Heavy Quark Fragmentation Functions for D-wave Quarkonium and Charmed Beauty Mesons

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Abstract

At the large transverse momentum region, the production of heavy-heavy bound-states such as charmonium, bottomonium, and $\bar{b}c$ mesons in high energy e^+e^- and hadronic collisions is dominated by parton fragmentation. We calculate the heavy quark fragmentation functions into the D-wave quarkonium and $\bar{b}c$ mesons to leading order in the strong coupling constant and in the non-relativistic expansion. In the $\bar{b}c$ meson case, one set of its D-wave states is expected to lie below the open flavor threshold. The total fragmentation probability for a $\bar{b}$ antiquark to split into the D-wave $\bar{b}c$ mesons is about

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2×10^{-5}, which implies that only 2% of the total pseudo-scalar ground state B_c comes from the cascades of these orbitally excited states.

I. INTRODUCTION

The production of heavy quark-antiquark bound states like charmonium, bottomonium, and the yet undiscovered charmed beauty mesons at high energy e^+e^- and hadronic machines can provide very interesting tests of perturbative QCD. In the charmonium case, our traditional wisdom [1] has been taken the following scenario: most of the J/ψ comes predominately from either the radiative decay of the P-wave χ_{cJ} states produced by the lowest order gluon fusion mechanism or the weak decays of B mesons, whereas the ψ' should be produced almost entirely from B meson decays. Similarly, most of the Υ should be produced from the radiative χ_{bJ} decay. Only until recently can this naive thinking be confronted by the wealth of high energy experimental data collected at the Tevatron. During the 1992-1993 run, the CDF detector recorded surprisingly large production rates of the prompt J/ψ, ψ', and χ_{cJ} which are orders of magnitude above the theoretical lowest order predictions [2]. In 1993 [3], it was realized that fragmentation can play an important role in quarkonium production at the large transverse momentum (p_T) region at the Tevatron. Parton fragmentation into quarkonium is formally a higher order process but nevertheless it can be enhanced at the sufficiently large transverse momentum region compared with the usual gluon fusion process [3]. A simple explanation of the enhancement is that the fragmenting partons are produced at high energies but with small invariant masses which can lead to the enhancement factor of powers of $(p_T/m_c)^2$ compared with the gluon fusion mechanism.

Another important theoretical development is the factorization formalism of Bodwin, Braaten, and Lepage [4] for the inclusive decay and production of heavy quarkonium. This new approach, which is based on non-relativistic QCD (NRQCD), allows us to go beyond the conventional color-singlet model [5] by considering a double series expansion of the decay or production rates in terms of the strong coupling constant and the relative velocity of the

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heavy quarks inside the bound state. This double series expansion leads us naturally to include the higher Fock states of the quarkonium, some of which have the heavy quark pair in the color-octet state. We will discuss more about this factorization formalism in the next section.

Most of the fragmentation functions for quarkonium that are relevant to phenomenology have now been calculated. Gluon fragmentation functions were calculated in Ref. [3] for the S-wave, Ref. [6,7] for the P-wave, and Ref. [8] for the $^1D_2$ state; heavy quark fragmentation functions were calculated in Ref. [9,10] for the S-wave, and Ref. [11–13] for the P-wave. Phenomenological applications of these fragmentation functions have been performed by a number of groups [14,15]. The current scenario of charmonium production at high transverse momentum is as follows: among all fragmentation contributions that are relevant to J/ψ production, the dominant ones come from (i) gluon fragmentation into a color-octet $^3S_1$ $c\bar{c}$ state followed by a double E1 transition into J/ψ, and (ii) gluon fragmentation into χ_{cJ} followed by the radiative decay $\chi_{cJ} \rightarrow J/\psi + \gamma$. The former mechanism is the most dominant one for ψ' production because of the absence of $\chi_{cJ}(2P)$ states below the $D\overline{D}$ threshold. The current experimental status with this new fragmentation insight has now been reviewed by many people in various occasions. We refer the readers to Refs. [14–17] for more details.

The charmed beauty $\bar{b}c$ mesons are expected to be observed soon at the Tevatron and preliminary limits on the production rate already exists [18]. Parton fragmentation has been applied phenomenologically to the production of $\bar{b}c$ mesons as well at the Tevatron [19,20]. The heavy quark fragmentation functions for $\bar{b}^* \rightarrow (\bar{b}c)$ were calculated in Ref. [21] for the S-wave and in Refs. [11,12] for the P-wave. The validity of the parton fragmentation has been questioned in some recent exact $\mathcal{O}(\alpha_s^4)$ calculations [22]. However, these calculations obtained different results and led to conflicting conclusions among themselves. It was only until the recent appearance of the results of Ref. [23] that this controversy can be settled down and the validity of the fragmentation approximation can be established in the production of $(\bar{b}c)$ mesons at the Tevatron. One can, therefore, use the fragmentation approximation with confidence in calculating the production cross sections and transverse

momentum spectra of the $(\bar{b}c)$ mesons, including the leading logarithmic corrections, induced gluon fragmentation contribution [20], and contributions from all orbitally excited states.

The purpose of this paper is to extend the previous calculations of heavy quark fragmentation functions to the D-wave case. The D-wave orbitally excited states are of interests phenomenologically. In the $(\bar{b}c)$ case, one set of the D-wave states is predicted by potential models to lie below the BD threshold. These states, once produced, will cascade into the pseudo-scalar ground state B_c via pion and/or photon emissions and thereby contribute to the inclusive production rate of B_c. The excited D-wave charmonium resonances have also been suggested [24] to resolve the ψ' surplus problem observed at CDF. In Section II, we briefly review the factorization model of Bodwin, Braaten, and Lepage [4] that can consistently factor out the long-distance physics and short-distance perturbative factors for the inclusive production and decay rates of heavy quarkonium. In Section III, we present the calculation of the D-wave fragmentation functions for both the unequal and equal mass cases within the spirit of the factorization model. We discuss our results in Section IV and conclude in Section V.

II. THE FACTORIZATION MODEL

According to the factorization formalism [4], the fragmentation function for a heavy quark Q to split into a quarkonium state X with longitudinal momentum fraction z can be written as

$$D_{Q \to X}(z, \mu) = \sum_n d_n(z, \mu) \langle \mathcal{O}_n^X \rangle , \qquad (1)$$

where $\mathcal{O}_n$ are local 4-fermion operators defined in NRQCD. The short-distance coefficients $d_n(z, \mu)$ are independent of the quarkonium state X. For a fragmentation scale μ of order of the heavy quark mass m_Q, the coefficients $d_n(z, \mu)$ can be calculated using perturbation theory in strong coupling constant $\alpha_s(2m_Q)$. The relative size of the matrix elements $\langle \mathcal{O}_n^X \rangle$ for a given state X can be estimated by how they scale with m_Q and with the typical relative

velocity v of the heavy quarks inside the quarkonium. Thus, the factorization formula Eq.(1) is a double expansion in α_s and v. To determine the relative importance of the various terms in this formula, one should take into account both the scaling in v of the matrix elements $\langle \mathcal{O}_n^X \rangle$ and the order of α_s in the coefficients $d_n(z, \mu)$. The leading term in v of this formula corresponds to the popular color-singlet model [5]. However, keeping only the leading term in this double expansion can sometimes lead to incomplete or even inconsistent results due to the presence of infrared divergences [4,25].

