3.54, and 3.56 to solve for n, n', and n'', respectively:

$$n = r \tag{3.57a}$$

$$n' = \frac{1}{b_2 v} \tag{3.57b}$$

$$n'' = -\frac{1}{b_2^2 v^3} \frac{\mathrm{d}v}{\mathrm{d}r} \tag{3.57c}$$

Inserting these identities into Eq. 3.41 transforms it into an equation in the canonical coordinate space. The second-order nonlinear power equation is reduced to a first-order nonlinear ODE in v:

$$\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{1}{r} v(r) = A b_2^2 r^{\gamma+1} v^3, \tag{3.58}$$

which is classified as a third-order Bernoulli ODE. This equation is independent of s, which is a direct result of the requirement that the canonical coordinates abide to the translation group given by Eq. 2.35. Ordinary differential equations of the Bernoulli-type are particularly convenient as they are nonlinear ODEs with analytical solutions. We may transform Eq. 3.58 into a linear equation by introducing the substitution: $u = v^{-2}$, which gives $v = \sqrt{u}$ and $\mathrm{d}v/\,\mathrm{d}r = -\frac{1}{2}u^{-3/2}\,\mathrm{d}u/\,\mathrm{d}r$. Making these substitutions, Eq. 3.58 reduces to the linear ODE:

$$\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{2}{r}u(r) = -2Ab_2^2 r^{\gamma+1}. \tag{3.59}$$

Equation 3.59 may be further reduced using r^{-2} as an integrating factor and then solved to find:

$$u(r) = \frac{1}{v^2} = \left(k_1 - \frac{2Ab_2^2}{\gamma}r^\gamma\right)r^2, \tag{3.60}$$

where k_1 is a constant. By solving for v and recalling $v = 1/(b_2 n')$ and $r = n$, we obtain a first-order nonlinear ODE satisfied by the reactor power:

$$\frac{\mathrm{d}n(t)}{\mathrm{d}t} = \pm n(t)\sqrt{\frac{k_1}{b_2^2} - \frac{2A}{\gamma}[n(t)]^\gamma}. \tag{3.61}$$

By inspection, Eq. 3.61 is invariant under time translation transformations and is consequently separable and may be integrated by quadrature to yield:

$$n(t) = \sqrt[\gamma]{\frac{\gamma k_1}{2Ab_2^2}}\,\mathrm{sech}^{2/\gamma}\left(\frac{\sqrt{k_1}}{2b_2}\left[\gamma b_2 k_2 - t\right]\right) \tag{3.62}$$

where k_2 is another constant. By setting $\gamma = 1$, we obtain the classical result obtained by Nordheim [3], Fuchs [4], and Hetrick [5] and may be rendered physically meaningful by using relevant initial conditions. By taking the first and second derivatives of Eq. 3.62 and inserting into Eq. 3.41, we find a condition on γ:

$$\gamma = \pm 1. \tag{3.63}$$

This reveals one additional analytical solution that was not made present in the classical derivations for temperature feedback of the form $\mathrm{d}\rho/\,\mathrm{d}t = -\alpha\kappa n$ and $\mathrm{d}\rho/\,\mathrm{d}t = -\alpha\kappa/(\omega^2 n)$. If one were attempting to verify a code that applies the NF model, they could simply set $\gamma = 1$ and use the necessary initial conditions to determine k_1 and k_2. We use the initial conditions: $n(t_o) = n_o$ and $\frac{\mathrm{d}n}{\mathrm{d}t}\big|_{t_o} = 0$, which is to say that the power reaches a maximum at t_o, and we find the following solutions:

$$n(t) = n_o\,\mathrm{sech}^2\left(\sqrt{\frac{\alpha\kappa n_o}{2\ell}}\,[t - t_o]\right), \qquad \gamma = 1 \tag{3.64a}$$

$$n(t) = n_o\cos^2\left(\sqrt{\frac{\alpha\kappa}{2\ell n_o}}\,[t - t_o]\right), \qquad \gamma = -1., \tag{3.64b}$$

The physics discussion is left to the footnote[1].

[1] We demonstrate the qualitative difference between the two solutions in Fig. 3.1, noting that the classical solution ($\gamma = 1$) decays at a lower rate than the reciprocal solution ($\gamma = -1$). This may be attributed to the corresponding feedback model for each case, $\rho'(t;\gamma = 1) \propto -n(t)$ versus $\rho'(t;\gamma = -1) \propto -1/n(t)$, where we see that as time passes and the power decreases, the reactivity rate for $\gamma = 1$ linearly decreases while the reactivity rate for $\gamma = -1$ tends to increase.

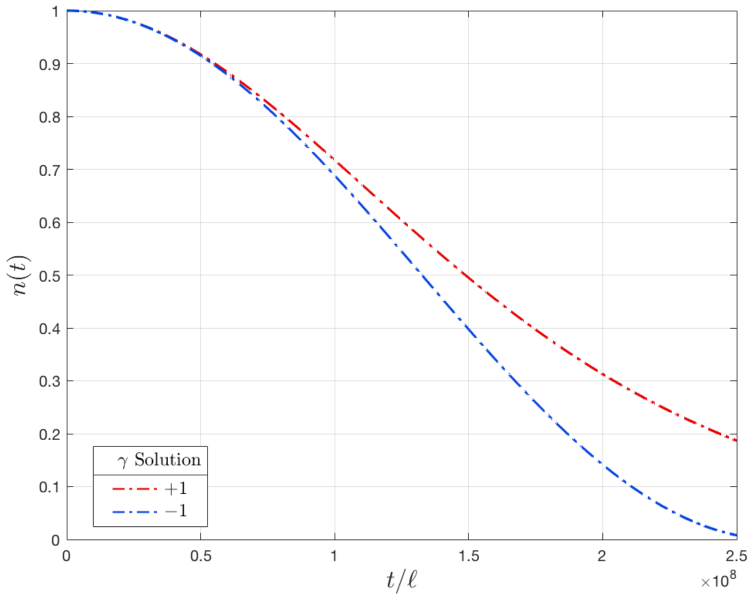


Figure 3.1: Comparison between the Nordheim-Fuchs solutions for the two permitted values of γ.

3.3.5 Solutions Admitted by the b_3-group

We now derive the canonical coordinates for the infinitesimal generator corresponding to the one-parameter b_3 symmetry group

$$\eta(t) = -\frac{\gamma b_3 t}{2} \tag{3.65a}$$

$$\phi(n) = b_3 n, \tag{3.65b}$$

whose infinitesimal generator is:

$$V = -\frac{\gamma b_3 t}{2}\frac{\partial}{\partial t} + b_3 n \frac{\partial}{\partial n}. \tag{3.66}$$

Then the canonical coordinates satisfy:

$$Vr = 0 = -\frac{\gamma b_3 t}{2}\frac{\partial r}{\partial t} + b_3 n \frac{\partial r}{\partial n} \tag{3.67a}$$

$$Vs = 1 = -\frac{\gamma b_3 t}{2}\frac{\partial s}{\partial t} + b_3 n \frac{\partial s}{\partial n}. \tag{3.67b}$$

We first obtain r by solving the characteristic system: $\frac{\mathrm{d}t}{\eta} = \frac{\mathrm{d}n}{\phi}$. The corresponding ODE is $\frac{\mathrm{d}n}{\mathrm{d}t} = -2n/\gamma t$. Upon integration and setting r to the constant of integration, we find:

$$r(t, n) = t^{2/\gamma} n. \tag{3.68}$$

Next, we obtain s by directly solving Eq. 3.67b, which yields

$$s(t, n) = k_1 t^{2/\gamma} n - \frac{2}{\gamma b_3}\ln(t), \tag{3.69}$$

where k_1 is a constant. Noting the partial derivatives for r and s,

$$\begin{aligned} r_t &= at^{a-1}n, \quad & s_t &= ak_1 t^{a-1}n - \frac{a}{b_3 t} \\ r_n &= t^a, & s_n &= k_1 t^a, \\ a &= \frac{2}{\gamma} \end{aligned} \tag{3.70}$$

we may form the constraint derivative, keeping Eq. 3.17 in mind,

$$v(r) = \frac{\mathrm{d}s}{\mathrm{d}r} = k_1 - \frac{a}{b_3 t^a (an + tn')}, \tag{3.71}$$

where we have simultaneously defined the reducing variable, $v(r)$. The next derivative is determined by utilizing Eqs. 3.18 and 3.19, where the second-order partial derivatives of r and s are:

$$\begin{aligned} r_{tt} &= a(a-1)t^{a-2}n, \quad & s_{tt} &= a(a-1)k_1 t^{a-2}n + \frac{a}{b_3 t^2} \\ r_{tn} &= at^{a-1}, & s_{tn} &= ak_1 t^{a-1} \\ r_{nn} &= 0, & s_{nn} &= 0. \end{aligned} \tag{3.72}$$

Combining the necessary derivatives, we find:

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{\mathrm{d}^2 s}{\mathrm{d}r^2} = \frac{a}{b_3 t^{2(a-1)}(an+tn')^3}\left(n'' + \frac{2a+1}{t}n' + \frac{a^2}{t^2}n\right). \tag{3.73}$$

From Eqs. 3.68, 3.71, and 3.73, we may write n, n', and n'' in terms of r, v, and t:

$$n = \frac{r}{t^a} \tag{3.74a}$$

$$n' = \frac{a}{t^{a+1}}\left[\frac{1}{b_3(k_1 - v)} - r\right] \tag{3.74b}$$

$$n'' = \frac{a}{t^{a+2}}\left[\frac{a}{b_3^2(k_1-v)^3}\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{2a+1}{b_3(k_1-v)} + (a+1)r\right]. \tag{3.74c}$$

Inserting these identities into Eq. 3.41, we obtain a first-order ODE satisfied by the constraint condition, $v(r)$,

$$\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{\gamma b_3^2 r}{2}\left(\frac{\gamma A r^\gamma}{2} + 1\right)\big(k_1 - v(r)\big)^3 + \frac{\gamma b_3}{2}\big(k_1 - v(r)\big)^2 + \frac{1}{r}\big(k_1 - v(r)\big). \tag{3.75}$$

Equation 3.75 is classified as Abel's Equation of the first kind due to it being cubic in its solution. General solutions are known for Abel's equation of the first kind [6, 7], and this particular equation's solution assumes the transcendental form:

$$k_2 + \sqrt{\frac{\gamma A r^\gamma}{2} + \left(1 - \frac{1}{b_3 r(k_1 - v(r))}\right)^2} = \sinh^{-1}\left(\sqrt{\frac{2}{\gamma A r^\gamma}}\left[1 - \frac{1}{b_3 r(k_1 - v(r))}\right]\right) \tag{3.76}$$

where k_2 is the new integration constant. By now reverting back to t, n, and n', we find the relationship:

$$k_2 + \frac{\gamma t}{2n(t)}\sqrt{\frac{2An^{\gamma+2}}{\gamma} + \left(\frac{\mathrm{d}n}{\mathrm{d}t}\right)^2} + \sinh^{-1}\left(\sqrt{\frac{\gamma}{2An^{\gamma+2}}}\cdot\frac{\mathrm{d}n}{\mathrm{d}t}\right) = 0. \tag{3.77}$$

In its current form, no analytical closed-form expressions for n' may be directly obtained from Eq. 3.77. This revelation provides three options, the first being to return to the canonical coordinate system, Eq. 3.67, and make functional assumptions on r and s (e.g., $r = r(t)$ or $s = s(n)$) to potentially elicit a target equation that is not transcendental in the solution. The second option is to write Eq. 3.77 as a surface equation, apply the once-prolonged infinitesimal generator to obtain coordinate functions from the resultant determining equations, and attempt to reduce the above equation to an algebraic equation for n and t with the corresponding canonical coordinates. Any attempt at pursuing option two will be done so in vain because the resulting reduced-order equation is simply a transcendental algebraic equation. This leads into option three, which is to treat the above transcendental ODE as an algebraic one in the derivative.

If we maintain the notion that an ODE is an algebraic equation, then it is clear that the solutions are said to be its zeros, or roots. With that in mind, the roots of the surface equation, Eq. 3.77, are then the values of n' that cause the equation to vanish. Examining the above, the square root function disappears for $n' = i\sqrt{2An^{\gamma+2}/\gamma}$ (for which we must then set $k_2 = -i\pi/2$ because $\sinh^{-1}(i) = i\pi/2$) and the inverse hyperbolic sine function is zero when $n' = 0$. The resultant equations corresponding to the roots of the surface equation are:

$$\frac{\mathrm{d}n}{\mathrm{d}t} = i\sqrt{\frac{2An^{\gamma+2}}{\gamma}} \tag{3.78a}$$

$$\frac{\mathrm{d}n}{\mathrm{d}t} = 0 \tag{3.78b}$$

$$0 = k_2 + \frac{\gamma t}{2n}\sqrt{\frac{2An^{\gamma+2}}{\gamma}}, \tag{3.78c}$$

where Eq. 3.78c results from inserting Eq. 3.78b into the surface equation. The respective solutions to the above are:

$$n(t) = \left(\frac{2}{\gamma}\right)^{2/\gamma}\left[k_3 + i\sqrt{\frac{2A}{\gamma}}t\right]^{-2/\gamma} \tag{3.79a}$$

$$n(t) = k_4 \tag{3.79b}$$

$$n(t) = \left(\frac{2k_2^2}{\gamma A t^2}\right)^{1/\gamma}, \tag{3.79c}$$

where k_3 and k_4 are constants. Equation 3.79a in its current form satisfies the original second-order nonlinear power equation for arbitrary k_3, thus, unlike the analytical solution given by Eq. 3.62 for the other NF symmetry group, we do not need to restrict γ to any specific values. We next eliminate k_3 with the initial condition $n(t_o) = n_o$, where n_o is some arbitrary initial operating power and $t \geq t_o$, which yields the expression

$$n(t) = n_o\left[1 + i\sqrt{\frac{\gamma A n_o^\gamma}{2}}(t - t_o)\right]^{-2/\gamma}. \tag{3.80}$$

We leave a discussion on physics considerations to the footnote[2].

We next consider Eqs. 3.79b and 3.79c, where we immediately see that $n = k_4$ corresponds to the trivial constant solution. Inserting Eq. 3.79c and its derivatives into the power equation tells us that $k_2 = i$ and the solution becomes:

$$n(t) = \left[-\frac{2}{\gamma A t^2} \right]^{1/\gamma}. \tag{3.85}$$

If the initial condition is once again $n(t_o) = n_o$, we find a condition that must be satisfied: $2 + \gamma A t_o^2 n_o^\gamma = 0$. Any number of combinations of γ and A may result in satisfying said condition and

[2]In order to relate Eq. 3.80 to a realistic power excursion, we need to eliminate the unit imaginary by assuming either $A < 0$ or $\gamma < 0$, and they must simultaneously have opposite signs. If we set $A = \alpha\kappa/\ell < 0$, we note that both the inverse heat capacity, κ, and the prompt neutron generation time, ℓ, are always positive. Therefore, we require $\alpha = -\alpha_T$, the negative of the temperature coefficient of reactivity, to be negative. This means $\alpha_T > 0$, which corresponds to systems with positive feedback. Interestingly, this provides an analytical solution in a regime that is not specified by the Nordheim-Fuchs model analysis given by Hetrick [5]. Proceeding, we then set $A = -|A|$ and, more specifically, $\alpha = -|\alpha|$ which provides the power behavior for a system with a positive reactivity feedback:

$$n(t) = n_o \left[1 - \sqrt{\frac{\gamma|\alpha|\kappa\omega^{1-\gamma} n_o^\gamma}{2\ell}} (t - t_o) \right]^{-2/\gamma}, \qquad \gamma > 0,\ A < 0. \tag{3.81}$$

Alternatively, if we assume $\gamma < 0$ and $A > 0$, we arrive at the expression

$$n(t) = n_o \left[1 - \sqrt{\frac{|\gamma|\alpha\kappa\omega^{1+|\gamma|}}{2\ell n_o^{|\gamma|}}} (t - t_o) \right]^{+2/|\gamma|}, \qquad \gamma < 0,\ A > 0. \tag{3.82}$$

We see that the solution given by Eq. 3.81 will diverge at a time given by:

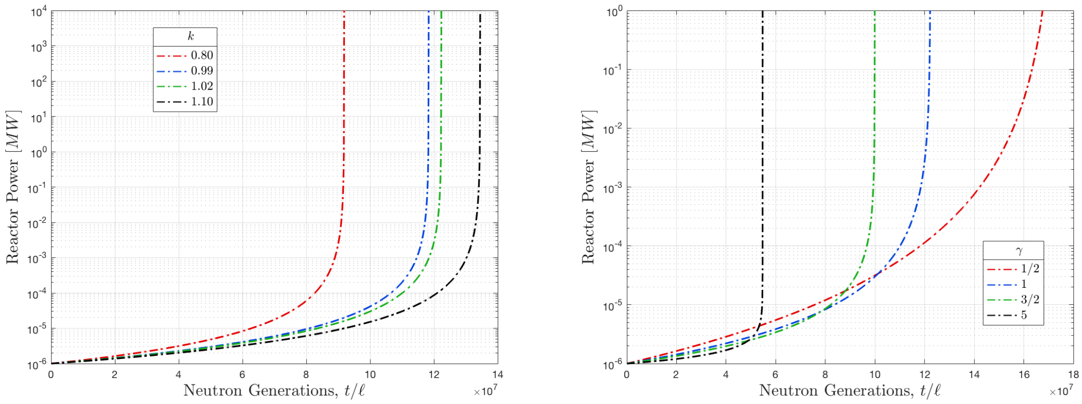
$$t_{div} = t_o + \sqrt{\frac{2}{\gamma|A|\omega^{1-\gamma} n_o^\gamma}}. \tag{3.83}$$

This tells us that the time of divergence will occur earlier for increasing n_o or for increasing $|A|$. As stated earlier, α and ℓ are held constant for a given material, then $|A|$ will only change with system size (and therefore criticality). In fact for increasing criticality, κ decreases which in turn causes $|A|$ to decrease, consequently t_{div} is expected to increase. This is somewhat of an oversimplification because this logic assumes a constant initial operation power regardless of system size, because the specific initial power (n_o/mass) will differ. Regardless, we continue and Fig. 3.2a illustrates the power profile for a spherical uranium system with a feedback coefficient of the form $\alpha = -|\alpha|$ for varying criticality values. As explained before, the divergence time occurs later for larger systems due to the increasing mass which requires more heat from the power produced to increase the temperature of the system. This can be gleaned from Eq. 3.81 by noting that as mass $m \to \infty$, $\kappa \to 0$, causing the denominator to vanish and $n(t) \to n_o$ for all time. We next observe in Fig. 3.2b that the value of γ in the reactivity feedback expression, $\rho' = -\alpha\kappa\omega^{1-\gamma} n^\gamma$, drastically affects the divergence time. Unsurprisingly, if we dampen feedback by decreasing γ, the reactor will take longer to respond to the initial reactivity insertion ρ_o. We have set $\omega = 1$ for all results shown.