One important feature in the factorization model is that the quarkonium state X is no longer considered as solely a $Q\overline{Q}$ pair but rather a superposition of Fock states. For example, the spin-triplet ^3D_J quarkonium states, denoted by δ_J^Q, have the following Fock state expansion,

$$|\delta_J^Q\rangle = O(1)|Q\overline{Q}(^3D_J, \underline{1})\rangle + O(v)|Q\overline{Q}(^3P_{J'}, \underline{8})g\rangle + O(v^2)|Q\overline{Q}(^3S_1, \underline{8} \text{ or } \underline{1})gg\rangle + \cdots \quad (2)$$

where the notations $\underline{1}$ and $\underline{8}$ refer, respectively, to the color-singlet and color-octet states of the $Q\overline{Q}$ pair. In the above Fock-state expansion there are also other $\mathcal{O}(v^2)$ states, e.g., $|Q\overline{Q}(^3D_J, \underline{8} \text{ or } \underline{1})gg\rangle$, but their production will be further suppressed by powers of v. Heavy quark fragmentation into D-wave quarkonium δ_J^Q can be deduced from the leading Feynman diagrams of $Q^* \to (Q\overline{Q})Q$ as shown in Fig.1 and Fig.2. In Fig.1, the $Q\overline{Q}$ pair can be either in the color-singlet ^3D_J state, in the color-octet $^3P_{J'}$, or in the color-octet or color-singlet $^3S_1$ states, whereas in Fig.2 it can only be in the color-octet $^3S_1$ state. The short-distance factors deduced from these diagrams are all of order α_s^2. Furthermore, all these diagrams are also of the same order in v. According to the NRQCD [4], the relevant local 4-fermion operators describing these three Fock states are $\mathcal{O}_1(^3D_J)$, $\mathcal{O}_8(^3P_{J'})$, and $\mathcal{O}_8(^3S_1)$ which have relative scalings like v^4, v^2, and v^0, respectively. However, the $Q\overline{Q}$ pair in the color-octet P-wave and S-wave states can evolve nonperturbatively into the physical D-wave quarkonium state δ_J^Q by emitting one and two soft gluon(s), respectively. Emitting a soft gluon costs a factor of v at the amplitude level in NRQCD and hence a factor of v^2 in the probability. Thus, with a non-relativistic normalization convention in the state, all three matrix elements

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$\langle \mathcal{O}_1^{\delta Q}(^3D_J) \rangle$, $\langle \mathcal{O}_8^{\delta Q}(^3P_{J'}) \rangle$, and $\langle \mathcal{O}_8^{\delta Q}(^3S_1) \rangle$ scale like $m_Q^3 v^7$. As a result, both Fig.1 and Fig.2 contribute to the fragmentation functions of order $\alpha_s^2 v^7$ and should all be taken in account for a consistent calculation. With appropriate normalization, the color-singlet matrix elements $\langle \mathcal{O}_1^{\delta Q}(^3D_J) \rangle$ as well as $\langle \mathcal{O}_1^{\delta Q}(^1D_2) \rangle$ for the spin-singlet D-wave state δ^Q can be related to the non-relativistic radial D-wave wave-function $\overline{R_D''(0)}$, which is the spin average of the spin-singlet and spin-triplet $R_D''(0)$'s, by

$$\langle \mathcal{O}_1^{\delta Q}(^1D_2) \rangle \approx \frac{75 N_c}{4\pi} |\overline{R_D''(0)}|^2 , \tag{3}$$

$$\langle \mathcal{O}_1^{\delta J}(^3D_J) \rangle \approx \frac{15(2J+1)N_c}{4\pi} |\overline{R_D''(0)}|^2 . \tag{4}$$

Potential models can be used to determine the value of the wave-function so that the NRQCD matrix elements for the color-singlet contributions are fixed. Unfortunately, potential models cannot determine the color-octet matrix elements since dynamical gluons are involved. However, from our experience of the P-wave quarkonium case [11], we do not expect the color-octet component to play a major role in the heavy quark fragmentation. We note that this is in sharp contrast with the gluon fragmentation in which a gluon can fragment into the color-octet $^3S_1$ state via the process $g^* \to Q\overline{Q}$. This process is of order α_s and is at least one power of α_s lesser than the leading color-singlet term in the gluon fragmentation. We also note that the color-singlet contribution of heavy quark fragmentation from Fig.1 is free of infrared divergences. On the other hand, infrared divergences will show up in the gluon fragmentation function into the spin-triplet D-wave quarkonium. It is necessary to include the color-octet contributions in order to achieve a finite and sensible perturbative answers for the gluon fragmentation functions into spin-triplet D-wave states. In what follows we will restrict ourselves to the color-singlet contribution in the heavy quark fragmentation into D-wave quarkonium. If the color-octet matrix elements can be determined in the future, it is straightforward to include their contributions since their corresponding short-distance coefficients can be extracted easily from the previous S-wave and P-wave calculations by modifying the color factors.

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The factorization model for the quarkonium system can be extended to the unequal mass case like the $(\bar{b}c)$ meson system. In what follows, we will denote the spin-singlet and the spin-triplet D-wave $(\bar{b}c)$ meson states by δ^{bc} and δ^{bc}_J, respectively. In this case, figure 2 is absent for the color-octet contribution.

III. HEAVY QUARK FRAGMENTATION FUNCTIONS INTO D-WAVE HEAVY-HEAVY MESONS

The general covariant formalism for calculating the production and decay rates of S-wave and P-wave heavy quarkonium in the non-relativistic expansion was developed some times ago [26]. It is straightforward to extend it to the case of unequal mass and higher orbital excitation. We shall present a covariant formalism for the production of D-wave meson $(Q\bar{q})$ by some unspecified short-distance processes. Let m_Q and m_q be the masses of the two quarks with $m_Q > m_q$, and introduce the mass parameters $r = m_q/(m_Q + m_q)$ and $\bar{r} = m_Q/(m_Q + m_q) = 1 - r$. In the leading non-relativistic approximation, the mass M of the meson is simply $m_Q + m_q$. The amplitude for producing the bound-state $(Q\bar{q})$ in a state with momentum P, total angular momentum J, total orbital angular momentum L, and total spin S is given by

$$A(P) = \sum_{L_z, S_z} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \, \Psi_{LL_z}(\mathbf{k}) \, \langle LL_z; SS_z | JJ_z \rangle \, \mathcal{M}(P, k) \,, \tag{5}$$

where

$$\mathcal{M}(P, k) = \mathcal{O}_\Gamma \, \Gamma_{SS_z}(P, k) \,. \tag{6}$$

$\mathcal{O}_\Gamma$ represents the short-distance interaction producing the Q and $\bar{q}$ in a specific relative orbital angular momentum, and in general is a product of Dirac matrices. The heavy quarks Q and $\bar{q}$ have momentum $rP + k$ and $\bar{r}P - k$, respectively, where $2k$ is the small relative momentum of the quarks inside the bound state. $\Psi_{LL_z}(\mathbf{k})$ is the Bethe-Salpeter wavefunction in the momentum space and is assumed to be a slow-varying smooth function of $\mathbf{k}$.

Sometimes $\mathcal{M}(P, k)$ is a trace by which the on-shell spinors $v(rP + k, s)$ and $\bar{u}(\bar{r}P - k, \bar{s})$ are brought together in the right order. $\Gamma_{SS_z}(P, k)$, up to second order in k, is given by

$$\Gamma_{SS_z}(P, k) = \sqrt{\frac{m_Q + m_q}{2m_Q m_q}} \sum_{s, \bar{s}} \left\langle \frac{1}{2} s; \frac{1}{2} \bar{s} \middle| SS_z \right\rangle v(rP + k, s) \, \bar{u}(\bar{r}P - k, \bar{s}) \, ,$$

$$\approx \sqrt{m_Q + m_q} \left(\frac{r\!\!\!/P + k\!\!\!/ - m_q}{2m_q} \right) \begin{pmatrix} \gamma^5 \\ -\epsilon\!\!\!/(P, S_z) \end{pmatrix} \left(\frac{\bar{r}\!\!\!/P - k\!\!\!/ + m_Q}{2m_Q} \right) , \tag{7}$$

where in the middle parenthesis γ^5 is for $S = S_z = 0$ (spin-singlet) and $-\epsilon\!\!\!/(P, S_z)$ is for $S = 1$ (spin-triplet).

Since k/M is a small quantity, we can expand $\mathcal{M}(P, k)$ around $k = 0$ in a Taylor expansion:

$$\mathcal{M}(P, k) = \mathcal{M}(P, 0) + k_\alpha \left. \frac{\partial \mathcal{M}(P, k)}{\partial k_\alpha} \right|_{k=0} + \frac{1}{2} k_\alpha k_\beta \left. \frac{\partial^2 \mathcal{M}(P, k)}{\partial k_\alpha k_\beta} \right|_{k=0} + \cdots \tag{8}$$

where the first, second, and third terms correspond to quantum numbers $L = 0$, 1, and 2 of the orbital angular momentum, and so forth. Thus for the S-wave ($L = 0$), P-wave ($L = 1$), and D-wave ($L = 2$) states, the amplitude $A(P)$ will depend on the radial wave-functions $R_S(0)$, $R'_P(0)$, and $R''_D(0)$ through the following relations:

$$\int \frac{d^3k}{(2\pi)^3} \, \Psi_{00}(\mathbf{k}) = \frac{R_S(0)}{\sqrt{4\pi}} \, ,$$

$$\int \frac{d^3k}{(2\pi)^3} \, \Psi_{1L_z}(\mathbf{k}) k_\alpha = -i\sqrt{\frac{3}{4\pi}} R'_P(0) \epsilon_\alpha(P, L_z) \, ,$$

$$\int \frac{d^3k}{(2\pi)^3} \, \Psi_{2L_z}(\mathbf{k}) k_\alpha k_\beta = \sqrt{\frac{15}{8\pi}} R''_D(0) \epsilon_{\alpha\beta}(P, L_z) \, , \tag{9}$$

where ϵ_α is the polarization vector for the spin-1 particle, and $\epsilon_{\alpha\beta}$ is the totally symmetric, traceless, and transverse second rank polarization tensor for the spin-2 particle. In the D-wave case, the amplitude becomes

$$A(P) = \frac{1}{2} \sqrt{\frac{15}{8\pi}} \epsilon_{\alpha\beta}(P, J_z) R''_D(0) \left. \frac{\partial^2 \mathcal{M}(k)}{\partial k_\alpha \partial k_\beta} \right|_{k=0} , \tag{10}$$

for the spin-singlet case, where $J_z = L_z$, and

$$A(P) = \frac{1}{2} \sqrt{\frac{15}{8\pi}} R''_D(0) \Pi^J_{\alpha\beta\rho}(P, J_z) \left. \frac{\partial^2 \mathcal{M}^\rho(k)}{\partial k_\alpha \partial k_\beta} \right|_{k=0} , \tag{11}$$

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where

$$\Pi^J_{\alpha\beta\rho}(P, J_Z) = \sum_{L_Z, S_Z} \epsilon_{\alpha\beta}(P, L_Z)\, \epsilon_\rho(P, S_Z)\langle 2L_Z; 1S_Z | JJ_Z\rangle\,, \tag{12}$$

for the spin-triplet case. Using the appropriate Clebsch-Gordan coefficients, we have [27]

$$\Pi^{J=1}_{\alpha\beta\rho}(P, J_Z) = -\sqrt{\frac{3}{20}}\left(\frac{2}{3}\mathcal{P}_{\alpha\beta}\,\epsilon_\rho(P, J_Z) - \mathcal{P}_{\alpha\rho}\,\epsilon_\beta(P, J_Z) - \mathcal{P}_{\beta\rho}\,\epsilon_\alpha(P, J_Z)\right)\,, \tag{13}$$

$$\Pi^{J=2}_{\alpha\beta\rho}(P, J_Z) = \frac{i}{M\sqrt{6}}\left(\epsilon_{\alpha\sigma}(P, J_Z)\epsilon_{\tau\beta\rho\sigma'}\,P^\tau g^{\sigma\sigma'} + \epsilon_{\beta\sigma}(P, J_Z)\epsilon_{\tau\alpha\rho\sigma'}\,P^\tau g^{\sigma\sigma'}\right)\,, \tag{14}$$

$$\Pi^{J=3}_{\alpha\beta\rho}(P, J_Z) = \epsilon_{\alpha\beta\rho}(P, J_Z)\,, \tag{15}$$

where

$$\mathcal{P}_{\alpha\beta} = -g_{\alpha\beta} + \frac{P_\alpha P_\beta}{M^2}\,, \tag{16}$$

and $\epsilon_{\alpha\beta\rho}(P, J_Z)$ is the totally symmetric, traceless, and transverse spin-3 polarization tensor. The polarization sums for $J = 1, 2,$ and 3 are given by the following familiar expressions [27]

$$\sum_{J_Z=-1}^{1} \epsilon_\alpha(P, J_Z)\epsilon_\beta^*(P, J_Z) = \mathcal{P}_{\alpha\beta}\,, \tag{17}$$

$$\sum_{J_Z=-2}^{2} \epsilon_{\alpha\beta}(P, J_Z)\epsilon_{\rho\sigma}^*(P, J_Z) = \frac{1}{2}\left(\mathcal{P}_{\alpha\rho}\mathcal{P}_{\beta\sigma} + \mathcal{P}_{\alpha\sigma}\mathcal{P}_{\beta\rho}\right) - \frac{1}{3}\mathcal{P}_{\alpha\beta}\mathcal{P}_{\rho\sigma}\,, \tag{18}$$

$$\begin{aligned}
\sum_{J_Z=-3}^{3} \epsilon_{\alpha\beta\gamma}(P, J_Z)\epsilon_{\rho\sigma\eta}^*(P, J_Z) = &\frac{1}{6}\Big(\mathcal{P}_{\alpha\rho}\mathcal{P}_{\beta\sigma}\mathcal{P}_{\gamma\eta} + \mathcal{P}_{\alpha\rho}\mathcal{P}_{\beta\eta}\mathcal{P}_{\gamma\sigma} + \mathcal{P}_{\alpha\sigma}\mathcal{P}_{\beta\rho}\mathcal{P}_{\gamma\eta}\\
&+\mathcal{P}_{\alpha\sigma}\mathcal{P}_{\beta\eta}\mathcal{P}_{\gamma\rho} + \mathcal{P}_{\alpha\eta}\mathcal{P}_{\beta\sigma}\mathcal{P}_{\gamma\rho} + \mathcal{P}_{\alpha\eta}\mathcal{P}_{\beta\rho}\mathcal{P}_{\gamma\sigma}\Big)\\
&-\frac{1}{15}\Big(\mathcal{P}_{\alpha\beta}\mathcal{P}_{\gamma\eta}\mathcal{P}_{\rho\sigma} + \mathcal{P}_{\alpha\beta}\mathcal{P}_{\gamma\sigma}\mathcal{P}_{\rho\eta} + \mathcal{P}_{\alpha\beta}\mathcal{P}_{\gamma\rho}\mathcal{P}_{\sigma\eta}\\
&+\mathcal{P}_{\alpha\gamma}\mathcal{P}_{\beta\eta}\mathcal{P}_{\rho\sigma} + \mathcal{P}_{\alpha\gamma}\mathcal{P}_{\beta\sigma}\mathcal{P}_{\rho\eta} + \mathcal{P}_{\alpha\gamma}\mathcal{P}_{\beta\rho}\mathcal{P}_{\sigma\eta}\\
&+\mathcal{P}_{\beta\gamma}\mathcal{P}_{\alpha\eta}\mathcal{P}_{\rho\sigma} + \mathcal{P}_{\beta\gamma}\mathcal{P}_{\alpha\sigma}\mathcal{P}_{\rho\eta} + \mathcal{P}_{\beta\gamma}\mathcal{P}_{\alpha\rho}\mathcal{P}_{\sigma\eta}\Big)\,. \tag{19}
\end{aligned}$$