For the type of reactor and reactivity feedback where Eq. 3.82 is applicable, we note that the reactor power goes to zero, or terminates, at a time given by:

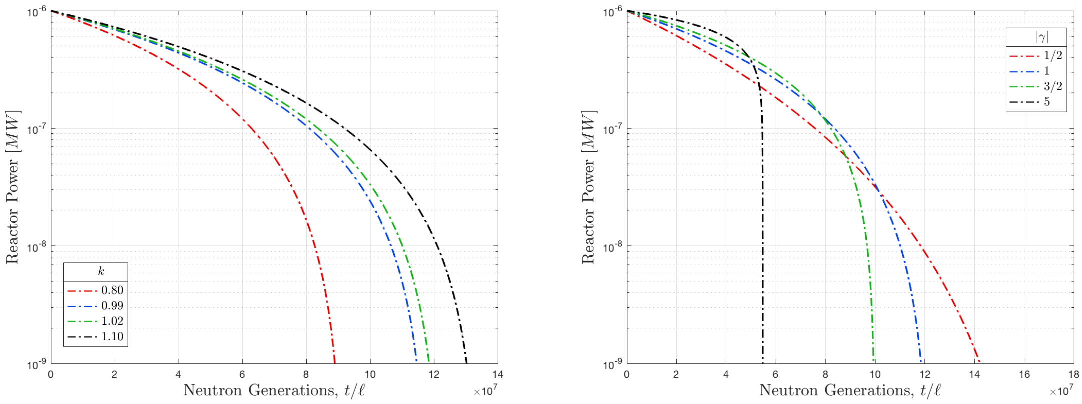
$$t_{term} = t_o + \sqrt{\frac{2n_o^{|\gamma|}}{|\gamma|A\omega^{1+|\gamma|}}}. \tag{3.84}$$

We next consider the exact same uranium system as described above, but now with $\alpha = +|\alpha|$, corresponding to a system with a negative temperature coefficient of reactivity. Figure 3.3a shows such a uranium system for varying criticality, where we clearly see that an increase in criticality results in a persistence of the reactor operation. Figure 3.3b shows the $k = 1.02$ case for varying $|\gamma|$ for a reactivity insertion rate of the form: $\rho' = \alpha\kappa\omega^{1+|\gamma|}/n^{|\gamma|}$, where we have again set $\omega = 1$. As $|\gamma|$ increases, the exponent $2/|\gamma|$ decreases while the $\sqrt{|\gamma|/2}$ factor increases. Noting the limit $\lim_{\gamma\to\infty} y^{1/\gamma} = 1$, it is clear that the $1 - \sqrt{|\gamma|/2}$ factor is what mathematically causes the power expression to terminate at earlier times.



(a) **(b)**

Figure 3.2: Reactor power as calculated by Eq. 3.81 for (a) systems of differing criticality with $\gamma = 1$ and (b) a system with $k = 1.02$ and differing γ feedback.



(a) **(b)**

Figure 3.3: Reactor power as calculated by Eq. 3.82 for (a) systems of differing criticality with $\gamma = -1$ and (b) a system with $k = 1.02$ and differing $|\gamma|$ feedback.

we leave such a discussion open for now. From said condition, we may eliminate $-2/(\gamma A) = t_o^2 n_o^\gamma$, yielding the solution:

$$n(t) = n_o \left(\frac{t_o}{t}\right)^{2/\gamma}. \tag{3.86}$$

The primary take-aways from this process are:

- We may reduce the second-order power equation to a first-order equation using canonical coordinates obtained from the one-parameter infinitesimal symmetry groups.
- For the canonical coordinates of the group $(\eta, \phi) = (b_2, 0)$, we reduced the equation to a

first-order Bernoulli equation that we were able to solve for directly, resulting in the classical $\text{sech}^2(\cdot)$ solution.

- For the second set of canonical coordinates for the symmetry group $(\eta, \phi) = (-\gamma b_3 t/2, b_3 n)$, we again reduced the second-order equation to a first-order Abel equation of the first kind. The solution to the Abel equation is transcendental in the first time-derivative of the reactor power, and solutions are then obtained by finding the roots of that transcendental equation, resulting in the solution for reactors with a positive coefficient of reactivity. This process showed how, even after finding a Lie group and reducing the equation, we might still arrive at an intractable equation. We got somewhat lucky that we could infer "roots" of the transcendental equation to derive ODEs, but this might not always be the case. It might be necessary to consider numerical solution or approximate solutions to the reduced equations (see Problem 3 in the exercises section below).

3.3.6 Other Lie Groups

Other possible Lie groups can be obtained by considering other reactivity models, outlined below. The authors implore the reader to test their understanding by using the methods outlined herein to find the solutions pertaining to these Lie groups. If you get stuck, consult the paper much of this chapter has been pulled from [8].

Arbitrary Order Polynomial Insertions

In this section we investigate the case of a reactivity insertion following an arbitrary order polynomial in time. We therefore set $A = 0$ and the corresponding reactivity rate expression is $\mathrm{d}\rho/\,\mathrm{d}t = \sum_{j=1}^{J} j a_j t^{j-1}$ and the power equation is

$$n(t)n'' = n^2 S_1 + (n')^2, \tag{3.87}$$

where the a_j are assumed to be known.

Now that $A = 0$, the coefficient system, Eq. 3.39, becomes:

$$0 = -\frac{\mathrm{d}^2 c_2}{\mathrm{d}t^2} + 2\frac{\mathrm{d}c_4}{\mathrm{d}t} \tag{3.88a}$$

$$0 = \frac{\mathrm{d}^2 c_3}{\mathrm{d}t^2} + \ln(n)\frac{\mathrm{d}^2 c_4}{\mathrm{d}t^2} - 2S_1\frac{\mathrm{d}c_2}{\mathrm{d}t} - S_2 c_2 + S_1 c_4. \tag{3.88b}$$

We may remove n-dependence from Eq. 3.88b by setting $c_4'' = 0$ which provides:

$$c_4(t) = m_4 t + b_4. \tag{3.89}$$

Solving Eq. 3.88a for $c_2(t)$ yields:

$$c_2(t) = m_4 t^2 + m_2 t + b_2. \tag{3.90}$$

We now solve for $c_3(t)$ from Eq. 3.88b to find:

$$\ell c_3(t) = m_4 \sum_{j=1}^{J} \frac{j a_j}{j+1} t^{j+2} + \sum_{j=1}^{J} \left(m_2 - \frac{b_4}{j+1} \right) a_j t^{j+1} + b_2 \sum_{j=1}^{J} a_j t^j + m_3 t + b_3. \tag{3.91}$$

Recalling Eq. 3.40, the coordinate functions for an arbitrary-order polynomial in time reactivity insertion are

$$\eta(t) = m_4 t^2 + m_2 t + b_2 \tag{3.92a}$$

$$\phi(t,n) = \frac{n}{\ell}\left[m_4 \sum_{j=1}^{J} \frac{j a_j}{j+1} t^{j+2} + \sum_{j=1}^{J}\left(m_2 - \frac{b_4}{j+1}\right) a_j t^{j+1} + b_2 S_0 + m_3 t + b_3\right] + n\ln(n)\left(m_4 t + b_4\right). \tag{3.92b}$$

The above is a 6-parameter symmetry group and we may proceed by applying the canonical coordinate method to each group individually. As it turns out, all of the symmetry groups provide the same analytical solution.

The Fuchs Ramp-Insertion Model

Consider the Fuchs ramp-insertion model, which assumes there is a linear reactivity insertion and also accounts for adiabatic temperature feedback. This model is appealing as it is a first-order combination of the previous two reactivity models, the Nordheim-Fuchs Model and Arbitrary Polynomial Insertion. The expression for the reactivity is:

$$\rho(t) = \rho_o - \alpha T(n; \gamma = 1) + a_1 t. \tag{3.93}$$

The power equation assumes the form:

$$n(t)\frac{\mathrm{d}^2 n}{\mathrm{d}t^2} = -\frac{\alpha\kappa}{\ell} n^3 + \frac{a_1}{\ell} n^2 + \left(\frac{\mathrm{d}n}{\mathrm{d}t}\right)^2. \tag{3.94}$$

Applying the appropriate restrictions of this model to the determining equations given by Eq. 3.39, we see that we must set $c_4(t) \equiv 0$. From this, the system reduces to:

$$0 = \frac{\mathrm{d}^2 c_2}{\mathrm{d}t^2} \tag{3.95a}$$

$$0 = \frac{\mathrm{d}^2 c_3}{\mathrm{d}t^2} - \frac{2a_1}{\ell}\frac{\mathrm{d}c_2}{\mathrm{d}t} + An\left[2\frac{\mathrm{d}c_2}{\mathrm{d}t} + c_3\right]. \tag{3.95b}$$

Equation 3.95a provides the solution $c_2(t) = m_2 t + b_2$. Inserting the derivative of c_2 into Eq. 3.95b provides a polynomial in n whose coefficients must simultaneously equal zero for the equation to be satisfied, yielding the system:

$$0 = 2m_2 + c_3(t) \tag{3.96a}$$

$$0 = \frac{\mathrm{d}^2 c_3}{\mathrm{d}t^2} - \frac{2a_1 m_2}{\ell}. \tag{3.96b}$$

Equation 3.96a tells us that $c_3(t)$ is a constant and is equal to $c_3 = -2m_2$. Thus, $\mathrm{d}^2 c_3/\,\mathrm{d}t^2 = 0$ in Eq. 3.96b and we are required to set $m_2 = 0$, thus $c_3 = 0$. This results in a single symmetry group corresponding to time-translation invariance:

$$\eta = b_2 \tag{3.97a}$$

$$\phi = 0. \tag{3.97b}$$

3.4 Exercises

1. Find the global transformations, $(\tilde{t}, \tilde{n})$, of the Nordheim-Fuchs Lie group, Eq. 3.48. Do you need to set b_2 and b_3 separately to zero before testing for invariance of the original equation (e.g., set $b_3 = 0$ and find the global transforms of the b_2-group)? Why or why not?

2. Find a solution admitted by the arbitrary-order polynomial Lie group, Eq. 5.57,

   (a) for the b_3-group (set all other parameters to zero).

   (b) Now for the m_4-group.

   (c) Now solve the original reactor power equation for this model,

   $$\frac{\mathrm{d}n}{\mathrm{d}t} = \left(\rho_o - \beta + \sum_j a_j t^j\right) \frac{n(t)}{\ell}. \tag{3.98}$$

   (d) Why does such an equation, which is separable, have so many Lie groups that provide the same solution?

   (e) Rewrite the Lie group in terms of the reactivity and its' derivatives (i.e., ρ, $\mathrm{d}\rho/\mathrm{d}t$, $\int \mathrm{d}t\rho(t)$).

3. Consider the Lie group for the Fuchs ramp insertion model, Eq. 3.97.

   (a) Apply the canonical coordinates method and obtain a reduced-order ODE for the reactor power.

   (b) Using the approximation $\ln(n) \approx n - 1 - \frac{1}{2}(n-1)^2$, solve this ODE and obtain an approximate solution, $n_{\text{approx}}(t)$, to the reactor power equation, Eq. 3.94.

   (c) Insert this solution into Eq. 3.94 to formulate an error, or residual, function. (Hint: determine the derivatives n'_{approx} and n''_{approx}, plug into Eq. 3.94, whatever does not cancel is the error).

   (d) Extra: numerically solve Eq. 3.94 with a convergence error of at least 10^{-6} (use an ODE solver in Matlab or Python). Compare the difference of this numerical solution with $n_{\text{approx}}(t)$. Is this difference small enough that a reactor's dynamics can be sufficiently approximated with the analytical approximate solution? Can the residual be used in some way with the approximate solution to circumvent using the numerical solution?

Chapter 4

Symmetry Analysis of Partial Differential Equations

4.1 Generalized Prolongation Formula for a System of PDEs

4.1.1 A System of First-Order PDEs

Consider the system of N first-order partial differential equations,

$$F_n(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = 0 \tag{4.1}$$

where $n = 1, \dots, N$, $\boldsymbol{x}$ is an I-length vector containing the independent variables, $\boldsymbol{u}$ is a J-length vector of the dependent variables (we switched from using y to u as that tends to be the practice in literature), and $\boldsymbol{p}$ contains all the first-order partials that appear in the system $\boldsymbol{F} = \boldsymbol{0}$, i.e.,

$$\boldsymbol{p} = \left(\frac{\partial u_1}{\partial x_1}, \cdots, \frac{\partial u_1}{\partial x_I}, \frac{\partial u_2}{\partial x_1}, \cdots, \frac{\partial u_2}{\partial x_I}, \cdots, \frac{\partial u_J}{\partial x_1}, \cdots, \frac{\partial u_J}{\partial x_I} \right). \tag{4.2}$$

If the system is square, $N = J$. As with ODEs, every derivative appearing in the system must contribute to maintaining the invariance of the system under a Lie group of transformations. For this reason, there must be a prolongation of the infinitesimal generator to account for each order of derivative and there is an coordinate function for each partial derivative that appears in the system. We show the general formula below.

The infinitesimal generator for Eq. 4.1 is given by:

$$V = \sum_{i=1}^{I} \eta_i(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial x_i} + \sum_{j=1}^{J} \phi^{(j)}(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial u_j} \tag{4.3}$$

and the first prolongation for the system of first-order PDEs is written as:

$$\boxed{\mathrm{pr}V = \sum_{i=1}^{I} \eta_i(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial x_i} + \sum_{j=1}^{J} \phi^{(j)}(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial u_j} + \sum_{i=1}^{I} \sum_{j=1}^{J} \zeta_i^{(j)}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}\right) \frac{\partial}{\partial u_i^j}} \tag{4.4}$$

where

$$u_i^j = \frac{\partial u_j}{\partial x_i} \tag{4.5a}$$

$$\mathrm{D}_i = \frac{\partial}{\partial x_i} + \sum_{k=1}^{J} u_i^k \frac{\partial}{\partial u^k}. \tag{4.5b}$$

and the coordinate function for the $u_i^j = \partial u_j / \partial x_i$ derivative is:

$$\boxed{\zeta_i^{(j)} = \mathrm{D}_i \phi^{(j)} - \sum_{k=1}^{I} u_k^j \, \mathrm{D}_i \eta_k} \tag{4.6}$$

Example: suppose $J = 1$ and $I = 2$, thus we have a single first-order PDE of two variables and then $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{u} = u$, $u_1 = \partial u/\partial x_1$, $u_2 = \partial u/\partial x_2$. Dropping the j superscript, the quantities of interest are:

$$\mathrm{pr}V = \eta_1(\boldsymbol{x}, u)\frac{\partial}{\partial x_1} + \eta_2(\boldsymbol{x}, u)\frac{\partial}{\partial x_2} + \phi(\boldsymbol{x}, u)\frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial u_1} + \zeta_2 \frac{\partial}{\partial u_2} \tag{4.7a}$$

$$\mathrm{D}_i = \frac{\partial}{\partial x_i} + \frac{\partial u}{\partial x_i}\frac{\partial}{\partial u} \tag{4.7b}$$

$$\begin{aligned} \zeta_1 &= \mathrm{D}_1\phi - u_1 \, \mathrm{D}_1\eta_1 - u_2 \, \mathrm{D}_1\eta_2 \\ &= \frac{\partial \phi}{\partial x_1} + \left(\frac{\partial \phi}{\partial u} - \frac{\partial \eta_1}{\partial x_1}\right)\frac{\partial u}{\partial x_1} - \frac{\partial \eta_1}{\partial u}\left(\frac{\partial u}{\partial x_1}\right)^2 - \left[\frac{\partial \eta_2}{\partial x_1} + \frac{\partial \eta_2}{\partial u}\frac{\partial u}{\partial x_1}\right]\frac{\partial u}{\partial x_2} \end{aligned} \tag{4.7c}$$

$$\begin{aligned} \zeta_2 &= \mathrm{D}_2\phi - u_1 \, \mathrm{D}_2\eta_1 - u_2 \, \mathrm{D}_2\eta_2 \\ &= \frac{\partial \phi}{\partial x_2} + \left(\frac{\partial \phi}{\partial u} - \frac{\partial \eta_2}{\partial x_2}\right)\frac{\partial u}{\partial x_2} - \frac{\partial \eta_2}{\partial u}\left(\frac{\partial u}{\partial x_2}\right)^2 - \left[\frac{\partial \eta_1}{\partial x_2} + \frac{\partial \eta_1}{\partial u}\frac{\partial u}{\partial x_2}\right]\frac{\partial u}{\partial x_1} \end{aligned} \tag{4.7d}$$

The reader should verify the ζ_i formulae for themselves.