After writing down the formalism for the general production of D-wave mesons, we shall specify the production mechanism. The specific production of D-wave $(Q\bar{q})$ mesons that we are considering is by the fragmentation of a heavy quark Q:

$$Q^*(q) \to Q\bar{q}(P, \underline{1}) + q(p')\,, \tag{20}$$

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where the off-shell Q^* is produced by a high energy source Γ. The fragmentation function is then given by the formula [9,21]:

$$D(z) = \frac{1}{16\pi^2} \int ds\theta \left(s - \frac{M^2}{z} - \frac{r^2 M^2}{1-z} \right) \lim_{q_0 \to \infty} \frac{\sum |A(P)|^2}{\sum |A_0|^2} ,\qquad(21)$$

where $\sum |A_0|^2 = N_c \text{Tr}(\Gamma \overline{\Gamma} \not{q})$ is the tree-level amplitude squared to create an on-shell Q quark with the same 3-momentum $\mathbf{q}$, and $\sum |A(P)|^2$ is the amplitude squared for producing $(Q\bar{q}) + q$ from the same source Γ. The general procedures to extract the fragmentation function have been described in detail in Ref. [9,21] and we shall be brief in what follows.

A. Unequal Mass Case

We shall first derive the fragmentation functions for a heavy quark Q into an unequal mass D-wave $(Q\bar{q})$ meson. The fragmentation of Q^* into a $Q\bar{q}$ meson is given by the process in Eq. (20), of which the leading Feynman diagram is given in Fig. 3. In reality, the above fragmentation process applies to $b^* \to \overline{\delta^{bc}} + c$ ($b^* \to \overline{\delta^{bc}_J} + c$) and also its charge conjugate $\bar{b}^* \to \delta^{bc} + \bar{c}$ ($\bar{b}^* \to \delta^{bc}_J + \bar{c}$). The amplitudes that are presented in the following are for $Q^* \to (Q\bar{q}) + q$, while the amplitudes for the conjugate process can be obtained by complex conjugation. However, the final expressions for the fragmentation functions are valid for both $Q^* \to (Q\bar{q}) + q$ and $\overline{Q^*} \to (\overline{Q}q) + \bar{q}$ fragmentation processes.

The amplitude $A(P)$ for producing the spin-singlet D-wave state is given in Eq. (10) with $\left. \frac{\partial^2 M}{\partial k_\alpha \partial k_\beta} \right|_{k=0}$ given by

$$\left. \frac{\partial^2 M}{\partial k_\alpha \partial k_\beta} \right|_{k=0} = N_{ij} \Delta_F \bar{u}(p') V^{\alpha\beta} \gamma_5 \Gamma ,\qquad(22)$$

where

$$N_{ij} = g_s^2 C_F \frac{\delta_{ij}}{\sqrt{3}} \frac{1}{4r\bar{r}M\sqrt{M}} \quad \text{with} \quad C_F = \frac{4}{3} ,\qquad(23)$$

$$V^{\alpha\beta} = V_1^{\alpha\beta} + (1 - 2r)V_2^{\alpha\beta} ,\qquad(24)$$

and

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$$V_1^{\alpha\beta} = \frac{32\bar{r}}{r^2} M \Delta_F^3 q^\alpha q^\beta (\slashed{P} + 2M)(\slashed{q} - \bar{r}M) + \frac{16\bar{r}}{r^2} M \Delta_F^2 A_F \, q^\alpha q^\beta \, \slashed{n} \, (\slashed{P} - M)$$

$$+ \frac{2}{r^2} \Delta_F A_F \slashed{n} \, (q^\alpha \gamma^\beta + q^\beta \gamma^\alpha)\slashed{P} + \frac{4\bar{r}}{r} M \Delta_F A_F^2 (q^\alpha n^\beta + q^\beta n^\alpha) \, \slashed{n} \, (\slashed{P} - M)$$

$$+ \frac{1}{r} A_F^2 \, \slashed{n} \, (n^\alpha \gamma^\beta + n^\beta \gamma^\alpha)\slashed{P} + 4\bar{r} M A_F^3 n^\alpha n^\beta \, \slashed{n} \, (\slashed{P} - M) \,, \tag{25}$$

$$V_2^{\alpha\beta} = -\frac{4}{r^2} M \Delta_F^2 (\gamma^\alpha q^\beta + \gamma^\beta q^\alpha)(\slashed{q} - \bar{r}M) - \frac{2}{r^2} M \Delta_F A_F \slashed{n}(\gamma^\alpha q^\beta + \gamma^\beta q^\alpha)$$

$$- \frac{1}{r} M A_F^2 \slashed{n}(\gamma^\alpha n^\beta + \gamma^\beta n^\alpha) \,, \tag{26}$$

with

$$\Delta_F = (s - \bar{r}^2 M^2)^{-1} \,, \tag{27}$$

$$A_F = (n \cdot (q - \bar{r}P))^{-1} \,. \tag{28}$$

In the above formulas, $s = q^2$ and we have chosen the axial gauge associated with the 4-vector $n^\mu = (1, 0, 0, -1)$. In this gauge, figure 3 is the only leading Feynman diagram where factorization between the short distance process and fragmentation becomes manifest! We square the amplitude $A(P)$ and sum over colors and helicities of q and $(Q\bar{q})$, and put them into Eq. (21). The heavy quark fragmentation function for the $^1D_2$ state thus obtained is given by

$$D_{Q \to Q\bar{q}(^1D_2)}(z) = \frac{\alpha_s^2 (2rM) |R_D''(0)|^2}{324\pi M^7} \frac{1}{r^6 \bar{r}^2} \frac{z(1-z)^2}{(1 - \bar{r}z)^{10}} \Big\{ 60 + 60(-7 + 8r - 4r^2)z$$

$$+ 5(267 - 600r + 580r^2 - 160r^3 + 64r^4)z^2$$

$$+ 10(-255 + 825r - 1130r^2 + 596r^3 - 88r^4 - 64r^5)z^3$$

$$+ (3225 - 13050r + 22585r^2 - 16872r^3 + 5036r^4 - 576r^5 + 1152r^6)z^4$$

$$- 4\bar{r}(690 - 2535r + 4200r^2 - 2279r^3 + 968r^4 - 524r^5 - 160r^6)z^5$$

$$+ \bar{r}^2(1545 - 4890r + 8525r^2 - 2176r^3 + 3076r^4 - 320r^5 + 320r^6)z^6$$

$$- 2\bar{r}^3(255 - 660r + 1475r^2 + 174r^3 + 740r^4 + 120r^5)z^7$$

$$+ 5\bar{r}^4(15 - 30r + 107r^2 + 56r^3 + 80r^4)z^8 \Big\} \,. \tag{29}$$

For the spin-triplet case, the amplitude $A(P)$ is given in Eq. (11) with $\left.\frac{\partial^2 \mathcal{M}^\rho}{\partial k_\alpha \partial k_\beta}\right|_{k=0}$ given by

$$\frac{\partial^2 \mathcal{M}^\rho(k)}{\partial k_\alpha \partial k_\beta}\bigg|_{k=0} = N_{ij}\Delta_F \bar{u}(p')V^{\alpha\beta\rho}\Gamma \; , \tag{30}$$

where

$$V^{\alpha\beta\rho} = V_1^{\alpha\beta\rho} + (1-2r)V_2^{\alpha\beta\rho} \; , \tag{31}$$