Example: suppose $J = 3$ and $I = 2$, then we have a system of three first-order PDEs in two variables. Then $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{u} = (u_1, u_2, u_3)$, $u_1^1 = \partial u_1/\partial x_1$, $u_2^1 = \partial u_1/\partial x_2$, and so on. Some equations of interest are:

$$\begin{aligned} \mathrm{pr}V = {} & \eta_1 \frac{\partial}{\partial x_1} + \eta_2 \frac{\partial}{\partial x_2} + \phi^{(1)}\frac{\partial}{\partial u_1} + \phi^{(2)}\frac{\partial}{\partial u_2} + \phi^{(3)}\frac{\partial}{\partial u_3} + \zeta_1^{(1)}\frac{\partial}{\partial u_1^1} + \zeta_1^{(2)}\frac{\partial}{\partial u_1^2} + \zeta_1^{(3)}\frac{\partial}{\partial u_1^3} \\ & + \zeta_2^{(1)}\frac{\partial}{\partial u_2^1} + \zeta_2^{(2)}\frac{\partial}{\partial u_2^2} + \zeta_2^{(3)}\frac{\partial}{\partial u_2^3} \end{aligned} \tag{4.8a}$$

$$\begin{aligned} \mathrm{D}_i &= \frac{\partial}{\partial x_i} + \frac{\partial u_1}{\partial x_i}\frac{\partial}{\partial u_1} + \frac{\partial u_2}{\partial x_i}\frac{\partial}{\partial u_2} + \frac{\partial u_3}{\partial x_i}\frac{\partial}{\partial u_3} \\ &= \frac{\partial}{\partial x_i} + u_i^1\frac{\partial}{\partial u_1} + u_i^2\frac{\partial}{\partial u_2} + u_i^3\frac{\partial}{\partial u_3} \end{aligned} \tag{4.8b}$$

$$\begin{aligned} \zeta_1^{(1)} &= \mathrm{D}_1\phi^{(1)} - u_1^1 \, \mathrm{D}_1\eta_1 - u_2^1 \, \mathrm{D}_1\eta_2 \\ &= \frac{\partial \phi^{(1)}}{\partial x_1} + \frac{\partial \phi^{(1)}}{\partial u_1}u_1^1 + \frac{\partial \phi^{(1)}}{\partial u_2}u_2^1 + \frac{\partial \phi^{(1)}}{\partial u_3}u_3^1 - \left[\frac{\partial \eta_1}{\partial x_1} + \frac{\partial \eta_1}{\partial u_1}u_1^1 + \frac{\partial \eta_1}{\partial u_2}u_1^2 + \frac{\partial \eta_1}{\partial u_3}u_1^3\right]u_1^1 \\ &\quad - \left[\frac{\partial \eta_2}{\partial x_1} + \frac{\partial \eta_2}{\partial u_1}u_1^1 + \frac{\partial \eta_2}{\partial u_2}u_1^2 + \frac{\partial \eta_2}{\partial u_3}u_1^3\right]u_2^1 \end{aligned} \tag{4.8c}$$

$$\zeta_1^{(2)} = \mathrm{D}_1\phi^{(2)} - u_1^2 \, \mathrm{D}_1\eta_1 - u_2^2 \, \mathrm{D}_1\eta_2 \tag{4.8d}$$

$$\zeta_1^{(3)} = \mathrm{D}_1\phi^{(3)} - u_1^3 \, \mathrm{D}_1\eta_1 - u_2^3 \, \mathrm{D}_1\eta_2 \tag{4.8e}$$

$$\zeta_2^{(1)} = \mathrm{D}_2\phi^{(1)} - u_1^1\,\mathrm{D}_2\eta_1 - u_2^1\,\mathrm{D}_2\eta_2 \tag{4.8f}$$

$$\zeta_2^{(2)} = \mathrm{D}_2\phi^{(2)} - u_1^2\,\mathrm{D}_2\eta_1 - u_2^2\,\mathrm{D}_2\eta_2 \tag{4.8g}$$

$$\zeta_2^{(3)} = \mathrm{D}_2\phi^{(3)} - u_1^3\,\mathrm{D}_2\eta_1 - u_2^3\,\mathrm{D}_2\eta_2 \tag{4.8h}$$

where the other $\zeta_i^{(j)}$ follow in the same manner. Because the u_i and u_i^j are assumed known in the context of the symmetry determining equations, one might need to rearrange the above formula in order to simplify the system when enforcing invariance. For a system of first-order PDEs, the symmetry determining equations are obtained via:

$$\mathrm{pr}V\boldsymbol{F}\Big|_{\boldsymbol{F}=\mathbf{0}} = 0 \tag{4.9}$$

4.1.2 A System of K-Order PDEs

If we now consider a system of N K-order PDEs, $K \geq 2$,

$$\boldsymbol{F}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}^{(1)}, \ldots \boldsymbol{p}^{(K)}\right) = \mathbf{0} \tag{4.10}$$

where $\boldsymbol{p}^{(k)}$ is the vector containing all k-order partial derivatives of the solution vector $\boldsymbol{u}$:

$$\boldsymbol{p}^{(k)} = \left(\frac{\partial^k u_1}{\partial x_{\ell_1}}, \cdots, \frac{\partial^k u_1}{\partial x_{\ell_I}}, \cdots \frac{\partial^k u_J}{\partial x_{\ell_1}}, \cdots, \frac{\partial^k u_J}{\partial x_{\ell_I}}\right) \tag{4.11}$$

where, using multi-index notation $\ell_m = (a_1, a_2, \ldots a_I)$, $m = 1, 2, \ldots, M$ is a vector containing the powers a_m such $k = \sum_{m=1}^{J} a_m$ is satisfied and we may write

$$\partial x_{\ell_j} = \partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_I^{a_I} \tag{4.12}$$

and therefore M is the combinatorial maximum for which the set a_m can sum to k.

The infinitesimal generator for Eq. 4.10 is:

$$V = \sum_{i=1}^{I} \eta_i(\boldsymbol{x}, \boldsymbol{u})\frac{\partial}{\partial x_i} + \sum_{j=1}^{J} \phi^{(j)}(\boldsymbol{x}, \boldsymbol{u})\frac{\partial}{\partial u_j} \tag{4.13}$$

and the K^{th}-prolongation of the infinitesimal generator for the system of PDEs is written as:

$$\boxed{\mathrm{pr}^{(K)}V = \sum_{i=1}^{I} \eta_i(\boldsymbol{x}, \boldsymbol{u})\frac{\partial}{\partial x_i} + \sum_{j=1}^{J} \phi^{(j)}(\boldsymbol{x}, \boldsymbol{u})\frac{\partial}{\partial u_j} + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} \zeta_{\{i_k\}}^{(k,j)}\left(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}, \ldots, \boldsymbol{p}^{(k)}\right)\frac{\partial}{\partial u_{\{i_k\}}^{(k,j)}}} \tag{4.14}$$

where

$$\{i_k\} = i_1 i_2 \cdots i_k \tag{4.15a}$$

$$u_{\{i_k\}}^{(k,j)} = \frac{\partial^k u_j}{\partial x_{\ell_k}} \tag{4.15b}$$

$$\mathrm{D}_i = \frac{\partial}{\partial x_i} + \sum_{j=1}^{J}\sum_{k=1}^{K} u_{\{i_k\}}^{(k,j)}\frac{\partial}{\partial u_{\{i_{k-1}\}}^{(k,j)}} \tag{4.15c}$$

where it is implied that we sum over all mixed partials that sum to k in the total differentiation operator. The coordinate function for the $u_i^j = \partial u_j / \partial x_i$ derivative is:

$$\boxed{\zeta_{\{i_k\}}^{(k,j)} = \mathrm{D}_{i_k} \zeta_{\{i_{k-1}\}}^{(k-1,j)} - \sum_{\ell=1}^{I} u_{\{i_{k-1}\}\ell}^{j} \, \mathrm{D}_{i_k} \eta_\ell} \tag{4.16}$$

Example: suppose $J = 1$, $I = 2$, and $K = 2$, then we have a single second-order PDE in two variables. Then $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{u} = u$. Some equations of interest are:

$$u_i = \frac{\partial u}{\partial x_i}, \quad u_{i_1 i_2} = \frac{\partial^2 u}{\partial x_{i_1} \partial x_{i_2}} \tag{4.17a}$$

$$\zeta_1^{(1)} = \frac{\partial \phi}{\partial x_1} + \left(\frac{\partial \phi}{\partial u} - \frac{\partial \eta_1}{\partial x_1} \right) u_1 - \frac{\partial \eta_1}{\partial u} (u_1)^2 - \left[\frac{\partial \eta_2}{\partial x_1} + \frac{\partial \eta_2}{\partial u} u_1 \right] u_2 \tag{4.17b}$$

$$\zeta_2^{(1)} = \frac{\partial \phi}{\partial x_2} + \left(\frac{\partial \phi}{\partial u} - \frac{\partial \eta_2}{\partial x_2} \right) u_2 - \frac{\partial \eta_2}{\partial u} (u_2)^2 - \left[\frac{\partial \eta_1}{\partial x_2} + \frac{\partial \eta_1}{\partial u} u_2 \right] u_1 \tag{4.17c}$$

$$\begin{aligned} \zeta_{11}^{(2)} = & \frac{\partial^2 \phi}{\partial x_1^2} + \left[2 \frac{\partial^2 \phi}{\partial x_1 \partial u} - \frac{\partial^2 \eta_1}{\partial x_1^2} \right] u_1 - \frac{\partial^2 \eta_2}{\partial x_1^2} u_2 + \left[\frac{\partial \phi}{\partial u} - 2 \frac{\partial \eta_1}{\partial x_1} \right] u_{11} - 2 \frac{\partial \eta_2}{\partial x_1} u_{12} \\ & + \left[\frac{\partial^2 \phi}{\partial u^2} - 2 \frac{\partial^2 \eta_1}{\partial x_1 \partial u} \right] (u_1)^2 - 2 \frac{\partial^2 \eta_2}{\partial x_1 \partial u} u_1 u_2 - \frac{\partial^2 \eta_1}{\partial u^2} (u_1)^3 - \frac{\partial^2 \eta_1}{\partial u^2} (u_1)^2 u_2 \\ & - 3 \frac{\partial \eta_1}{\partial u} u_1 u_{11} - 3 \frac{\partial \eta_2}{\partial u} u_2 u_{11} - 2 \frac{\partial \eta_2}{\partial u} u_1 u_{12} \end{aligned} \tag{4.17d}$$

$$\begin{aligned} \zeta_{12}^{(2)} = & \zeta_{21}^{(2)} \\ = & \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + \left[\frac{\partial^2 \phi}{\partial x_1 \partial u} - \frac{\partial^2 \eta_2}{\partial x_1 \partial x_2} \right] u_2 + \left[\frac{\partial^2 \phi}{\partial x_2 \partial u} - \frac{\partial^2 \eta_1}{\partial x_1 \partial x_2} \right] u_1 - \frac{\partial \eta_2}{\partial x_1} u_{22} \\ & + \left[\frac{\partial \phi}{\partial u} - \frac{\partial \eta_1}{\partial x_1} - \frac{\partial \eta_2}{\partial x_2} \right] u_{12} - \frac{\partial \eta_1}{\partial x_2} u_{11} - \frac{\partial^2 \eta_2}{\partial x_1 \partial u} (u_2)^2 \\ & + \left[\frac{\partial^2 \phi}{\partial u^2} - \frac{\partial^2 \eta_1}{\partial x_1 \partial u} - \frac{\partial^2 \eta_2}{\partial x_2 \partial u} \right] u_1 u_2 - \frac{\partial^2 \eta_1}{\partial x_2 \partial u} (u_1)^2 - \frac{\partial^2 \eta_2}{\partial u^2} u_1 (u_2)^2 - \frac{\partial^2 \eta_1}{\partial u^2} (u_1)^2 u_2 \\ & - 2 \frac{\partial \eta_2}{\partial u} u_2 u_{12} - 2 \frac{\partial \eta_1}{\partial u} u_1 u_{12} - \frac{\partial \eta_1}{\partial u} u_2 u_{11} - \frac{\partial \eta_2}{\partial u} u_1 u_{22} \end{aligned} \tag{4.17e}$$

$$\begin{aligned} \zeta_{22}^{(2)} = & \frac{\partial^2 \phi}{\partial x_2^2} + \left[2 \frac{\partial^2 \phi}{\partial x_2 \partial u} - \frac{\partial^2 \eta_2}{\partial x_2^2} \right] u_2 - \frac{\partial^2 \eta_1}{\partial x_2^2} u_1 + \left[\frac{\partial \phi}{\partial u} - 2 \frac{\partial \eta_2}{\partial x_2} \right] u_{22} - 2 \frac{\partial \eta_1}{\partial x_2} u_{12} \\ & + \left[\frac{\partial^2 \phi}{\partial u^2} - 2 \frac{\partial^2 \eta_2}{\partial x_2 \partial u} \right] (u_2)^2 - 2 \frac{\partial^2 \eta_1}{\partial x_2 \partial u} u_1 u_2 - \frac{\partial^2 \eta_2}{\partial u^2} (u_2)^3 - \frac{\partial^2 \eta_2}{\partial u^2} u_1 (u_2)^2 \\ & - 3 \frac{\partial \eta_2}{\partial u} u_2 u_{22} - 3 \frac{\partial \eta_1}{\partial u} u_1 u_{22} - 2 \frac{\partial \eta_1}{\partial u} u_2 u_{12} \end{aligned} \tag{4.17f}$$

Once expressions for the prolonged coordinate functions are known, we may then obtain the symmetry determining equations as:

$$\mathrm{pr}^{(K)} V \boldsymbol{F} \Big|_{\boldsymbol{F}=\boldsymbol{0}} = 0, \tag{4.18}$$

the solution of which is the Lie group $(\boldsymbol{\eta}, \boldsymbol{\phi})$, where $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_I)$ and $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_J)$.

4.2 Devising New Coordinates with a Lie Group

Now that we no longer have a single independent variable and single dependent variable as we had in ODEs, we cannot effectively use the canonical coordinates method to find the transformed coordinate system given the Lie group of transformations $(\boldsymbol{\eta}, \boldsymbol{\phi})$. We can still, in a somewhat formulaic manner, construct the desired transformed coordinate system using the characteristic system.

At no loss of generality, we consider the case where $J = 1$ and therefore we have a single PDE of I variables. By constructing the "parameter-invariant" Lagrange-Charpit equations, i.e. the characteristic system, we have:

$$\frac{\mathrm{d}x_1}{\eta_1(\boldsymbol{x},u)} = \frac{\mathrm{d}x_2}{\eta_2(\boldsymbol{x},u)} = \cdots = \frac{\mathrm{d}x_I}{\eta_I(\boldsymbol{x},u)} = \frac{\mathrm{d}u}{\phi(\boldsymbol{x},u)}, \tag{4.19}$$

from which a set of similarity variables, $\boldsymbol{\xi}(\boldsymbol{x}, y)$, and reducing variable, $v(\boldsymbol{\xi})$, may be obtained. Note that $(\boldsymbol{\xi}, v(\boldsymbol{\xi}))$ is akin to the $(r, s(r))$ canonical coordinates from the previous chapters. As an example, one similarity variable, ξ_1, would be obtained by integrating the first and second members of Eq. 4.19 over the relevant domain $\mathcal{D}$ and solving for the constant of integration:

$$\xi_1(\boldsymbol{x}, y) \equiv \text{constant} = \int_{\mathcal{D}} \left(\frac{\mathrm{d}x_1}{\eta_1} - \frac{\mathrm{d}x_2}{\eta_2} \right) \tag{4.20}$$

where, again, this is exactly what we did to obtain the canonical coordinate r for ODEs. Similarly, the reducing variable, $v(\boldsymbol{\xi})$ is found by integrating the last member and another (strategically chosen) member and solving for the constant of integration:

$$v(\boldsymbol{\xi}) \equiv \text{constant} = \int_{\mathcal{D}} \left(\frac{\mathrm{d}y}{\phi} - \frac{\mathrm{d}x_i}{\eta_i} \right). \tag{4.21}$$

The original PDE may then be written in terms of $\boldsymbol{\xi}$ and $v(\boldsymbol{\xi})$, reducing the number of independent variables from I to $I - 1$.

4.3 Example: A Single Second-Order PDE of Two variables, the Heat Equation

Consider the well-known homogeneous heat equation, a linear second-order PDE in two variables:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \tag{4.22}$$

or written as a surface equation:

$$F(x, t, u, u_t, u_{xx}) = 0 = u_t - u_{xx}, \tag{4.23}$$

where we adopt the shorthand $u_t = \partial u/\partial t$ and $u_{xx} = \partial^2 u/\partial x^2$.

4.3.1 Symmetry Determining Equations

From this surface equation, we may write down the infinitesimal generator and the twice-prolonged infinitesimal generator as:

$$V = \eta^{(x)}(x,t,u)\frac{\partial}{\partial x} + \eta^{(t)}(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial \phi} \tag{4.24a}$$

$$\mathrm{pr}^{(2)}V = V + \zeta_t^{(1)}(x,t,u,u_t,u_{xx})\frac{\partial}{\partial u_t} + \zeta_{xx}^{(2)}(x,t,u,u_t,u_{xx})\frac{\partial}{\partial u_{xx}}, \tag{4.24b}$$

where we have added the superscripts (x) and (t) to indicate the coordinate function being attached to the respective derivative; as indicated, this does not preclude these coordinate functions from being functions of any combination of the original independent and dependent variable space. We know other ζ functions, e.g. $\zeta_x^{(1)}$ or $\zeta_{tt}^{(2)}$, will not be present in the final symmetry determining equations because the surface equation does not explicitly contain those quantities, e.g. $\frac{\partial F}{\partial u_x} = 0$ or $\frac{\partial F}{\partial u_{tt}} = 0$. Noting the partials,

$$\begin{aligned} &\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial t} = 0, \quad \frac{\partial F}{\partial u} = 0, \\ &\frac{\partial F}{\partial u_t} = 1, \quad \frac{\partial F}{\partial u_{xx}} = -1, \end{aligned} \tag{4.25}$$

we find

$$\mathrm{pr}^{(2)}VF = \zeta_t^{(1)} - \zeta_{xx}^{(2)}. \tag{4.26}$$

From Eqs. 4.17b and 4.17f, we find expressions for ζ:

$$\zeta_t^{(1)} = \frac{\partial\phi}{\partial t} + \left(\frac{\partial\phi}{\partial u} - \frac{\partial\eta^{(t)}}{\partial t}\right)u_t - \frac{\partial\eta^{(t)}}{\partial u}(u_t)^2 - \left[\frac{\partial\eta^{(x)}}{\partial t} + \frac{\partial\eta^{(x)}}{\partial u}u_t\right]u_x \tag{4.27a}$$

$$\begin{aligned} \zeta_{xx}^{(2)} = &\frac{\partial^2\phi}{\partial x^2} + \left[2\frac{\partial^2\phi}{\partial x\partial u} - \frac{\partial^2\eta^{(x)}}{\partial x^2}\right]u_x - \frac{\partial^2\eta^{(t)}}{\partial x^2}u_t + \left[\frac{\partial\phi}{\partial u} - 2\frac{\partial\eta^{(x)}}{\partial x}\right]u_{xx} - 2\frac{\partial\eta^{(t)}}{\partial x}u_{xt} \\ &+ \left[\frac{\partial^2\phi}{\partial u^2} - 2\frac{\partial^2\eta^{(x)}}{\partial x\partial u}\right](u_x)^2 - 2\frac{\partial^2\eta^{(t)}}{\partial x\partial u}u_t u_x - \frac{\partial^2\eta^{(x)}}{\partial u^2}(u_x)^3 - \frac{\partial^2\eta^{(x)}}{\partial u^2}u_t(u_x)^2 \\ &- 3\frac{\partial\eta^{(x)}}{\partial u}u_x u_{xx} - 3\frac{\partial\eta^{(t)}}{\partial u}u_t u_{xx} - 2\frac{\partial\eta^{(t)}}{\partial u}u_x u_{xt}. \end{aligned} \tag{4.27b}$$

Next we enforce $F = 0$ by substituting

$$u_{xx} = u_t \text{ when } F = 0$$

in the above and, after some algebra, we obtain the symmetry determining equation for the homogeneous heat equation:

$$\boxed{\begin{aligned} \mathrm{pr}^{(2)}VF\Big|_{F=0} = 0 = \mathcal{H}\phi &- \left[\mathcal{H}\eta^{(t)} - 2\eta_x^{(x)}\right]u_t - \left[\mathcal{H}\eta^{(x)}2\phi_{xu}\right]u_x \\ &+ 2\left[\eta_{xu}^{(t)} + \eta_u^{(x)}\right]u_x u_t + 2\left[\eta_x^{(t)} + \eta_u^{(t)}u_x\right]u_{xt} \\ &+ \left[(u_x + u_t)\,\eta_{uu}^{(x)} - \phi_{uu} + 2\eta_{xu}^{(x)}\right](u_x)^2 \end{aligned}} \tag{4.28}$$

where we have defined the "heat equation operator" as:

$$\mathcal{H} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \tag{4.29}$$

and we use the partial derivative shorthand notation on the coordinate functions.

There are a number of known solutions to the heat equation without initial and boundary conditions to constrain it. We will go over one solution so the reader can see the process where the process differs from the previous PDE example (recall in that example, the Lie group we devised was immediately known because that PDE's general solution is given by the Method of Characteristics). We want to give the reader an additional example when such a convenient solution is not readily available.

4.3.2 A Solution Permitted by a Lie Group

A known solution to the heat equation's symmetry determining equations is given by the Lie group:

$$\eta^{(x)} = 0 \tag{4.30a}$$
$$\eta^{(t)} = 1 \tag{4.30b}$$
$$\phi(u) = au \tag{4.30c}$$

where a is a constant. Clearly, $\mathcal{H}\phi = a\mathcal{H}u = 0$ from Eq. 4.22, and all other coordinate function derivatives appearing in Eq. 4.55 are zero, thus this is indeed a Lie group of transformations of the heat equation. The global transformations are given by:

$$\frac{\partial\widetilde{x}}{\partial\epsilon} = \eta^{(x)} \quad \rightarrow \quad \widetilde{x}(x;\epsilon) = x \tag{4.31a}$$
$$\frac{\partial\widetilde{t}}{\partial\epsilon} = \eta^{(t)} \quad \rightarrow \quad \widetilde{t}(t;\epsilon) = t + \epsilon \tag{4.31b}$$
$$\frac{\partial\widetilde{u}}{\partial\epsilon} = \phi(\widetilde{u}) \quad \rightarrow \quad \widetilde{u}(u;\epsilon) = u\,\mathrm{e}^{a\epsilon}, \tag{4.31c}$$

where we used the terminal conditions $\widetilde{x}(\epsilon = 0) = x$, $\widetilde{t}(\epsilon = 0) = t$, and $\widetilde{u}(\epsilon = 0) = u$. From Eq. 4.30, the characteristic system is readily obtained:

$$\frac{\mathrm{d}x}{0} = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}u}{au} = \mathrm{d}\epsilon. \tag{4.32}$$

We may determine a new coordinate system $(x, t, u) \rightarrow (r, s, w)$ using solution curves given by the characteristic system. We will find the coordinates from the following equations:

$$r \text{ from } \frac{\mathrm{d}x}{\mathrm{d}t} = 0 \tag{4.33a}$$
$$s \text{ from } \frac{\mathrm{d}t}{\mathrm{d}\epsilon} = 1 \tag{4.33b}$$
$$w \text{ from } \frac{\mathrm{d}u}{\mathrm{d}t} = au. \tag{4.33c}$$

Note that it should not matter which new coordinate comes from which characteristic member as the end result should always give the same solution, but sometimes it is easier to deal with one set of equivalent coordinates over another (the reader is implored to test this themselves). Performing the necessary integrations on the above set of ODEs and setting the constant of integration to the desired coordinate, we find:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \rightarrow x = \text{constant} = r(x) \tag{4.34a}$$
$$\frac{\mathrm{d}t}{\mathrm{d}\epsilon} = 1 \rightarrow \int \mathrm{d}t = \int \mathrm{d}\epsilon = t = \epsilon + c = s(t) \tag{4.34b}$$
$$\frac{\mathrm{d}u}{\mathrm{d}t} = au \rightarrow \int \frac{\mathrm{d}u}{u} = a\int \mathrm{d}t = \ln(u) = at + c \rightarrow u = w\,\mathrm{e}^{at} \tag{4.34c}$$

where we note that we <u>do not</u> want the new coordinates to be functions of the group parameter, ϵ. This is because the Lie group parameter is reserved for mappings between transformations and should not appear in the new coordinate system itself but, rather, is a parameter between surfaces within that old or new coordinate system. With that, the Lie group of transformations provides the new coordinate system:

$$r(x) = x \tag{4.35a}$$
$$s(t) = t \tag{4.35b}$$
$$w(t, u) = u\,\mathrm{e}^{-at}\,. \tag{4.35c}$$

Next, we want to determine the new coordinate system's characteristic system as it will provide information on the behavior of the Lie group in the new coordinate system (this will hopefully make sense in a few steps!). The process of finding the new coordinate system's characteristic system is broken down into the steps:

1. Take Eq. 4.35, simply put tildes on every variable
2. Use Eq. 4.31 to go to the original coordinates
3. Use Eq. 4.35 to write back in rsw-space.

We demonstrate each step below:

1. Take Eq. 4.35, simply put tildes on every variable:

$$\widetilde{r}(\widetilde{x}) = \widetilde{x} \tag{4.36a}$$
$$\widetilde{s}(\widetilde{t}) = \widetilde{t} \tag{4.36b}$$
$$\widetilde{w}(\widetilde{t}, \widetilde{u}) = \widetilde{u}\,\mathrm{e}^{-a\widetilde{t}} \tag{4.36c}$$

2. Use Eq. 4.31 to go to the original coordinates

$$\widetilde{r}(\widetilde{x}) = \widetilde{x} = x \tag{4.37a}$$
$$\widetilde{s}(\widetilde{t}) = \widetilde{t} = t + \epsilon \tag{4.37b}$$
$$\widetilde{w}(\widetilde{t}, \widetilde{u}) = \widetilde{u}\,\mathrm{e}^{-a\widetilde{t}} = u\,\mathrm{e}^{a\epsilon}\,\mathrm{e}^{-at-a\epsilon} = u\,\mathrm{e}^{-at} \tag{4.37c}$$

3. Use Eq. 4.35 to write back in rsw-space.

$$\widetilde{r}(\widetilde{x}) = \widetilde{x} = x = r \tag{4.38a}$$
$$\widetilde{s}(\widetilde{t}) = \widetilde{t} = t + \epsilon = s + \epsilon \tag{4.38b}$$
$$\widetilde{w}(\widetilde{t}, \widetilde{u}) = \widetilde{u}\,\mathrm{e}^{-a\widetilde{t}} = u\,\mathrm{e}^{-at} = w \tag{4.38c}$$

Thus, we may write:

$$\widetilde{r} = r \tag{4.39a}$$
$$\widetilde{s} = s + \epsilon \tag{4.39b}$$

$$\widetilde{w} = w \tag{4.39c}$$

and we can see (by now, from experience!) the coordinate functions that give these global transformations are:

$$\eta^{(r)} = 0 \tag{4.40a}$$

$$\eta^{(s)} = 1 \tag{4.40b}$$

$$\phi^{(w)} = 0 \tag{4.40c}$$

and the characteristic system in the new coordinate system is

$$\boxed{\frac{\mathrm{d}r}{0} = \frac{\mathrm{d}s}{1} = \frac{\mathrm{d}w}{0} = \mathrm{d}\epsilon.} \tag{4.41}$$

This characteristic system will become very important in a few steps.

Next we convert the original heat equation into the new coordinate system. This is done by first substituting Eq. 4.35c into Eq. 4.22 to find:

$$\begin{aligned} 0 &= \frac{\partial(w\,\mathrm{e}^{at})}{\partial t} - \frac{\partial^2(w\,\mathrm{e}^{at})}{\partial x^2} \\ &= \mathrm{e}^{at}\left[aw + \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2}\right] \end{aligned} \tag{4.42}$$

Then, we find:

$$0 = aw + \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2}. \tag{4.43}$$

Next, we need to convert the (x, t) partial derivatives to (r, s) partials. Note:

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial t} \tag{4.44a}$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x}, \tag{4.44b}$$

and from Eqs. 4.35a and 4.35b, we know $\frac{\partial r}{\partial t} = 0$, $\frac{\partial r}{\partial x} = 1$, $\frac{\partial s}{\partial t} = 1$, and $\frac{\partial s}{\partial x} = 0$. Thus,

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial s} \tag{4.45a}$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \tag{4.45b}$$

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial r}\right) = \frac{\partial^2 w}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 w}{\partial s \partial r}\frac{\partial s}{\partial x}, \\ &= \frac{\partial^2 w}{\partial r^2}. \end{aligned} \tag{4.45c}$$

The heat equation in the new coordinate system is:

$$0 = aw(r, s) + \frac{\partial w}{\partial s} - \frac{\partial^2 w}{\partial r^2} \tag{4.46}$$

This equation is of course more complicated than the original equation as it now has an additional term the original did not have. However, we can simplify this equation using the characteristic system, Eq. 4.41. First, we note the total derivative of w:

$$\frac{\mathrm{d}w}{\mathrm{d}\epsilon} = \frac{\partial w}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}\epsilon} + \frac{\partial w}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}\epsilon}, \tag{4.47}$$

which is thought of as a directional derivative in the direction $\mathrm{d}\epsilon$. From the characteristic system, Eq. 4.41, we can form the derivatives (by simply rearranging the members):

$$\frac{\mathrm{d}w}{\mathrm{d}\epsilon} = 0, \quad \frac{\mathrm{d}r}{\mathrm{d}\epsilon} = 0, \quad \frac{\mathrm{d}s}{\mathrm{d}\epsilon} = 1,$$

which reduces Eq. 4.47 to provide the crucial identity:

$$\boxed{\frac{\partial w}{\partial s} = 0 \quad \text{in the direction } \mathrm{d}\epsilon\text{, thus} \quad w(r,s) \to w(r)} \tag{4.48}$$

Thus, by simply considering Eq. 4.46 along the direction $\mathrm{d}\epsilon$, we can use the identity given by Eq. 4.48 to yield the simplified equation:

$$\boxed{0 = aw(r) - \frac{\mathrm{d}^2 w}{\mathrm{d}r^2} \quad \text{along direction } \mathrm{d}\epsilon} \tag{4.49}$$

Note that w is only a function of r along the direction $\mathrm{d}\epsilon$ and therefore the partial derivatives become ordinary derivatives. Equation 4.49 is a standard second-order linear ODE with constant coefficients, thus we may make the assumption of the solution as:

$$w(r) \propto \mathrm{e}^{\lambda r}, \tag{4.50}$$

then Eq. 4.49 becomes:

$$0 = \mathrm{e}^{\lambda r}\left(a - \lambda^2\right). \tag{4.51}$$

Therefore we find:

$$\lambda = \pm\sqrt{a}, \tag{4.52}$$

and we can write the solution as:

$$w(r) = A\,\mathrm{e}^{\sqrt{a}r} + B\,\mathrm{e}^{-\sqrt{a}r}. \tag{4.53}$$

We note the special case when $a = 0$, then $\mathrm{d}^2 w/\,\mathrm{d}r^2 = 0$ can be integrated twice to yield:

$$w(r) = Ar + B, \quad a = 0. \tag{4.54}$$

Finally, recalling Eq. 4.34c, i.e. $u = w\,\mathrm{e}^{at}$, we find the desired solution of the heat equation in our original coordinate system:

$$\boxed{u(x,t) = \begin{cases} A\,\mathrm{e}^{at-\sqrt{a}x} + B\,\mathrm{e}^{at+\sqrt{a}x}, & \text{when } a \neq 0 \\ Ax + b, & \text{when } a = 0 \end{cases}} \tag{4.55}$$

To summarize, the process of solving the PDE, given a Lie group, is to use the Lie group to transform the to a new coordinate system. The new coordinate system will have its own global transformations with a different characteristic system. This new characteristic system is then used to determine the directional derivative, which will simplify the equation in the new coordinate system. Sometimes we might have to repeat this process more than once (see the example in the next chapter), but each successive iteration should reduce the order or reduce the number of independent variables. This process is clearly arduous in this example, but it may be the only way to find solutions to much more complicated problems.