and

$$
\begin{aligned}
V_1^{\alpha\beta\rho} &= \frac{4}{r}\Delta_F \Big[(\gamma^\beta g^{\rho\alpha} + \gamma^\alpha g^{\rho\beta}) - \frac{1}{r}\Delta_F \Big((q^\alpha(\gamma^\beta\gamma^\rho - 2\bar{r}g^{\rho\beta}) + q^\beta(\gamma^\alpha\gamma^\rho - 2\bar{r}g^{\rho\alpha}))\,\slashed{P} \\
&\quad + 2M\,(q^\alpha g^{\rho\beta} + q^\beta g^{\rho\alpha}) \Big) + \frac{8\bar{r}}{r}M^2\Delta_F^2 q^\alpha q^\beta\,\gamma^\rho \Big](\slashed{q} + \bar{r}M) \\
&\quad + \frac{1}{r}A_F\,\slashed{q}\Big[2(\gamma^\alpha g^{\rho\beta} + \gamma^\beta g^{\rho\alpha}) \\
&\quad + A_F\Big(n^\alpha((\gamma^\beta\gamma^\rho - 2rg^{\rho\beta})\slashed{P} + 2rMg^{\rho\beta}) + n^\beta((\gamma^\alpha\gamma^\rho - 2rg^{\rho\alpha})\slashed{P} + 2rMg^{\rho\alpha}) \Big) \\
&\quad + \frac{2}{r}\Delta_F\Big(q^\alpha((\gamma^\beta\gamma^\rho - 2rg^{\rho\beta})\slashed{P} + 2rMg^{\rho\beta}) + q^\beta((\gamma^\alpha\gamma^\rho - 2rg^{\rho\alpha})\slashed{P} + 2rMg^{\rho\alpha}) \Big) \Big] \\
&\quad - 4\bar{r}MA_F\Big[\frac{4}{r^2}q^\alpha q^\beta\Delta_F^2 + \frac{1}{r}\Delta_F A_F\,(q^\alpha n^\beta + q^\beta n^\alpha) + A_F^2\,n^\alpha n^\beta\Big]\,\slashed{q}(\slashed{P} - M)\gamma^\rho \; , \tag{32}
\end{aligned}
$$

$$V_2^{\alpha\beta\rho} = \frac{M}{r}A_F\,\slashed{q}\Big[\frac{2}{r}\Delta_F\left(q^\alpha\gamma^\beta + q^\beta\gamma^\alpha\right) + A_F\left(n^\alpha\gamma^\beta + n^\beta\gamma^\alpha\right)\Big]\gamma^\rho \; . \tag{33}$$

We then square the amplitude and sum over the colors and helicities of q and $(Q\bar{q})$ using the polarization sum formulas in Eqs. (17) – (19). The fragmentation functions thus obtained are

$$
\begin{aligned}
D_{Q\to Q\bar{q}(^3D_1)}(z) &= \frac{\alpha_s^2(2rM)|R_D''(0)|^2}{3240\pi M^7}\frac{1}{r^6\bar{r}^2}\frac{z(1-z)^2}{(1-\bar{r}z)^{10}} \\
&\quad \times \Big\{ 60(1+4r)^2 - 60(1+4r)(7 + 36r - 28r^2)z \\
&\quad + 5(261 + 2652r + 6128r^2 - 12576r^3 + 7360r^4)z^2 \\
&\quad + 10(-237 - 2589r - 8704r^2 + 27328r^3 - 28608r^4 + 11360r^5)z^3 \\
&\quad + (2775 + 31350r + 153935r^2 - 637872r^3 + 897816r^4 - 611056r^5 + 183552r^6)z^4 \\
&\quad - 4\bar{r}(540 + 6585r + 48660r^2 - 162629r^3 + 199928r^4 - 118084r^5 + 28400r^6)z^5 \\
&\quad + \bar{r}^2(1095 + 13830r + 137935r^2 - 376916r^3 + 385696r^4 - 192320r^5 + 36800r^6)z^6 \\
&\quad - 2\bar{r}^3(165 + 2100r + 26395r^2 - 55836r^3 + 48360r^4 - 26720r^5 + 3360r^6)z^7 \\
&\quad + 5\bar{r}^4(9 + 114r + 1717r^2 - 2424r^3 + 2476r^4 - 1872r^5 + 192r^6)z^8 \Big\} \; , \tag{34}
\end{aligned}
$$

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$$D_{Q \to Q\bar{q}(^3D_2)}(z) = \frac{\alpha_s^2(2rM)|R_D''(0)|^2}{648\pi M^7} \frac{1}{r^6\bar{r}^2} \frac{z(1-z)^2}{(1-\bar{r}z)^{10}} \Big\{ 180 + 180(-7 + 4r)z$$

$$+ 5(777 - 852r + 464r^2 - 64r^3 + 128r^4)z^2$$

$$+ 10(-693 + 1087r - 1126r^2 + 624r^3 - 112r^4 - 128r^5)z^3$$

$$+ (7875 - 15650r + 23295r^2 - 22564r^3 + 9112r^4 + 128r^5 + 2304r^6)z^4$$

$$- 4\bar{r}(1470 - 2005r + 4740r^2 - 4613r^3 + 2296r^4 - 488r^5 - 320r^6)z^5$$

$$+ \bar{r}^2(2835 - 2050r + 12035r^2 - 8332r^3 + 5952r^4 - 960r^5 + 640r^6)z^6$$

$$- 2\bar{r}^3(405 - 40r + 2565r^2 - 602r^3 + 1000r^4 - 320r^5)z^7$$

$$+ 5\bar{r}^4(21 + 10r + 205r^2 + 52r^3 + 84r^4)z^8 \Big\} , \tag{35}$$

$$D_{Q \to Q\bar{q}(^3D_3)}(z) = \frac{2\alpha_s^2(2rM)|R_D''(0)|^2}{405\pi M^7} \frac{1}{r^6} \frac{z(1-z)^2}{(1-\bar{r}z)^{10}} \Big\{ 90 + 90(-7 + 2r)z$$

$$+ 5(399 - 174r + 200r^2)z^2 + 10(-378 + 193r - 437r^2 + 70r^3)z^3$$

$$+ (4725 - 2900r + 8390r^2 - 1918r^3 + 2268r^4)z^4$$

$$+ 2(-1995 + 1700r - 4930r^2 + 1617r^3 - 2387r^4 + 350r^5)z^5$$

$$+ (2205 - 2830r + 7940r^2 - 5194r^3 + 4914r^4 - 930r^5 + 1000r^6)z^6$$

$$- 2\bar{r}(360 - 325r + 1680r^2 - 749r^3 + 1540r^4 + 295r^5 + 90r^6)z^7$$

$$+ 5\bar{r}^2(21 - 14r + 133r^2 - 56r^3 + 189r^4 - 18r^5 + 18r^6)z^8 \Big\} . \tag{36}$$

All the above fragmentation functions are valid for the fragmentation of $\bar{b} \to \delta^{bc}$ and δ_J^{bc}, as well as for $b \to \overline{\delta^{bc}}$ and $\overline{\delta_J^{bc}}$, where $J = 1, 2, 3$.