4.4 Exercises

1. Prove that the symmetry criterion is satisfied for the Lie group we studied given by Eq. 4.30 (i.e., use the global transformations, Eqs. 4.31, and show that the heat equation is left invariant under the Lie group of transformations).

2. Consider the heat equation,

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \tag{4.56}$$

   Find the solution permitted by the Lie group $(\widetilde{x}, \widetilde{t}, \widetilde{u}) = (x+\alpha\epsilon, t+\epsilon, u)$, where α is a constant.

3. Consider the Inviscid Burgers's equation,

$$\frac{\partial u(x,t)}{\partial t} + u\frac{\partial u(x,t)}{\partial x} = 0 \tag{4.57}$$

   (a) Derive the symmetry determining equation(s).

   (b) Assuming $\eta^{(x)} = m(x)t + b(x)$ and $\eta^{(t)} = M(t)x + B(t)$, use the determining equations to to determine the coordinate functions, including ϕ (you should end up with an eight-parameter Lie group).

4. Consider the 2-D wave equation,

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} - \frac{\partial^2 u(x,y,t)}{\partial x^2} - \frac{\partial^2 u(x,y,t)}{\partial y^2} = 0 \tag{4.58}$$

   (a) Derive the symmetry determining equation(s)

   (b) Find the solution permitted by the Lie group $(\widetilde{x}, \widetilde{y}, \widetilde{t}, \widetilde{u}) = (x\cos\epsilon - y\sin\epsilon, x\sin\epsilon + y\cos\epsilon, t, u)$

Chapter 5

Symmetry Analysis of Integro-Differential Equations

As we have seen in the previous chapters, when one starts the symmetry analysis, one is tasked with obtaining the symmetry determining equations- from which the Lie groups may be obtained and a path to order-reduction and/or solution is paved. This process can be called the traditional Lie algorithm (TLA), often referred to as Ovsiannikov's method. The TLA requires local invariance to effectively derive the symmetry determining equations. An issue arises when an equation contains both local and nonlocal operators, such as integro-differential equations, delay differential equations, and stochastic differential equations, and the algorithm becomes overly cumbersome for a variety of reasons and we quote several authorities on their perspectives. From Ibragimov et al. in [9] (Chapter 5),

> The main obstacle for the application of Lie's infinitesimal technique to integro-differential equations is their nonlocality. These equations do not have a frame locally defined in the space... of differential functions. Consequently, the algorithmic approach based on the definition of a symmetry group as a group of transformations leaving the frame unalterable... does not apply to integro-differential equations.

From this, the "space of differential functions" precludes the presence of nonlocal functions as they are, by definition, not purely differential. As a consequence, the frame (i.e., the manifold defined by the IDE) is not guaranteed to be left invariant and therefore the TLA is not applicable. If one were to apply the TLA cast in the language of differential forms [10] onto an IDE (as was done in [11]), Fushchich, Shtelen, and Serov observe [12],

> ...we note that the standard Lie algorithm is inapplicable to the IDE... The symmetry properties of [an IDE] can be studied by means of method of differential forms, but in this case one faces a problem of unwieldy calculations which become really enormous when the order of the differential operator... and the number of components... increase. This circumstance essentially restricts the applicability of the method of differential forms.

The realization of this insufficiency has resulted in two classes of solution methods for obtaining the symmetry determining equations of equations with nonlocal operators: indirect and direct methods. In this chapter, we provide the foundations form deriving the SDEs of IDEs and then provide an example.

5.1 Another Way to Derive the Symmetry Determining Equations

We now present the Grigoriev-Meleshko Method (GMM) [13] for obtaining the symmetry determining equations for an integro-differential equation (IDE) (which can be applied to ODEs and PDEs as well). Consider the IDE defined as:

$$\Phi(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = 0 \tag{5.1}$$

where $\boldsymbol{x}$ and $\boldsymbol{u}$ are vectors containing the dependent and independent variables, respectively, and $\boldsymbol{p}$ is a vector of the derivatives. For the ensuing illustration of the GMM, we do not treat integral operators as independent quantities the same way we have been treating derivatives as independent variables defining the over-arching surface equation of the ODEs and PDEs. Continuing, suppose there is a one-parameter Lie group of transformations, $T_\epsilon = (\boldsymbol{\alpha}, \boldsymbol{\beta})$, whose action transforms $\boldsymbol{x} \to \widetilde{\boldsymbol{x}}$, $\boldsymbol{u} \to \widetilde{\boldsymbol{u}}$ and leaves Φ invariant, i.e.,

$$\widetilde{\boldsymbol{x}}(\boldsymbol{x}.\boldsymbol{u};\epsilon) = \boldsymbol{\alpha}(\boldsymbol{x}, \boldsymbol{u};\epsilon) \tag{5.2a}$$

$$\widetilde{\boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{u};\epsilon) = \boldsymbol{\beta}(\boldsymbol{x}, \boldsymbol{u};\epsilon) \tag{5.2b}$$

$$\Phi(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{p}}) = \Phi(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = 0. \tag{5.2c}$$

Here ϵ is a real parameter called the group parameter and we consider only point transformations.

First, let us consider the process of obtaining the determining equations for a purely local equation as we did in every previous chapters. Normally, when Eq. 5.1 is a local equation, we define the infinitesimal operator V, apply V to Φ, and enforce invariance such that Eq. 5.2c is satisfied. This yields the determining equation(s):

$$V\Phi(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})\big|_{\Phi=0} = 0, \tag{5.3}$$

where for argument's sake V is sufficiently prolonged.

For an equation consisting of local and nonlocal operators, this derivation process breaks down because V is defined to enforce infinitesimal (i.e., local, differential) invariance. This is to say that V cannot act on nonlocal operators because said operators feed nonlocal information into the local domain and the requirement of local invariance is not guaranteed under the action of Eq. 5.3. For this reason, such a direct application of Eq. 5.3 onto the IDE Φ will prove nigh. Thus, we instead consider the more fundamental idea of what an admitted Lie group is and how it may be used to derive the desired determining equations for IDEs.

Predicated on the satisfaction of the conditions given by Eqs. 5.2, there is therefore an admitted Lie group $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}})$ with corresponding coordinate functions $(\boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{u}))$. The relationship between $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}})$ and $(\boldsymbol{\eta}, \boldsymbol{\phi})$ is expressed with the Lie series:

$$\widetilde{\boldsymbol{x}} = \boldsymbol{x} + \epsilon\boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{u}) + \mathcal{O}(\epsilon^2), \tag{5.4a}$$

$$\widetilde{\boldsymbol{u}} = \boldsymbol{u} + \epsilon\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{u}) + \mathcal{O}(\epsilon^2), \tag{5.4b}$$

such that

$$\boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{u}) = \left.\frac{\mathrm{d}\widetilde{\boldsymbol{x}}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \qquad \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{u}) = \left.\frac{\mathrm{d}\widetilde{\boldsymbol{u}}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \tag{5.5}$$

which is no different from how we defined the Lie series and the coordinate functions in previous chapters. From this we may appropriately define our infinitesimal generator:

$$V = \boldsymbol{\eta} \cdot \partial_{\boldsymbol{x}} + \boldsymbol{\phi} \cdot \partial_{\boldsymbol{u}} \tag{5.6}$$

where $\partial_{\boldsymbol{x}}$ and $\partial_{\boldsymbol{u}}$ are vectors containing all partial derivatives of the independent and dependent variables, respectively. Equation 5.6 can be written in our previous notation as:

$$V = \sum_i \eta^{(i)} \frac{\partial}{\partial x_i} + \sum_j \phi^{(j)} \frac{\partial}{\partial u_j} \tag{5.7}$$

Equation 5.6 is also consistent with the definition for the ODE and PDE cases. By now taking the Lie derivative of Φ, the following relationship may be established using Eqs. 5.4b - 5.6:

$$\left.\frac{\mathrm{d}\Phi(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{p}})}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = V\Phi(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})\big|_{\Phi=0} = 0, \tag{5.8}$$

and therefore we gain an additional invariance condition identity:

$$\boxed{\left.\frac{\mathrm{d}\Phi(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{p}})}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = 0.} \tag{5.9}$$

Equation 5.9 is another definition of the determining equations as ascribed by Eq. 5.8. This identity is more suited to our needs when Φ is an IDE as it allows us to derive the required relationships between the coordinate functions when nonlocal operators are present in the original surface equation. As was seen in the previous chapters for the determining equations of a purely local equation, this is not different from those determining equations because Eq. 5.8 holds for any equation (where V is sufficiently prolonged).

The process of obtaining the determining equations for an IDE is as follows. We *assume* there is a solution to the IDE and therefore there is an admitted Lie group. Thus, condition 5.2c is satisfied and we may write $\Phi(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{p}})$ immediately (by simply replacing all variables $\boldsymbol{x}, \boldsymbol{u}$ with $\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}}$) which is now parameterized by ϵ. We then evaluate the Lie derivative given by Eq. 5.9 which provides the determining equations which must now be solved to obtain explicit representations of $\boldsymbol{\eta}, \boldsymbol{\phi}$.

If the determining equations are solved for a non-trivial solution, say for the simple case where $(\boldsymbol{x}, \boldsymbol{u}) = (x, u)$ for which $(\boldsymbol{\eta}, \boldsymbol{\phi}) = (\eta, \phi)$, we may then write down the characteristic system:

$$\frac{\mathrm{d}x}{\eta(x,u)} = \frac{\mathrm{d}u}{\phi(x,u)}, \tag{5.10}$$

which are clearly equivalent to all other Lagrange-Charpit equations we have written previously. Thus, once the determining equations are solved, the ensuing process of obtaining similarity variables for reduction and/or solution is the same, no matter the type of equation we are dealing with.

5.2 Example: The 1-D Neutron Transport Equation

5.2.1 The Physics Model

The form of the neutron transport equation (NTE) that we consider is for a one-dimensional slab (planar geometry) with monoenergetic prompt neutrons, constant material properties, isotropic scattering, and no inhomogeneous source. The NTE is then:

$$\frac{1}{v}\frac{\partial \psi}{\partial t} + \mu \frac{\partial \psi}{\partial x} + \sigma \psi(x, \mu, t) = \frac{\sigma c}{2} \int_{-1}^{1} \mathrm{d}\mu' \psi(x, \mu', t), \tag{5.11}$$

where

- $\psi(x, \mu, t)$ is the angular neutron flux passing through the point x with velocity v at time t in the direction $\mu = \cos(\theta)$, where θ is the off-axis angle,
- $\sigma = \sigma_t$ is the total macroscopic cross section and c is the average number of neutrons emitted per collision, defined as

$$c = \frac{\sigma_s + \overline{\nu}\sigma_f}{\sigma}. \tag{5.12}$$

  Here σ_s and σ_f are the scatter and fission macroscopic cross sections, respectively, and $\overline{\nu}$ is the average number of neutrons emitted per fission.

Equation 5.11 is a first-order integro-differential equation in three variables. We perform a symmetry analysis on this equation in a more general form in [14], but here we show how we can obtain a solution to this difficult equation via symmetry analysis.

5.2.2 Symmetry Determining Equations of the NTE

We begin the Lie group analysis by defining the transport system as:

$$\Phi(\boldsymbol{x}, \psi, \boldsymbol{p}) = 0, \tag{5.13}$$

where $\boldsymbol{x} = (x, \mu, t)$ and $\boldsymbol{p} = (\partial\psi/\partial x, \partial\psi/\partial t)$. We now suppose there is a one-parameter group of transformations, $T_\epsilon = (\boldsymbol{\alpha}(\boldsymbol{x}, \psi; \epsilon), \beta(\boldsymbol{x}, \psi; \epsilon))$, whose action transforms $\boldsymbol{x} \to \widetilde{\boldsymbol{x}}$, $\psi \to \widetilde{\psi}$ and $\boldsymbol{p} \to \widetilde{\boldsymbol{p}}$ that leaves the system invariant, i.e.,

$$\widetilde{\boldsymbol{x}}(\boldsymbol{x}, \psi; \epsilon) = \boldsymbol{\alpha}(\boldsymbol{x}, \psi; \epsilon) \tag{5.14a}$$

$$\widetilde{\psi}(\boldsymbol{x}, \psi; \epsilon) = \beta(\boldsymbol{x}, \psi; \epsilon) \tag{5.14b}$$

$$\Phi(\widetilde{\boldsymbol{x}}, \widetilde{\psi}, \widetilde{\boldsymbol{p}}) = \Phi(\boldsymbol{x}, \psi, \boldsymbol{p}) = 0. \tag{5.14c}$$

Equation 5.14c is referred to as the symmetry criterion as it states that, under the action of T_ϵ, the original system is left invariant. We will use the second definition of a Lie group as being an invariant transformation of solutions to solutions and thus the symmetry determining equations are defined from the following:

$$\left.\frac{\mathrm{d}\Phi(\widetilde{\boldsymbol{x}}, \widetilde{\psi}, \widetilde{\boldsymbol{p}})}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = 0. \tag{5.15}$$

Continuing, we may define the admitted transformation group following Eq. 5.14a:

$$\widetilde{\boldsymbol{x}}(\boldsymbol{x}, \psi; \epsilon) = \boldsymbol{\alpha}(\boldsymbol{x}, \psi; \epsilon) = \left\langle \widetilde{x}(\boldsymbol{x}, \psi; \epsilon), \widetilde{\mu}(\boldsymbol{x}, \psi; \epsilon), \widetilde{t}(\boldsymbol{x}, \psi; \epsilon) \right\rangle \tag{5.16a}$$

$$\widetilde{\psi}(\boldsymbol{x}, \psi; \epsilon) = \beta(\boldsymbol{x}, \psi; \epsilon) \tag{5.16b}$$

where it is understood that $\boldsymbol{\alpha}$ and β act on $\boldsymbol{x}$ and ψ, respectively, and the resulting transformations satisfy the symmetry criterion given by Eq. 5.14c. From Eqs. 5.16, we see $T_\epsilon = [\boldsymbol{\alpha}, \beta]$ maps the point (x, μ, t, ψ) in $x\mu t\psi$-space to a new point $(\widetilde{x}, \widetilde{\mu}, \widetilde{t}, \widetilde{\psi})$ in $x\mu t\psi$-space. If Φ is invariant under the transformation T_ϵ, then T_ϵ forms a Lie group of transformations.

The coordinate functions of the infinitesimal transformations are related to T_ϵ by:

$$\eta^{(j)}(\boldsymbol{x}, \psi) = \left.\frac{\partial\alpha^{(j)}}{\partial\epsilon}\right|_{\epsilon=0}, \quad j = 1, 2, 3 \tag{5.17a}$$

$$\phi(\boldsymbol{x}, \psi) = \left.\frac{\partial\beta}{\partial\epsilon}\right|_{\epsilon=0} \tag{5.17b}$$

where j refers to the independent variable index of $\boldsymbol{x} = \langle x, \mu, t\rangle$, $b = \widetilde{\psi}(\boldsymbol{x}, \psi; \epsilon)$ is the transformed neutron flux.