For a given principal quantum number n the $^1D_2$ and $^3D_2$ states constructed in the LS coupling scheme are mixed in general to form the physical states 2^+ and $2^{+'}$ defined as

$$|2^{+'}\rangle = \cos\theta|^1D_2\rangle - \sin\theta|^3D_2\rangle ,$$
$$|2^+\rangle = \sin\theta|^1D_2\rangle + \cos\theta|^3D_2\rangle , \tag{37}$$

where θ is the mixing angle. Thus, the fragmentation functions into the physical states 2^+ and $2^{+'}$ are

$$D_{Q \to Q\bar{q}(2^{+'})}(z) = \cos^2\theta D_{Q \to Q\bar{q}(^1D_2)}(z) + \sin^2\theta D_{Q \to Q\bar{q}(^3D_2)}(z) - \sin\theta\cos\theta D_{\text{mix}}(z) ,$$
$$D_{Q \to Q\bar{q}(2^+)}(z) = \sin^2\theta D_{Q \to Q\bar{q}(^1D_2)}(z) + \cos^2\theta D_{Q \to Q\bar{q}(^3D_2)}(z) + \sin\theta\cos\theta D_{\text{mix}}(z) . \tag{38}$$

Therefore, we also need to calculate the fragmentation function $D_{\text{mix}}(z)$. We note that in the case of charmonium it is the charge-conjugation quantum number C that prevents the mixing of $c\bar{c}(^1D_2)$ and $c\bar{c}(^3D_2)$ states, because $c\bar{c}(^1D_2)$ has $C = +1$ while $c\bar{c}(^3D_2)$ has $C = -1$. In the unequal-mass case the meson does not have this quantum number C to prevent the mixing. The mixing fragmentation function is obtained by calculating the interference term $A(^1D_2)A^*(^3D_2) + A^*(^1D_2)A(^3D_2)$ and sum over the colors and helicities of the particles in the final state as before. The mixing fragmentation function thus obtained is

$$
\begin{aligned}
D_{\text{mix}}(z) = {}& -\frac{5\alpha_s^2(2rM)|R_D''(0)|^2}{54\sqrt{6}\pi M^7} \frac{1}{r^6\bar{r}^2} \frac{z(1-z)^2}{(1-\bar{r}z)^8}\Big\{12 + 12(-5 + 4r - 2r^2)z \\
& + (129 - 198r + 156r^2 + 64r^3)z^2 + 4(-39 + 85r - 76r^2 + 8r^3 - 36r^4)z^3 \\
& + 2(57 - 156r + 133r^2 - 130r^3 + 198r^4 + 48r^5)z^4 \\
& - 4\bar{r}(12 - 27r + 4r^2 - 51r^3 - 2r^4 - 14r^5)z^5 \\
& + \bar{r}^2(9 - 16r - 11r^2 - 62r^3 - 4r^4)z^6\Big\} .
\end{aligned}
\tag{39}
$$

The fragmentation probabilities for these states can be obtained by integrating the fragmentation functions over z,

$$
\begin{aligned}
\int_0^1 D_{Q\to Q\bar{q}(^1D_2)}(z)dz = {}& \frac{10N}{21r^7\bar{r}^{10}}\Big[\bar{r}(793 + 3712r + 42160r^2 + 123220r^3 + 253315r^4 \\
& + 126622r^5 + 9778r^6 + 1180r^7 - 80r^8) \\
& + 315r(11 + 40r + 280r^2 + 580r^3 + 701r^4 + 152r^5 + 16r^6)\log r\Big],
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
\int_0^1 D_{Q\to Q\bar{q}(^3D_1)}(z)dz = {}& \frac{N}{7r^7\bar{r}^{10}}\Big[\bar{r}(209 + 457r + 28224r^2 + 479388r^3 + 1136191r^4 \\
& - 358043r^5 - 430066r^6 - 113308r^7 + 12528r^8) \\
& + 105r(5 + 60r + 1772r^2 + 8644r^3 + 4759r^4 - 5632r^5 - 2108r^6 \\
& - 368r^7 + 64r^8)\log r\Big],
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
\int_0^1 D_{Q\to Q\bar{q}(^3D_2)}(z)dz = {}& \frac{5N}{21r^7\bar{r}^{10}}\Big[\bar{r}(1127 - 700r + 67880r^2 + 320552r^3 + 393317r^4 \\
& + 191990r^5 + 118742r^6 + 4712r^7 - 160r^8) \\
& + 315r(9 + 44r + 604r^2 + 1260r^3 + 879r^4 + 532r^5 + 156r^6)\log r\Big],
\end{aligned}
$$

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$$\int_0^1 D_{Q \to Q\bar{q}(^3D_3)}(z)dz = \frac{8N}{21r^7\bar{r}^{10}} \left[\bar{r}(2191 + 4828r + 100786r^2 + 160027r^3 + 288799r^4 \right.$$

$$+ 428008r^5 - 63854r^6 + 87913r^7 + 14422r^8)$$

$$+ 630r(15 + 20r + 343r^2 + 126r^3 + 861r^4 + 112r^5 + 63r^6$$

$$\left. + 78r^7 + 6r^8)\log r \right], \tag{43}$$

$$\int_0^1 D_{\mathrm{mix}}(z)dz = -\frac{10\sqrt{6}N}{7r^7\bar{r}^{10}} \left[\bar{r}(153 + 358r + 3214r^2 - 13796r^3 - 15021r^4 \right.$$

$$+ 7176r^5 + 9696r^6 + 240r^7)$$

$$\left. + 105r(5 + 10r - 20r^2 - 180r^3 - 37r^4 + 110r^5 + 36r^6)\log r \right], \tag{44}$$

where $N = \alpha_s^2(2rM)|R_D''(0)|^2/(3240\pi M^7)$. We note that the running scale used in the strong coupling constant α_s is set at $2rM$, which is the minimal virtuality of the exchanged gluon [9].

B. Equal Mass Case

Heavy quark fragmentation functions for the D-wave quarkonium can be obtained simply by setting $r = 1/2$ and $\theta = 0$ in the previous formulas for the unequal mass case. For convenience we also present their explicit formulas in the following.

$$D_{Q \to \delta Q}(z) = \frac{8\alpha_s^2(2m_Q)|R_D''(0)|^2}{81\pi m_Q^7} \frac{z(1-z)^2}{(2-z)^{10}} \left(3840 - 15360z + 30720z^2 - 37120z^3 \right.$$

$$\left. + 35328z^4 - 29344z^5 + 18344z^6 - 5848z^7 + 775z^8 \right), \tag{45}$$

$$D_{Q \to \delta_1 Q}(z) = \frac{8\alpha_s^2(2m_Q)|R_D''(0)|^2}{405\pi m_Q^7} \frac{z(1-z)^2}{(2-z)^{10}} \left(17280 - 103680z + 321120z^2 - 551840z^3 \right.$$

$$\left. + 546744z^4 - 314752z^5 + 112402z^6 - 24594z^7 + 2915z^8 \right), \tag{46}$$

$$D_{Q \to \delta_2 Q}(z) = \frac{16\alpha_s^2(2m_Q)|R_D''(0)|^2}{81\pi m_Q^7} \frac{z(1-z)^2}{(2-z)^{10}} \left(2880 - 14400z + 37360z^2 - 58240z^3 \right.$$

$$\left. + 58604z^4 - 38372z^5 + 16517z^6 - 4014z^7 + 445z^8 \right), \tag{47}$$

$$D_{Q \to \delta_3 Q}(z) = \frac{8\alpha_s^2(2m_Q)|R_D''(0)|^2}{405\pi m_Q^7} \frac{z(1-z)^2}{(2-z)^{10}} \left(11520 - 69120z + 231680z^2 - 488960z^3 \right.$$

$$\left. + 675136z^4 - 592288z^5 + 309688z^6 - 80736z^7 + 8285z^8 \right). \tag{48}$$

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We can also obtain the fragmentation probabilities by integrating over z:

$$\int_0^1 dz D_{Q\to\delta_0^Q}(z) = \frac{8\alpha_s^2(2m_Q)|R_D''(0)|^2}{81\pi m_Q^7}\left(\frac{776677}{21} - 53355\log 2\right) , \tag{49}$$

$$\int_0^1 dz D_{Q\to\delta_1^Q}(z) = \frac{2\alpha_s^2(2m_Q)|R_D''(0)|^2}{405\pi m_Q^7}\left(547127 - 789300\log 2\right) , \tag{50}$$

$$\int_0^1 dz D_{Q\to\delta_2^Q}(z) = \frac{16\alpha_s^2(2m_Q)|R_D''(0)|^2}{81\pi m_Q^7}\left(\frac{2889265}{168} - 24810\log 2\right) , \tag{51}$$

$$\int_0^1 dz D_{Q\to\delta_3^Q}(z) = \frac{8\alpha_s^2(2m_Q)|R_D''(0)|^2}{405\pi m_Q^7}\left(\frac{5182432}{21} - 356025\log 2\right) . \tag{52}$$

IV. DISCUSSIONS

We have seen that the covariant formalism used to calculate the D-wave fragmentation functions is quite cumbersome and tedious. The complicated results are therefore needed to be cross-checked by other means. To be sure we have recalculated the D-wave fragmentation functions for the equal mass case from scratch and we did end up with the same results as what we have in the above for $r = 1/2$. Two additional nontrivial checks are performed by using the Braaten-Levin spin-counting rule [28] and the heavy-quark spin-counting rule which we will discuss in turns.

A. Braaten-Levin Spin-counting Rule

The fragmentation function $D_{i\to H}(z)$ for a parton i splitting into a hadron H is related to the distribution function $f_{i/H}(x)$ of finding the parton i inside the hadron H by the analytic continuation [29,30]

$$f_{i/H}(x) = xD_{i\to H}\left(\frac{1}{x}\right) . \tag{53}$$

The results of perturbative fragmentation functions allow us to study the perturbative tail of the distribution functions of the heavy quark inside the heavy mesons as well. From our explicit calculations, we see that $f_{i/H}(x)$ has a pole located at $x = \bar{r}$. The pole is cut off

by non-perturbative effects related to the formation of bound states of the $(Q\bar{q})$ pair. This pole is of order 6, 8 and 10 for S-, P-, and D-wave states, respectively. In the general case of L-waves we expect this pole is of order $6 + 2L$. Therefore, we can expand $f(x)$ as a Laurent series,

$$f(x) = \frac{a_n(r)}{(x - \bar{r})^n} + \frac{a_{n-1}(r)}{(x - \bar{r})^{n-1}} + \text{less singular terms} , \tag{54}$$

with $n = 6 + 2L$ for the general L-waves.

The Braaten-Levin rule states that the leading r-dependent coefficients $a_n(r)$ satisfy the simple spin-counting. Specifically, we have

$$a_6(^1S_0) : a_6(^3S_1) = 1 : 3 \quad \text{for S-waves,} \tag{55}$$

$$a_8(^1P_1) : a_8(^3P_0) : a_8(^3P_1) : a_8(^3P_2) = 3 : 1 : 3 : 5 \quad \text{for P-waves,} \tag{56}$$

$$a_{10}(^1D_2) : a_{10}(^3D_1) : a_{10}(^3D_2) : a_{10}(^3D_3) = 5 : 3 : 5 : 7 \quad \text{for D-waves,} \tag{57}$$

and so on.

The applicability of these counting rules has been demonstrated for the S- and P-wave cases in Ref. [11]. For the D-wave case we can expand the distribution functions, which are obtained from the D-wave fragmentation functions by Eq. (53), around $x = \bar{r}$ and obtain the first two terms in the Laurent series:

$$f_{Q/\delta^{Q\bar{q}}}(x) = \frac{30720N(r\bar{r})^2}{(x - \bar{r})^{10}} + \frac{320N(576r - 275)r\bar{r}}{(x - \bar{r})^9} + \cdots , \tag{58}$$

$$f_{Q/\delta_1^{Q\bar{q}}}(x) = \frac{18432N(r\bar{r})^2}{(x - \bar{r})^{10}} + \frac{192N(576r - 275)r\bar{r}}{(x - \bar{r})^9} + \cdots , \tag{59}$$

$$f_{Q/\delta_2^{Q\bar{q}}}(x) = \frac{30720N(r\bar{r})^2}{(x - \bar{r})^{10}} + \frac{320N(576r - 275)r\bar{r}}{(x - \bar{r})^9} + \cdots , \tag{60}$$

$$f_{Q/\delta_3^{Q\bar{q}}}(x) = \frac{43008N(r\bar{r})^2}{(x - \bar{r})^{10}} + \frac{448N(576r - 275)r\bar{r}}{(x - \bar{r})^9} + \cdots , \tag{61}$$

where $N = \alpha_s^2|R_D''(0)|^2/(3240\pi M^7)$. Therefore, the leading coefficients a_{10}'s indeed satisfy the spin-counting ratio $5 : 3 : 5 : 7$. Actually, we found that not only the first term but also

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the second term in the Laurent series of $f(x)$ obey the Braaten-Levin spin-counting rules, while the third term does not. We conjecture that the Braaten-Levin spin-counting rule can be applied to the first two coefficients in the expansion (54) for all L-wave (L = 0, 1, 2, ...) cases.

B. Heavy-Light Limit

Hadrons containing a single heavy quark exhibit heavy quark symmetry in the limit $m_Q/\Lambda_{QCD} \to \infty$. In the limit of $m_Q/\Lambda_{QCD} \to \infty$, both the heavy quark spin $\vec{S}_Q$ and the total spin $\vec{J}$ of a heavy hadron containing a single heavy quark Q become good quantum numbers. This implies that in the spectroscopy of the hadron containing a single heavy quark Q, the angular momentum of the light degrees of freedom $\vec{J}_l = \vec{J} - \vec{S}_Q$ is also a good quantum number. We refer collectively to all the degrees of freedom in the heavy-light hadron other than the heavy quark as the light degrees of freedom. For heavy-light ($Q\bar{q}$) mesons, $\vec{J}_l = \vec{S}_q + \vec{L}$ where $\vec{S}_q$ is the spin of the light quark q and $\vec{L}$ is the orbital angular momentum. Thus, hadronic states can be labeled simultaneously by the eigenvalues j and j_l of the total spin $\vec{J}$ and the angular momentum of the light degrees of freedom $\vec{J}_l$, respectively. In general [31], the spectrum of hadrons containing a single heavy quark has, for each j_l, a degenerate doublet with total spins $j_+ = j_l + 1/2$ and $j_- = j_l - 1/2$. (For the case of $j_l = 0$, the total spin must be 1/2.) For D-wave heavy-light mesons, j_l can either be 3/2 or 5/2. Thus $(j_-, j_+) = (1, 2)$ and $(2, 3)$ for $j_l = 3/2$ and $5/2$, respectively. As a result, we expect to have two distinct doublets $(^3D_1, 2^{+'})$ and $(2^+, {}^3D_3)$ in the limit of $m_Q/m_q \to \infty$, i.e. $r \to 0$. In this limit, the mixing coefficients in Eq. (37) can be determined by the Clebsch-Gordan coefficients in the tensor product of a spin 1/2 state and a spin 2 state with the following result:

$$|2^{+'}\rangle = \sqrt{\frac{2}{5}}|^1D_2\rangle + \sqrt{\frac{3}{5}}|^3D_2\rangle , \tag{62}$$

$$|2^+\rangle = -\sqrt{\frac{3}{5}}|^1D_2\rangle + \sqrt{\frac{2}{5}}|^3D_2\rangle , \tag{63}$$

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i.e., we are transforming the states $^1D_2$ and $^3D_2$ in the LS coupling scheme to the states 2^+ and $2^{+'}$ in the jj coupling scheme.