We now suppose there is indeed a solution to Eq. 5.13 given by

$$\psi_o(\boldsymbol{x}),$$

which essentially means we are parameterizing ψ by $\boldsymbol{x}$. Thus $\widetilde{\boldsymbol{x}}$ is known as

$$\widetilde{\boldsymbol{x}} = \boldsymbol{\alpha}\left(\boldsymbol{x}, \psi_o(\boldsymbol{x}); \epsilon\right) \tag{5.18}$$

and can be inverted to find

$$\boldsymbol{x} = \hat{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{x}}; \epsilon) \tag{5.19}$$

where we note that there is now no dependence on $\widetilde{\psi}$ in the inversion $\hat{\boldsymbol{\alpha}}$. We also note that $\hat{a}^{(j)}$ refers to the j-th independent variable, parameterized by ϵ, contained in $\boldsymbol{x} = \hat{\boldsymbol{\alpha}}$. Since the transformation group, Eqs. 5.16, is admitted by Eqs. 5.13 and transforms the solution $\psi_o(\boldsymbol{x})$ to another solution, the family of functions

$$\widetilde{\psi} = \widetilde{\psi}_o(\boldsymbol{x}; \epsilon) = \beta\Big(\hat{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{x}}; \epsilon), \psi_o\big(\hat{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{x}}; \epsilon)\big); \epsilon\Big) \tag{5.20}$$

also satisfy Eq. 5.13 for any value of ϵ, i.e., written explicitly:

$$\Phi(\widetilde{\boldsymbol{x}}; \epsilon) = 0 = -\frac{1}{v}\frac{\partial\widetilde{\psi}_o(\widetilde{\boldsymbol{x}}; \epsilon)}{\partial\widetilde{t}} - \widetilde{\mu}\frac{\partial\widetilde{\psi}_o(\widetilde{\boldsymbol{x}}; \epsilon)}{\partial\widetilde{x}} - \sigma\widetilde{\psi}_o\left(\widetilde{\boldsymbol{x}}; \epsilon\right) + \frac{\sigma c}{2}\int_{-1}^{1} \mathrm{d}\mu'\widetilde{\psi}_o\left(\widetilde{x}, \mu', \widetilde{t}; \epsilon\right) \tag{5.21}$$

We note that $\Phi(\widetilde{\boldsymbol{x}}; \epsilon)$ is not dependent on $\widetilde{\psi}(\boldsymbol{x}; \epsilon) = \widetilde{\psi}_o$ because we have parameterized these quantities by the original independent variable phase-space $\boldsymbol{x} = \hat{\boldsymbol{\alpha}}$, as seen in Eq. 5.20. We may now apply Eq. 5.15 to derive the symmetry determining equations by taking the ϵ derivative and evaluating at $\epsilon = 0$. We note that the second definition of a Lie group is a statement that a Lie group transforms *solutions to solutions* and therefore an ϵ-derivative will only apply to the transformed solutions $\widetilde{\psi}_o$ and not on the transformed independent variables $\widetilde{\boldsymbol{x}}$. With this in mind, we find:

$$\begin{aligned}\frac{\mathrm{d}\Phi(\widetilde{\boldsymbol{x}}; \epsilon)}{\mathrm{d}\epsilon} = 0 = &-\frac{1}{v}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left\{\frac{\partial\widetilde{\psi}_o(\widetilde{\boldsymbol{x}}; \epsilon)}{\partial\widetilde{t}}\right\} - \widetilde{\mu}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left\{\frac{\partial\widetilde{\psi}_o(\widetilde{\boldsymbol{x}}; \epsilon)}{\partial\widetilde{x}}\right\} - \sigma\frac{\mathrm{d}\widetilde{\psi}_o\left(\widetilde{\boldsymbol{x}}; \epsilon\right)}{\mathrm{d}\epsilon} \\ &+ \frac{\sigma c}{2}\int_{-1}^{1} \mathrm{d}\mu'\frac{\mathrm{d}\widetilde{\psi}_o\left(\widetilde{x}, \mu', \widetilde{t}; \epsilon\right)}{\mathrm{d}\epsilon}\end{aligned} \tag{5.22}$$

From the chain rule, we write the ϵ-derivative of $\widetilde{\psi}_o$:

$$\frac{\mathrm{d}\widetilde{\psi}_o(\widetilde{\boldsymbol{x}}; \epsilon)}{\mathrm{d}\epsilon} = \frac{\partial\widetilde{\psi}_o}{\partial\epsilon} + \frac{\partial\widetilde{\psi}_o}{\partial\psi_o}\sum_{j=1}^{J}\frac{\partial\psi_o}{\partial\hat{\alpha}^{(j)}}\frac{\partial\hat{\alpha}^{(j)}}{\partial\epsilon} \tag{5.23}$$

Equation 5.23 may be further simplified by first noting the Lie Series:

$$\widetilde{\boldsymbol{x}} = \boldsymbol{x} + \epsilon\boldsymbol{\eta} + \mathcal{O}\left(\epsilon^2\right) = \hat{\boldsymbol{\alpha}} + \epsilon\boldsymbol{\eta} + \mathcal{O}\left(\epsilon^2\right) \tag{5.24a}$$

$$\widetilde{\psi}_o = \psi_o + \epsilon\phi + \mathcal{O}\left(\epsilon^2\right) \tag{5.24b}$$

from which we may invert Eq. 5.24a by solving for $\hat{\boldsymbol{\alpha}}$:

$$\hat{\boldsymbol{\alpha}} = \widetilde{\boldsymbol{x}} - \epsilon\boldsymbol{\eta} - \mathcal{O}\left(\epsilon^2\right). \tag{5.25}$$

To be more explicit, the inverted components of Eq. 5.25, $\hat{\alpha} = (x, \mu, t)$, are:

$$x = \widetilde{x} - \epsilon\eta^{(x)} - \mathcal{O}\left(\epsilon^2\right) \tag{5.26a}$$
$$\mu = \widetilde{\mu} - \epsilon\eta^{(\mu)} - \mathcal{O}\left(\epsilon^2\right) \tag{5.26b}$$
$$t = \widetilde{t} - \epsilon\eta^{(t)} - \mathcal{O}\left(\epsilon^2\right). \tag{5.26c}$$

Thus, the partial derivatives of interest are:

$$\frac{\partial\widetilde{\psi}_o}{\partial\epsilon} = \phi(\boldsymbol{x}) \tag{5.27a}$$
$$\frac{\partial\widetilde{\psi}_o}{\partial\psi_o} = 1 \tag{5.27b}$$
$$\frac{\partial\hat{\alpha}^{(j)}}{\partial\epsilon} = -\eta^{(j)}(\boldsymbol{x}). \tag{5.27c}$$

By evaluating Eqs. 5.23 at $\epsilon = 0$, we may define the quantity

$$\Upsilon(\boldsymbol{x}) = \left.\frac{\mathrm{d}\widetilde{\psi}_o(\boldsymbol{x};\epsilon)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \phi(\boldsymbol{x}) - \sum_{j=1}^{3}\eta^{(j)}(\boldsymbol{x})\frac{\partial\psi_o(\boldsymbol{x})}{\partial x_j}. \tag{5.28}$$

The quantity Υ is known as the Lie-Bäcklund operator for the solution and it is known that every Lie-Bäcklund operator has an equivalent Lie point operator [26]. This means that the ensuing SDEs written in terms of the Lie-Bäcklund operator, Υ, can be written in terms of the infinitesimal transformation coordinate functions. Finally, by evaluating Eqs. 5.22 at $\epsilon = 0$, we arrive at the symmetry determining equation of the NTE:

$$\left.\frac{\mathrm{d}\Phi(\widetilde{\boldsymbol{x}};\epsilon)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = 0 = -\frac{1}{v}\frac{\partial\Upsilon}{\partial t} - \mu\frac{\partial\Upsilon}{\partial x} - \sigma\Upsilon(x,\mu,t) + \frac{\sigma c}{2}\int_{-1}^{1}\mathrm{d}\mu'\Upsilon(x,\mu',t), \tag{5.29}$$

As expected, the symmetry determining equation is itself an integrodifferential equation in Υ and, in the form presented in Eq. 5.29, is no easier to solve than the original equation. We have introduced four unknown functions ($\boldsymbol{\eta}$ and ϕ) that must satisfy the equation in Υ.

The next step is to write the symmetry determining equations explicitly in terms of the coordinate functions $\boldsymbol{\eta}$ and ϕ. This is done by simply inserting Eq. 5.28 into Eqs. 5.29 and performing the necessary algebra. For brevity, we define the operators:

$$\mathcal{H}f = -\mathcal{L}f + \mathcal{S}f \tag{5.30a}$$
$$\mathcal{L}f = \mathcal{T}f + \sigma f \tag{5.30b}$$
$$\mathcal{T}f = \frac{1}{v}\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} \tag{5.30c}$$
$$\mathcal{S}f = \frac{\sigma c}{2}\int_{-1}^{1}\mathrm{d}\mu' f(\mu') \tag{5.30d}$$

Proceeding, we have:

$$\left.\frac{\mathrm{d}\Phi(\widetilde{\boldsymbol{x}};\epsilon)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = 0 = \mathcal{H}[\phi(\boldsymbol{x})] - \sum_{j=1}^{3}\mathcal{H}\left[\eta^{(j)}(\boldsymbol{x})\frac{\partial\psi_o(\boldsymbol{x})}{\partial x_j}\right] \tag{5.31}$$

Expanding some of the operators and collecting terms, we have:

$$0 = \mathcal{H}[\phi(\boldsymbol{x})] + \sum_j \left\{ \frac{\partial \psi_o}{\partial x_j} \mathcal{L}\left[\eta^{(j)}\right] - \mathcal{S}\left[\eta^{(j)} \frac{\partial \psi_o}{\partial x_j}\right] + \eta^{(j)} \mathcal{T}\left[\frac{\partial \psi_o}{\partial x_j}\right] \right\} \tag{5.32}$$

We next wish to eliminate the second-order derivatives of ψ_o appearing in Eq. 5.32. The second-order derivatives appear in the $\mathcal{T}[\partial\psi_o/\partial x_j]$ of Eq. 5.32, which can be solved for using Eq. 5.30c in combination with the NTE (i.e., solve for $\mathcal{T}\psi_o$ using $\mathcal{H}\psi_o = 0$) to find:

$$\mathcal{T}\left[\frac{\partial \psi_o}{\partial x_j}\right] = \frac{\partial}{\partial x_j} \mathcal{T}[\psi_o] = \frac{\partial}{\partial v_j}\left[-\sigma\psi_o + \mathcal{S}[\psi_o]\right] \tag{5.33a}$$

Recalling the components of $\boldsymbol{x} = \langle x, \mu, t\rangle$, the operation $\partial/\partial x_j$ is appropriately given by the components of the total derivatives $\psi_o(\boldsymbol{x})$. From Eq. 5.33a, we may reduce the sum containing the second-order derivatives in Eq. 5.32 to:

$$\begin{aligned}\sum_j \eta^{(j)} \frac{\partial \mathcal{T}[\psi_o]}{\partial x_j} = {} & \eta^{(x)}\left[-\sigma\frac{\partial \psi_o}{\partial x} + \frac{\partial}{\partial x}\mathcal{S}[\psi_o]\right] + \eta^{(\mu)}\left[-\sigma\frac{\partial \psi_o}{\partial \mu} + \frac{\partial}{\partial \mu}\mathcal{S}[\psi_o]\right] \\ & + \eta^{(t)}\left[-\sigma\frac{\partial \psi_o}{\partial t} + \frac{\partial}{\partial t}\mathcal{S}[\psi_o]\right].\end{aligned} \tag{5.34}$$

As a reminder, in general the coordinate functions pertaining to the original independent variables may be functions of the entire independent phase space, i.e. $\eta^{(j)} = \eta^{(j)}(\boldsymbol{x}) = \eta^{(j)}(x,\mu,t)$. The derivatives operating on the $\mathcal{S}[\psi_o]$ terms require explicit treatment. For the spatial and angular derivatives, we have:

$$\frac{\partial}{\partial x}\mathcal{S}[\psi_o] = \frac{\sigma c}{2}\int_{-1}^{1} \mathrm{d}\mu' \frac{\partial \psi_o(x,\mu',t)}{\partial x} \tag{5.35a}$$

$$\frac{\partial}{\partial \mu}\mathcal{S}[\psi_o] = 0 \quad \text{(for this model, but not in general)} \tag{5.35b}$$

Inserting Eq. 5.34 into Eq. 5.32, we arrive at our desired symmetry determining equation

$$\begin{aligned}0 = {} & \mathcal{H}[\phi(\boldsymbol{x})] + \sum_j \left\{ \frac{\partial \psi_o}{\partial x_j}\mathcal{L}\left[\eta^{(j)}\right] - \mathcal{S}\left[\eta^{(j)}\frac{\partial \psi_o}{\partial x_j}\right]\right\} + \eta^{(x)}\left[-\sigma\frac{\partial \psi_o}{\partial x} + \frac{\partial}{\partial x}\mathcal{S}[\psi_o]\right] \\ & - \eta^{(\mu)}\sigma\frac{\partial \psi_o}{\partial \mu} + \eta^{(t)}\left[-\sigma\frac{\partial \psi_o}{\partial t} + \frac{\partial}{\partial t}\mathcal{S}[\psi_o]\right].\end{aligned} \tag{5.36a}$$

We note that $\mathcal{S}[\eta^{(j)}\partial\psi_o/\partial v_j]$ is not readily simplified, so we leave them in this more general form. Expanding the sum over j, and further expanding the $\mathcal{L}\eta^{(j)}$, we find some term cancellation to arrive at our desired determining equation, written explicitly in terms of the coordinate functions:

$$\boxed{\begin{aligned}0 = {} & -\frac{1}{v}\frac{\partial \phi}{\partial t} - \mu\frac{\partial \phi}{\partial x} - \sigma\phi + \frac{\sigma c}{2}\int_{-1}^{1}\mathrm{d}\mu'\phi + \left[\frac{1}{v}\frac{\partial \eta^{(x)}}{\partial t} + \mu\frac{\partial \eta^{(x)}}{\partial x}\right]\frac{\partial \psi_o}{\partial x} \\ & - \frac{\sigma c}{2}\int_{-1}^{1}\mathrm{d}\mu'\eta^{(x)}\frac{\partial \psi_o}{\partial x} + \left[\frac{1}{v}\frac{\partial \eta^{(\mu)}}{\partial t} + \mu\frac{\partial \eta^{(\mu)}}{\partial x}\right]\frac{\partial \psi_o}{\partial \mu} - \frac{\sigma c}{2}\int_{-1}^{1}\mathrm{d}\mu'\eta^{(\mu)}\frac{\partial \psi_o}{\partial \mu'} \\ & + \left[\frac{1}{v}\frac{\partial \eta^{(t)}}{\partial t} + \mu\frac{\partial \eta^{(t)}}{\partial x}\right]\frac{\partial \psi_o}{\partial t} - \frac{\sigma c}{2}\int_{-1}^{1}\mathrm{d}\mu'\eta^{(t)}\frac{\partial \psi_o}{\partial t} \\ & + \frac{\sigma c}{2}\eta^{(x)}\int_{-1}^{1}\mathrm{d}\mu'\frac{\partial \psi_o}{\partial x} + \frac{\sigma c}{2}\eta^{(t)}\int_{-1}^{1}\mathrm{d}\mu'\frac{\partial \psi_o}{\partial t}\end{aligned}} \tag{5.37}$$

where dependence on μ' is implied for all integrands. Equation 5.37 is the most simplified form of the determining equations without making further restrictions on the functionality of the coordinate functions.

5.2.3 Solution to the Determining Equation and the Associated Lie Groups

For the sake of showing how to obtain a solution to the symmetry determining equation and the neutron transport equation, we can assume a solution corresponds to a space- and time-translation invariance coupled with solution-scaling invariance. These types of invariant forms, translation and scaling, imply the following forms of the coordinate functions: $\eta^{(x)} = f$, $\eta^{(t)} = b$, $\phi = a\psi$, where a, b, and f are arbitrary constants. Inserting these into Eq. 5.37 gives a determining equation satisfied by $\eta^{(\mu)}$. Using the fact that $\mathcal{H}[\phi] = a\mathcal{H}[\psi] = 0$ gives:

$$0 = \left[\frac{1}{v}\frac{\partial \eta^{(\mu)}}{\partial t} + \mu\frac{\partial \eta^{(\mu)}}{\partial x}\right]\frac{\partial \psi}{\partial \mu} - \frac{\sigma c}{2}\int_{-1}^{1} \mathrm{d}\mu' \eta^{(\mu)}\frac{\partial \psi}{\partial \mu'}. \tag{5.38}$$

An easy solution to this equation is the trivial solution: $\eta^{(\mu)} = 0$; this solution implies the identity transformation, which we state below. We will use this solution and proceed by stating a solution to the symmetry determining equations 5.37, i.e. the Lie group is:

$$\eta^{(x)} = f \tag{5.39a}$$
$$\eta^{(\mu)} = 0 \tag{5.39b}$$
$$\eta^{(t)} = b \tag{5.39c}$$
$$\phi = a\psi. \tag{5.39d}$$

These coordinate functions have the associated infinitesimal generator and characteristic system:

$$V = f\frac{\partial}{\partial x} + b\frac{\partial}{\partial t} + a\psi\frac{\partial}{\partial \psi}, \tag{5.40a}$$
$$\frac{\mathrm{d}x}{f} = \frac{\mathrm{d}\mu}{0} = \frac{\mathrm{d}t}{b} = \frac{\mathrm{d}\psi}{a\psi} = \mathrm{d}\epsilon, \tag{5.40b}$$

where ϵ is the group parameter.

The group of global transformations, that leave the NTE invariant, corresponding to the Lie group 5.39 is obtained by solving the system of equations:

$$\frac{\partial \widetilde{x}(x;\epsilon)}{\partial \epsilon} = \eta^{(x)} = f \tag{5.41a}$$
$$\frac{\partial \widetilde{\mu}(\mu;\epsilon)}{\partial \epsilon} = \eta^{(\mu)} = 0 \tag{5.41b}$$
$$\frac{\partial \widetilde{t}(t;\epsilon)}{\partial \epsilon} = \eta^{(t)} = b \tag{5.41c}$$
$$\frac{\partial \widetilde{\psi}(\psi;\epsilon)}{\partial \epsilon} = \phi = a\psi, \tag{5.41d}$$

with respective initial conditions: $\widetilde{x}(x;\epsilon = 0) = x$, $\widetilde{\mu}(\mu;\epsilon = 0) = \mu$, $\widetilde{t}(t;\epsilon = 0) = t$, $\widetilde{\psi}(\psi;\epsilon = 0) = \psi$. Solving each of the above yields the group of global transformations:

$$\widetilde{x}(x;\epsilon) = x + f\epsilon \tag{5.42a}$$

$$\widetilde{\mu}(\mu;\epsilon) = \mu \tag{5.42b}$$
$$\widetilde{t}(t;\epsilon) = t + b\epsilon \tag{5.42c}$$
$$\widetilde{\psi}(\psi;\epsilon) = \psi\,\mathrm{e}^{a\epsilon} \tag{5.42d}$$

We will define a new coordinate system that maps $(x\mu t\psi)$-space to $(qrsw)$-space by solving members of Eq. 5.40b and we will be able to construct a new characteristic system using the global transforms Eqs. 5.42a - 5.42d. The member equations we solve will will be $\frac{\mathrm{d}x}{\mathrm{d}\epsilon}$, $\frac{\mathrm{d}\mu}{\mathrm{d}x}$, $\frac{\mathrm{d}t}{\mathrm{d}x}$, and $\frac{\mathrm{d}\psi}{\mathrm{d}t}$. Integrating these equations and solving for the constant of integration defines a variable in the new coordinate system. We first solve the $\frac{\mathrm{d}x}{\mathrm{d}\epsilon}$ equation to determine q:

$$\frac{\mathrm{d}x}{\mathrm{d}\epsilon} = f \to q = x = f\epsilon + c_1. \tag{5.43}$$

We note that we set the new variable q to include the group parameter ϵ because we do not need the new variables to be parameterized by ϵ. Solving the remaining members, we find:

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = 0 \to r = \mu \tag{5.44a}$$
$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{b}{f} \to s = \frac{t}{b} - \frac{x}{f} \tag{5.44b}$$
$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{a\psi}{b} \to w = \psi\,\mathrm{e}^{-at/b}\,. \tag{5.44c}$$

Note that we could have solved other members of Eq. 5.40b, such as $\frac{\mathrm{d}\psi}{\mathrm{d}x}$ to find $w = \psi\exp(-ax/f)$, and such choices are arbitrary as we will end up with the same solution. Admittedly, some choices make the ensuing analysis easier, and we did in fact chose to solve $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ for this reason. Equations 5.43 - 5.44c define the new coordinate system in $(qrsw)$-space.

We next determine the global transformations of the new coordinates, $(\widetilde{q}, \widetilde{r}, \widetilde{s}, \widetilde{w})$. This is done in three steps, by (1) taking the definitions Eqs. 5.43 - 5.44c and placing tildes over all variables in the definitions, then (2) using Eqs. 5.42 to recover the original coordinates, then (3) rearranging to write the expressions in the new coordinates. We demonstrate below, where each line is one of the outlined steps:

$$\begin{aligned}\widetilde{q} &= \widetilde{x} \\ &= x + f\epsilon \\ &= q + f\epsilon\end{aligned} \tag{5.45a}$$
$$\begin{aligned}\widetilde{r} &= \widetilde{\mu} \\ &= \mu \\ &= r\end{aligned} \tag{5.45b}$$
$$\begin{aligned}\widetilde{s} &= \frac{\widetilde{t}}{b} - \frac{\widetilde{x}}{f} \\ &= \frac{t + b\epsilon}{b} - \frac{x + f\epsilon}{f} \\ &= s\end{aligned} \tag{5.45c}$$
$$\begin{aligned}\widetilde{w} &= \widetilde{\psi}\,\mathrm{e}^{-a\widetilde{t}/b} \\ &= \psi\,\mathrm{e}^{a\epsilon}\,\mathrm{e}^{-a(t+b\epsilon)/b} \\ &= w.\end{aligned} \tag{5.45d}$$

Thus we have a translation transformation in q and the identity transformation for r, s, and w. This allows us to immediately construct a new characteristic system:

$$\boxed{\frac{\mathrm{d}q}{f} = \frac{\mathrm{d}r}{0} = \frac{\mathrm{d}s}{0} = \frac{\mathrm{d}w}{0} = \mathrm{d}\epsilon.} \tag{5.46}$$

Using the transformation group, Eqs. 5.43 - 5.44c, we may transform the NTE into an equivalent equation in $(qrsw)$-space, then we will be able to simplify this new transport equation using the characteristic system Eq. 5.46. First we will need the identities for $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial x}$, which can be determined using the chain rule to find:

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial q}{\partial t}\frac{\partial w}{\partial q} + \frac{\partial r}{\partial t}\frac{\partial w}{\partial r} + \frac{\partial s}{\partial t}\frac{\partial w}{\partial s} \\ &= \frac{1}{b}\frac{\partial w}{\partial s}\end{aligned} \tag{5.47a}$$

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\partial q}{\partial x}\frac{\partial w}{\partial q} + \frac{\partial r}{\partial x}\frac{\partial w}{\partial r} + \frac{\partial s}{\partial x}\frac{\partial w}{\partial s} \\ &= \frac{\partial w}{\partial q} - \frac{1}{f}\frac{\partial w}{\partial s},\end{aligned} \tag{5.47b}$$

where, in going from the first to the second line, we utilized the partial derivatives of q, r, and s via Eqs. 5.43 - 5.44b. Next, we write the ψ derivatives of the NTE in terms of the new coordinates using Eqs. 5.47a - 5.47b:

$$\begin{aligned}\frac{\partial \psi}{\partial t} &= \mathrm{e}^{at/b}\left[\frac{a}{b}w + \frac{\partial w}{\partial t}\right] \\ &= \mathrm{e}^{at/b}\left[\frac{a}{b}w + \frac{1}{b}\frac{\partial w}{\partial s}\right]\end{aligned} \tag{5.48a}$$

$$\begin{aligned}\frac{\partial \psi}{\partial x} &= \mathrm{e}^{at/b}\frac{\partial w}{\partial x} \\ &= \mathrm{e}^{at/b}\left[\frac{\partial w}{\partial q} - \frac{1}{f}\frac{\partial w}{\partial s}\right].\end{aligned} \tag{5.48b}$$

Finally, we insert Eqs. 5.48a - 5.48b as well as Eqs. 5.43 - 5.44c into the NTE, Eq. 5.11, cancel and collect terms to find:

$$\left[\frac{1}{vb} - \frac{r}{f}\right]\frac{\partial w}{\partial s} + r\frac{\partial w}{\partial q} + \left[\frac{a}{vb} + \sigma\right]w(q, r, s) = \frac{\sigma c}{2}\int_{-1}^{1}\mathrm{d}r' w(q, r', s). \tag{5.49}$$

As was mentioned, Eq. 5.49 is an equivalent NTE to Eq. 5.11 in the new coordinate system defined by Eqs. 5.43 - 5.44c. In its current form, this new transport equation has not provided any useful information as it is still an integro-differential equation with two derivatives and one integral. However, we are in a position to simplify this equation by considering the directional derivative with respect to the group parameter, $\frac{\mathrm{d}w}{\mathrm{d}\epsilon}$. Using the characteristic system, Eq. 5.46, and the chain rule, we have:

$$\boxed{\frac{\mathrm{d}w}{\mathrm{d}\epsilon} = 0 = \frac{\partial w}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}\epsilon} + \frac{\partial w}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}\epsilon} + \frac{\partial w}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}\epsilon}.} \tag{5.50}$$

Now assembling directional derivatives of Eq. 5.46:

$$\frac{\mathrm{d}q}{\mathrm{d}\epsilon} = f \tag{5.51a}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\epsilon} = 0 \tag{5.51b}$$

$$\frac{\mathrm{d}s}{\mathrm{d}\epsilon} = 0, \tag{5.51c}$$

and inserting these into Eq. 5.50 yields a useful identity along the directional ϵ:

$$\boxed{\frac{\partial w}{\partial q} = 0 \quad \text{along direction } \epsilon} \tag{5.52}$$

thus

$$\boxed{w(q, r, s) \to w(r, s) \text{ along direction } \epsilon} \tag{5.53}$$

and Eq. 5.49 simplifies now to:

$$\boxed{\left[\frac{1}{vb} - \frac{r}{f}\right]\frac{\partial w}{\partial s} + \left[\frac{a}{vb} + \sigma\right] w(r, s) = \frac{\sigma c}{2}\int_{-1}^{1} \mathrm{d}r' w(r', s).} \tag{5.54}$$

Equation 5.54 is the NTE in (rsw)-space along the direction ϵ. We have reduced the equation from being a function of three independent variables down to an *equivalent* equation of two independent variables via the transformation group Eqs. 5.43 - 5.44c. Clearly, Eq. 5.54 is still somewhat difficult to solve as it is still an integro-differential equation. As an aside, had we solved for w using $\mathrm{d}\psi/\,\mathrm{d}x$, we would have found the same equation, except the coefficient of w would be $ar/f + \sigma$. There would then be two r variables in the equation, so we opted to use the "cleaner" route of solving for w using $\mathrm{d}\psi/\,\mathrm{d}t$.

We further reduce this new equation, Eq. 5.54, by casting the (rsw)-space to another new coordinate system, say $(\mathcal{RSW})$-space. Instead of deriving the symmetry determining equations for Eq. 5.54 to find a symmetry group, we can simply infer a symmetry group. Thus, we simply state the coordinate functions we proceed with:

$$\eta^{(r)} = 0 \tag{5.55a}$$

$$\eta^{(s)} = 1 \tag{5.55b}$$

$$\phi^{(w)} = \gamma w, \tag{5.55c}$$

where γ is some scaling parameter to be determined. The corresponding global transformation group is:

$$\widetilde{r}_2 = r \tag{5.56a}$$

$$\widetilde{s}_2 = s + \epsilon_2 \tag{5.56b}$$

$$\widetilde{w}_2 = w\, \mathrm{e}^{\gamma\epsilon_2}, \tag{5.56c}$$

where we add the subscript to distinguish this transformation group from Eqs. 5.45 as we are now mapping between another parameterized coordinate system. Thus the infinitesimal generator and characteristic system are:

$$V_2 = \frac{\partial}{\partial s} + \gamma w \frac{\partial}{\partial w}, \tag{5.57a}$$

$$\frac{\mathrm{d}r}{0} = \frac{\mathrm{d}s}{1} = \frac{\mathrm{d}w}{\gamma w} = \mathrm{d}\epsilon_2. \tag{5.57b}$$

To determine the new coordinate system, we solve the members of Eq. 5.57b: $\frac{\mathrm{d}r}{\mathrm{d}s}$, $\frac{\mathrm{d}s}{\mathrm{d}\epsilon_2}$, and $\frac{\mathrm{d}w}{\mathrm{d}s}$, from which we solve for the constants of integration to yield $\mathcal{R}, \mathcal{S}$, and $\mathcal{W}$; doing so gives:

$$\mathcal{R} = r \tag{5.58a}$$
$$\mathcal{S} = s \tag{5.58b}$$
$$\mathcal{W} = w\,\mathrm{e}^{-\gamma s}\,. \tag{5.58c}$$

As was done in Eqs. 5.45, we may determine the global transformations of the new coordinates:

$$\widetilde{\mathcal{R}} = \mathcal{R} \tag{5.59a}$$
$$\widetilde{\mathcal{S}} = \mathcal{S} + \epsilon_2 \tag{5.59b}$$
$$\widetilde{\mathcal{W}} = \mathcal{W}, \tag{5.59c}$$

and we have a characteristic system for $(\mathcal{RSW})$-space:

$$\boxed{\frac{\mathrm{d}\mathcal{R}}{0} = \frac{\mathrm{d}\mathcal{S}}{1} = \frac{\mathrm{d}\mathcal{W}}{0} = \mathrm{d}\epsilon_2.} \tag{5.60}$$

We can now transform Eq. 5.54 from (rsw)-space to $(\mathcal{RSW})$-space using the derivatives:

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial s} &= \frac{\partial \mathcal{R}}{\partial s}\frac{\partial \mathcal{W}}{\partial \mathcal{R}} + \frac{\partial \mathcal{S}}{\partial s}\frac{\partial \mathcal{W}}{\partial \mathcal{S}} \\ &= \frac{\partial \mathcal{W}}{\partial \mathcal{S}}\end{aligned} \tag{5.61a}$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \mathrm{e}^{\gamma s}\left[\gamma \mathcal{W} + \frac{\partial \mathcal{W}}{\partial s}\right] \\ &= \mathrm{e}^{\gamma s}\left[\gamma \mathcal{W} + \frac{\partial \mathcal{W}}{\partial \mathcal{S}}\right],\end{aligned} \tag{5.61b}$$

where we used the partial derivatives of Eqs. 5.58a and 5.58b to go from the first to the second line of Eq. 5.61a. Inserting Eqs. 5.58a - 5.58c and Eq. 5.61b into Eq. 5.54 and simplifying yields another equivalent transport equation in $(\mathcal{RSW})$-space:

$$\left[\frac{1}{vb} - \frac{\mathcal{R}}{f}\right]\frac{\partial \mathcal{W}}{\partial \mathcal{S}} + \left[\frac{\gamma + a}{vb} + \sigma - \frac{\gamma}{f}\mathcal{R}\right]\mathcal{W}(\mathcal{R},\mathcal{S}) = \frac{\sigma c}{2}\int_{-1}^{1} \mathrm{d}\mathcal{R}'\mathcal{W}(\mathcal{R}',\mathcal{S}). \tag{5.62}$$

Again, we simplify this equation by considering the directional derivative $\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}\epsilon_2}$, which may be constructed from the characteristic system, Eq. 5.60. Applying the chain rule, we find

$$\boxed{\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}\epsilon_2} = 0 = \frac{\partial \mathcal{W}}{\partial \mathcal{R}}\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\epsilon_2} + \frac{\partial \mathcal{W}}{\partial \mathcal{S}}\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}\epsilon_2}.} \tag{5.63}$$

From the characteristic system, we also know $\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\epsilon_2} = 0$ and $\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}\epsilon_2} = 1$, which provides the condition:

$$\boxed{\frac{\partial \mathcal{W}}{\partial \mathcal{S}} = 0 \quad \text{along direction } \epsilon_2} \tag{5.64}$$

thus

$$\boxed{\mathcal{W}(\mathcal{R},\mathcal{S}) \to \mathcal{W}(\mathcal{R}) \text{ along direction } \epsilon_2.} \tag{5.65}$$

Finally, we find a reduced equation in one-variable:

$$\left[\frac{\gamma + a}{vb} + \sigma - \frac{\gamma}{f}\mathcal{R}\right]\mathcal{W}(\mathcal{R}) = \frac{\sigma c}{2}\int_{-1}^{1} \mathrm{d}\mathcal{R}'\mathcal{W}(\mathcal{R}'). \tag{5.66}$$

We can simplify the form of the coefficient on the LHS by considering the exponent of Eq. 5.44c: at/b. In order for this quantity to be dimensionless, we set

$$a = -\sigma \tag{5.67a}$$

$$b = 1/v \tag{5.67b}$$

$$\gamma = \sigma f \gamma_1 \tag{5.67c}$$

here we have included a -1 in a for convenience and have also made the assumption that γ is proportional to σ and f, where γ_1 is some value to be determined. Inserting Eqs. 5.67a - 5.67c into Eq. 5.66 and isolating the LHS, we find:

$$\boxed{\mathcal{W}(\mu) = \frac{c/2}{(f-\mu)\gamma_1}\int_{-1}^{1} \mathrm{d}\mu'\mathcal{W}(\mu'),} \tag{5.68}$$

where we have applied the double-identity mapping $\mathcal{R} = r = \mu$ to recover μ. Equation 5.68 is classified as a homogeneous Fredholm integral equation of the Second Kind with an asymmetric non-degenerate singular kernel [15]. The kernel, $K(\mu, \mu') = \frac{c/2}{(f-\mu)\gamma_1}$, is asymmetric because $K(\mu, \mu') \neq K(\mu', \mu)$. It is non-degenerate because it cannot be written as a finite sum of the form $K(\mu, \mu') \neq \sum_{n=0}^{N} g_1(\mu)g_2(\mu')$ (note the kernel can be written as a product of two functions $g_1(\mu)g_2(\mu')$, the issue is that the sum is not finite). Also, the kernel is singular at the point $\mu = f$, which implies any solutions $\mathcal{W}$ lie in the complex plane.

Though it appears relatively simple in form, Eq. 5.68 requires rigorous mathematics to completely solve. The typical method for solving Fredholm equations is the Liouville-Neumann Series, also known as the Neumann Series Method[1], but that method only applies to inhomogeneous Fredholm equations. A formal solution to Eq. 5.68 requires a spectral analysis of the Fredholm operator, defined by the RHS of Eq. 5.68, and a treatment of the solution as a Schwarz distribution [17], i.e. a generalized function or simply a distribution, to which the Singular Eigenfunction Expansion Method may be applied [18, 19, 20, 21, 22]. For our purposes, it suffices to simply state the solution as we have succeeded in our goal of deriving the Lie group corresponding to the Case solution. To that end, we may further define γ_1 and f:

$$f = \alpha \tag{5.69a}$$

$$\gamma_1 = -ik, \tag{5.69b}$$

where α is a real eigenvalue and k is a fixed real number and the solution is indexed accordingly: $\mathcal{W}_{\alpha,k}(\mu)$. Thus, we obtain exactly the same equation that Case originally obtained:

$$\mathcal{W}_{\alpha,k}(\mu) = \frac{ic}{2k}\frac{1}{(\alpha-\mu)}\int_{-1}^{1} \mathrm{d}\mu'\mathcal{W}_{\alpha,k}(\mu'). \tag{5.70}$$

We note that the "i" appears in Case's solution from a Laplace transform in time while "k" shows up from a Fourier transform in space (see Case and Zweifel, Sec. 7.2.A and 7.3.A [18]).

[1] See Ash Sec. 2.10, Eqs. 2.144 - 2.146 for an example where Liouville-Neumann Series is applicable because the equation is inhomogeneous [16].

Finally, we can map our two coordinate systems back to the original coordinate system using Eqs. 5.58a - 5.58c to go from $(\mathcal{RSW}) \rightarrow (qrsw)$ and Eqs. 5.43 - 5.44c to go from $(qrsw) \rightarrow (x\mu t\psi)$. Solving for ψ using Eq. 5.44c, i.e.

$$\psi = w\,\mathrm{e}^{at/b}, \tag{5.71}$$

we may use Eq. 5.58c to replace w with $\mathcal{W}$ to find

$$\begin{aligned}\psi &= \mathcal{W}(\mathcal{R})\,\mathrm{e}^{\gamma s}\,\mathrm{e}^{at/b}\\ &= \mathcal{W}(\mu)\,\mathrm{e}^{\gamma(t/b - x/f)}\,\mathrm{e}^{at/b},\end{aligned} \tag{5.72}$$

where we used $\mathcal{R} = \mu$ and Eq. 5.44b to revert s back to t and x. Finally, by inserting Eqs. 5.67a - 5.67c and 5.69a - 5.69b, we find the solution:

$$\psi(x, \mu, t) = \mathrm{e}^{ik\sigma x}\,\mathrm{e}^{-(1+i\alpha k)v\sigma t}\,\mathcal{W}_{\alpha,k}(\mu), \tag{5.73}$$

where it is understood that one must further solve the Fredholm equation, Eq. 5.70, to fully obtain μ-dependence and doing so is beyond the scope of this paper.

We now discuss the physical meaning of the Lie groups used in this section to arrive at Case's solution. The first Lie group given by Eq. 5.42 is restated here using the inferred values given by Eqs. 5.67 and 5.69:

$$\begin{aligned}\widetilde{x}(x;\epsilon) &= x + \alpha\epsilon\\ \widetilde{\mu}(\mu;\epsilon) &= \mu\\ \widetilde{t}(t;\epsilon) &= t + \epsilon/v\\ \widetilde{\psi}(\psi;\epsilon) &= \psi\,\mathrm{e}^{-\sigma\epsilon}\,.\end{aligned}$$

This transformation group is a simultaneous space-translation, time-translation, and solution-scaling transformation, while remaining in the original μ-space. The combination of time-translation and solution-scaling results in the consolidated semi-solution given by Eq. 5.44c, i. e. $w = \psi\exp\{+v\sigma t\}$ which implies, for invariance, that a change in space and time of $+\alpha\epsilon$ and $+\epsilon/v$, respectively[2], must be met by a scaling of the solution by a factor $\exp\{-\sigma\epsilon\}$. The new, less-one dimension along the group-parameter characteristic curve, coordinate system is given by:

$$\langle r, s, w\rangle = \left\langle \mu, vt - \frac{x}{\alpha}, \psi\,\mathrm{e}^{v\sigma t}\right\rangle, \tag{5.75}$$

where we see s acts as a Galilean boost, or a shear, while w scales the original solution ψ with a factor $\exp\{v\sigma t\}$. Often we see the combination of space- and time-translation invariance in the construction of traveling wave solutions. This Galilean boost combined with a scaling of solution suggests that the traveling wave, i.e. the neutron flux information of the solution, is either amplified or dampened by neutrons moving at speed v in a medium characterized by the interaction probability σ. The traveling wave is exponentially dampened when the system is subcritical and will exponentially amplify for supercritical systems. In the new coordinate system given by the Lie group of transformations, space and time are bundled into a single dimension and similarly the flux with time providing the reduced transport equation given by Eq. 5.54 (where the coefficient of w is zero when using Eqs. 5.67a and 5.67b), restated here:

$$\left[1 - \frac{r}{\alpha}\right]\frac{\partial w}{\partial s} = \frac{\sigma c}{2}\int_{-1}^{1}\mathrm{d}r'\,w(r', s).$$

[2]The parameter α has an implicit dependence on the physical parameters (e. g. v, σ, c) when the Fredholm equation is solved.

This transport equation has a Lie group that is the "same" Lie group of transformations given by Eq. 5.56, such that we have a Lie group of leaving the angle identical, s-translation (translation in the shear $vt - x/\alpha$) with solution scaling:

$$\begin{aligned}\widetilde{r}_2(r;\epsilon_2) &= r\\ \widetilde{s}_2(s;\epsilon_2) &= s+\epsilon_2\\ \widetilde{w}_2(w;\epsilon_2) &= w\,\mathrm{e}^{-ik\alpha\sigma\epsilon_2}\,.\end{aligned}$$

Thus a translation in s by $+\epsilon_2$ requires a scaling of the solution by the factor $\exp\{-ik\alpha\sigma\epsilon_2\}$, which can be thought of as a rotation of $k\alpha\sigma\epsilon_2$ radians in the complex plane. As before, k and α have constraints derived from solving the Fredholm equation for the angular dependence. The new reduced coordinate system is

$$\langle\mathcal{R},\mathcal{W}\rangle = \left\langle r, w\,\mathrm{e}^{-ik\alpha\sigma(vt-x/\alpha)}\right\rangle, \tag{5.77}$$

which results in the final Fredholm equation along the characteristic curve ϵ_2. Thus the second Lie group reduces the transport variables into a complex exponential for the original angle dependence.

5.3 Exercises

1. Consider a flux of neutrons, that initially have an energy E_o, passing through a material composed of atoms with atomic mass A. The flux that has undergone n collisions, $F_n(u)$, will have a lethargy $u = \ln(E_o/E)$ with $E_o \geq E$. The equation describing the n-th collision flux is:

$$\begin{aligned}F_n(u) = S(u) &+ g(u)\int_{u-\delta}^{n\delta}\mathrm{d}u' f(u')F_{n-1}(u')\\ &+ g(u)\int_{n\delta}^{u}\mathrm{d}u' f(u')F_n(u')\end{aligned} \tag{5.78}$$

   where

$$g(u) = \mathrm{e}^{-u} \tag{5.79a}$$

$$f(u) = \gamma(u)\frac{\mathrm{e}^{u}}{1-\alpha}. \tag{5.79b}$$

   Also, $S(u)$ is an arbitrary source producing neutrons with lethargy u, $\delta = \ln(1/\alpha)$ is the max lethargy a neutron can gain in a single collision, $\alpha = (A-1)^2/(A+1)^2$, and $\gamma(u) = \Sigma_s(u)/\Sigma_t(u)$ is an arbitrary function of the scattering and total macroscopic cross sections of the medium.

   (a) Derive the symmetry determining equation for this integral equation
   (b) Show that the Lie group below satisfies the determining equation:

$$(\eta,\phi) = \left(0, c\exp\left\{\int_{n\delta}^{u}\mathrm{d}u'\mathcal{A}(u')\right\}\right), \tag{5.80}$$

   where $\mathcal{A}(u) = \frac{1}{g(u)}\frac{\mathrm{d}g}{\mathrm{d}u} + f(u)g(u)$.
   (c) Determine the solution permitted by this Lie group

(d) Find the global transformations of this Lie group and prove invariance of the surface equation.

(e) Comment on the functional form of this Lie group, i.e. specifically, what is the connection between this Lie group's form and the integrating factor?

Chapter 6

Conclusions

Throughout these lectures, we have provided a framework for the systematic implementation of Lie Group Theory to solve varying types of differential equations. We hope the reader can follow along, step-by-step, in solving their own problems. This work is clearly focused on the utility, or application, of LGT, but there are many texts that provide deeper insight, in the pure mathematical sense, to how LGT truly is a "generalized integration theory." The motivated reader is pointed to the texts [23, 24, 25, 26, 27, 28, 29, 30] for such an academic journey. To conclude these lectures, we will simply summarize the overarching themes and steps that are present in the solution process via LGT. These steps, are:

1. Write the equation under investigation as a surface equation, such that $F(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = 0$.

2. Derive the symmetry determining equations using the formula:

$$\mathrm{pr}^{(K)} F(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})\Big|_{F=0} = \left.\frac{\mathrm{d}F(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{p}})}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = 0. \tag{6.1}$$

   We tend to use $\mathrm{pr}^{(K)} F(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})|_{F=0} = 0$ for differential equations that contain purely local operators, and we are required to use $\frac{\mathrm{d}F(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{p}})}{\mathrm{d}\epsilon}|_{\epsilon=0} = 0$ for equations with nonlocal operators. Note that we can use $\frac{\mathrm{d}F(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{p}})}{\mathrm{d}\epsilon}|_{\epsilon=0} = 0$ on all equations, regardless of the presence of nonlocal operators.

3. Find solutions to the symmetry determining equations, which are known as Lie groups. This is of course easier said than done, but usually we can determine whether or not F permits the basic forms of Lie groups (e.g. translation, scaling) by inspection. A "sanity check" at this stage is to derive the global transformations,

$$\frac{\partial \widetilde{\boldsymbol{x}}}{\partial \epsilon} = \boldsymbol{\eta}(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}}), \qquad \widetilde{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{u}; \epsilon = 0) = \boldsymbol{x} \tag{6.2a}$$

$$\frac{\partial \widetilde{\boldsymbol{u}}}{\partial \epsilon} = \boldsymbol{\phi}(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}}), \qquad \widetilde{\boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{u}; \epsilon = 0) = \boldsymbol{u}, \tag{6.2b}$$

   and to ensure that the symmetry criterion is satisfied by inserting the global transforms into the original equation

$$F(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = F(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{p}}) = 0. \tag{6.3}$$

4. Given a Lie group, we construct the characteristic system,

$$\frac{\mathrm{d}x_1}{\eta^{(1)}(\boldsymbol{x},\boldsymbol{u})} = \cdots = \frac{\mathrm{d}x_I}{\eta^{(I)}(\boldsymbol{x},\boldsymbol{u})} = \frac{\mathrm{d}u_1}{\phi^{(1)}(\boldsymbol{x},\boldsymbol{u})} = \cdots = \frac{\mathrm{d}u_J}{\phi^{(J)}(\boldsymbol{x},\boldsymbol{u})} = \mathrm{d}\epsilon, \tag{6.4}$$

   to systematically determine a new coordinate system that we can then transform our original variable space into to either simplify the surface equation or outright solve it. For ODEs, we are able to use canonical coordinates which guarantee that, at a minimum, the equation in the new variable space will be separable. For PDEs, we have a less systematic approach, and we must strategically solve members of the characteristic system.

   Perhaps the most crucial detail in this step is to use the fact that the surface equation, once converted to the new variable space, will only simplify when we "evaluate it" along the direction $\mathrm{d}\epsilon$; this process allows us to construct an identity for the total parametric derivative $\frac{\mathrm{d}w}{\mathrm{d}\epsilon}$ (where w is the transformed variable corresponding to u) using the characteristic system. From which we may determine relations between the partials and the new coordinate space's parametric dependence on ϵ, which will result in a reduction of order of the equation. See the discussion leading up to Eq. 4.48 for the heat equation and Eqs. 5.52 and 5.64 for the neutron transport equation example.

5. Once the equation in the new coordinate system is solved, we can revert back to the original variable space to obtain the desired solution.

Given these steps, one can explore the mathematical structure of an equation and, based on the permitted Lie groups, make deeper observations on the nature of the equation.

Now onward!

Bibliography

[1] Albright, E.J., McHardy, J.D., Ramsey, S.D., and Schmidt, J.H., "Symmetry Analysis of Differential Equations: A Primer," Los Alamos Technical Report, LA-14502 (2018).

[2] Stephani, H., *Differential equations: Their solution using symmetries*, Cambridge University Press, Cambridge, 1989.

[3] Nordheim, L.W., "Pile Kinetics," MDDC-35. 1946.

[4] Fuchs, K., "Efficiency for Very Slow Assembly," LA-596. 1946.

[5] Hetrick, D. L., *Dynamics of Nuclear Reactors.* University of Chicago Press. 1971.

[6] Panayotounakos, D.E. and Zarmpoutis, T.I., "Construction of Exact Parametric or Closed Form Solutions of Some Unsolvable Classes of Nonlinear ODEs," *Intl. J. of Mathematics and Mathematical Sci.*, 2011.

[7] Polyanin, A.D. and Zaitsev, V.F., *Handbook of Exact Solutions for Ordinary Differential Equations*, 2nd ed., Chapman & Hall/CRC, Boca Raton, 2003.

[8] O'Rourke, P.F., Ramsey, S.D., and Temple, B.A., "Lie Group Analysis of the Point-Reactor Neutron Kinetics Equations for Various Reactivity Models," *Annals of Nuclear Energy*, **160**, 2021.

[9] Ibragimov, N.H.(Editor), *CRC Handbook of Lie group analysis of differential equations*, Boca Raton, Florida, CRC Press, Inc., Vol.1, 1994; Vol.2, 1995; Vol.3, 1996.

[10] Harrison, B.K. and Estrabrook, F.B., "Geometric approach to invariance groups and solutions of partial differential system," *J. Math. Phys.*, **12**(4), 653-666 (1971).

[11] Fushchich, W.I. and Selehman, N.A., "Integro-differential equations invariant under group of Galilei, Poincare, Schrödinger, and conformal group," *Dokl. AN Ukr SSR, A*, No.5, 21-24 (1983).

[12] Fushchich, W.I., W.M. Shtelen, and N.I. Serov. *Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics.* 1st Ed. 1993. ed. 1993. Mathematics and Its Applications.

[13] Grigoriev, Y.N. and Meleshko, S.V., "Group Analysis of Kinetic Equations," *Russ. J. Numer. Anal. Math. Modelling*, **10**(5), 425-446. (1995).

[14] O'Rourke, P.F. and Ramsey, S.D., "Symmetry Determining Equations and Lie Groups of the Neutron Transport Equation," *Nuclear Science and Engineering*, (2024).

[15] Zemyan, S.M., 2012. *The Classical Theory of Integral Equations: A Concise Treatment*, Springer.

[16] Ash, M.S., 1979. *Nuclear Reactor Kinetics.* McGraw-Hill.

[17] Schwarz, L., 1951. *Théorie des distributions*, vol. 1–2, Hermann.

[18] Case, K.M. and Zweifel, P.F., 1967. *Linear Transport Theory.* Addison-Wesley.

[19] Davison, B., "Angular Distribution due to an Isotropic Point Source and Spherically Symmetric Eigensolutions of the Transport Equation," *Canadian Report MT-112* (1945).

[20] Wigner, E.P., "Mathematical Problems of Nuclear Reactor Theory," *Proc. Symp. Appl. Math.* (American Mathematical Society, Providence, Rhode Island, 1959) **11**, 89.

[21] Van Kampen, N.G., "On the Theory of Stationary Waves in Plasmas," *Physica*, **21**, 949 (1955).

[22] Case, K.M., "Recent Developments in Neutron Transport Theory," *Univ. of Michigan Memorial Phoenix Project Report* (1961).

[23] Lie, S., "Uber die Integration durch bestimmte Integrals von einer Klasse linearer Partieller Differential Gleichungen". *Arch. Math.*, **6**, 328-628. 1881.

[24] Ovsjannikov, L., *Group Analysis of Differential Equations.* Academic Press. 1982.

[25] Olver, P., *Applications of Lie Groups to Differential Equations*, Springer. 1986.

[26] Cantwell, B., *Introduction to Symmetry Analysis.* Cambridge University Press. 2002.

[27] Hydon, P., *Symmetry Methods for Differential Equations: A Beginner's Guide.* Cambridge University Press. 2000.

[28] Stephani, H., *Differential Equations: Their Solution Using Symmetries.* Cambridge University Press. 1989.

[29] Bluman, G. and Kumei, S., *Symmetries and Differential Equations*, Springer Science & Business Media, Berlin (2013).

[30] Bluman, G. and Anco, S., *Symmetry and Integration Methods for Differential Equations*, Springer Science & Business Media, Berlin (2008).