In their discussions of the heavy quark fragmentation functions within the context of Heavy Quark Effective Theory, Jaffe and Randall [32] showed that fragmentation functions can have a $1/m_Q$ expansion if expanded in terms of a more natural variable

$$y = \frac{\frac{1}{z} - \bar{r}}{r}, \tag{64}$$

rather than the usual fragmentation variable z, and the heavy-quark mass expansion is given as a power series in r,

$$D(y) = \frac{1}{r}a(y) + b(y) + \mathcal{O}(r), \tag{65}$$

where $a(y), b(y)$, *etc.* are functions of the variable y. The leading term $a(y)$ is constrained by the heavy quark spin-flavor symmetry while all the higher order terms contain spin-flavor symmetry breaking effects. One can recast our results for the D-wave fragmentation functions derived in Sec.III in the above form, by carefully expanding the powers of r and $(1 - \bar{r}z)$. The leading $1/r$ terms of the fragmentation functions are then given by

$$D_{Q\to\delta^{Qq}}(y) \to \frac{10N'(y-1)^2}{ry^{10}}\left(3072 - 2656y + 2056y^2 - 912y^3 + 660y^4 - 120y^5 + 75y^6\right),$$

$$D_{Q\to\delta_1^{Qq}}(y) \to \frac{3N'(y-1)^2}{ry^{10}}\left(6144 - 5312y + 1952y^2 - 1824y^3 + 1020y^4 - 120y^5 + 15y^6\right),$$

$$D_{Q\to\delta_2^{Qq}}(y) \to \frac{5N'(y-1)^2}{ry^{10}}\left(6144 - 5312y + 3392y^2 - 1824y^3 + 1220y^4 - 200y^5 + 105y^6\right),$$

$$D_{Q\to\delta_3^{Qq}}(y) \to \frac{112N'(y-1)^2}{ry^{10}}\left(384 - 332y + 347y^2 - 114y^3 + 95y^4 - 20y^5 + 15y^6\right),$$

$$D_{\text{mix}}(y) \to -\frac{50\sqrt{6}N'(y-1)^2}{ry^{10}}\left(144y^2 + 20y^4 - 8y^5 + 9y^6\right),$$

$$D_{Q\to2^{+'}}(y) \to \frac{5N'(y-1)^2}{ry^{10}}\left(6144 - 5312y + 1952y^2 - 1824y^3 + 1020y^4 - 120y^5 + 15y^6\right),$$

$$D_{Q\to2^+}(y) \to \frac{80N'(y-1)^2}{ry^{10}}\left(384 - 332y + 347y^2 - 114y^3 + 95y^4 - 20y^5 + 15y^6\right),$$

where $N' = \alpha_s^2|R_D''(0)|^2/(3240\pi(rM)^7)$. Thus, at leading-order of $1/m_Q$, we obtain the following spin-counting ratios

$$\frac{D_{Q \to \delta_1^{Qq}}(y)}{D_{Q \to 2^{+\prime}}(y)} \to \frac{3}{5} \, , \tag{66}$$

$$\frac{D_{Q \to 2^+}(y)}{D_{Q \to \delta_3^{Qq}}(y)} \to \frac{5}{7} \, , \tag{67}$$

as expected from heavy quark spin symmetry.

V. CONCLUSIONS

The detections of the D-wave orbitally excited states of heavy-heavy mesons are much more difficult than their ground states. So far, none of D-wave quarkonium has been identified. The basic reasons are (i) the production rates are small, (ii) small branching ratios in subsequent decays, and (iii) the very small efficiencies in subsequent levels of identification. From the potential model calculations, D-wave charmonium are likely to be above the $D\overline{D}$ threshold and hence they will decay predominantly into a pair of D mesons. In principle, D-wave quarkonium can be identified partially by their pure leptonic decay (e.g. $^3D_1 \to \mu^+\mu^-$), or by their subsequent decays into the lower-lying states. It is because of the subsequent levels of decays that reduce substantially the branching ratios and efficiencies in identifying these D-wave bound states. However, both the branching ratios and efficiencies are uncertain at this stage. In the following, we shall estimate the production rates of the D-wave charmonium, bottomonium, and $(\bar{b}c)$ mesons based upon the fragmentation functions obtained in Sec. III and IV.

The values of $|R_D^{\prime\prime}(0)|^2$ for the $(c\bar{c})$, $(b\bar{b})$, and $(\bar{b}c)$ systems were summarized nicely in Ref. [33] using different potential models. We choose the QCD-motivated potential of Buchmüller and Tye [34], and the corresponding values of $|R_D^{\prime\prime}(0)|^2$ for the $(c\bar{c})$, $(b\bar{b})$, and $(\bar{b}c)$ systems are 0.015, 0.637, and 0.055 GeV^7, respectively. These values are for the first set of the D-wave states. We note that the values of $|R_D^{\prime\prime}(0)|^2$ obtained using the Buchmüller-Tye potential are the smallest ones of the four potential models studied in Ref. [33].

In the following analysis, we will adopt the values $m_c = 1.5$ GeV and $m_b = 4.9$ GeV that were used in the fit of the Buchmüller-Tye potential to the bound state spectra [33].

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Using $\alpha_s(2m_c) \approx 0.253$, $|R_D''(0)|^2 = 0.055 \text{GeV}^7$, $r = m_c/(m_b + m_c) = 0.234$, and ignoring the possibly small mixing effects for the $(\bar{b}c)$ system, we show the D-wave fragmentation functions for $(\bar{b}c)$ mesons in Fig. 4. The fragmentation probabilities for a $\bar{b}$ antiquark into D-wave $(\bar{b}c)$ mesons can be obtained from Eqs. (40) – (43),

$$\int_0^1 D_{\bar{b} \to \delta^{bc}}(z)dz = 6.7 \times 10^{-6} \, , \tag{68}$$

$$\int_0^1 D_{\bar{b} \to \delta_1^{bc}}(z)dz = 1.7 \times 10^{-6} \, , \tag{69}$$

$$\int_0^1 D_{\bar{b} \to \delta_2^{bc}}(z)dz = 6.5 \times 10^{-6} \, , \tag{70}$$

$$\int_0^1 D_{\bar{b} \to \delta_3^{bc}}(z)dz = 8.5 \times 10^{-6} \, . \tag{71}$$

Therefore, the total probability is about 2.3×10^{-5}. We can compare this value with the corresponding probabilities of the S-wave and P-wave states. Using the same α_s and the values of $|R_S(0)|^2$ and $|R_P'(0)|^2$ calculated with the same Buchmüller-Tye potential, the total fragmentation probabilities of the 1S and 1P $(\bar{b}c)$ states are about 1×10^{-3} and 1.7×10^{-4}, respectively [11]. These total fragmentation probabilities show a similar suppression factor of order v^2 in going from 1S to 1P states and from 1P to 1D states, which are in accord with the velocity counting rules in NRQCD [4].

Next, for the D-wave charmonium we use $\alpha_s(2m_c) \approx 0.253$ and $|R_D''(0)|^2 = 0.015 \text{GeV}^7$. The D-wave fragmentation functions for charmonium are shown in Fig. 5, and the fragmentation probabilities are given by

$$\int_0^1 D_{c \to \delta^c}(z)dz = 3.1 \times 10^{-6} \, , \tag{72}$$

$$\int_0^1 D_{c \to \delta_1^c}(z)dz = 2.3 \times 10^{-6} \, , \tag{73}$$

$$\int_0^1 D_{c \to \delta_2^c}(z)dz = 3.6 \times 10^{-6} \, , \tag{74}$$

$$\int_0^1 D_{c \to \delta_3^c}(z)dz = 1.7 \times 10^{-6} \, . \tag{75}$$

The total probability is about 1.1×10^{-5}, to be compared with the probabilities of $P(c \to J/\psi) \approx 1.8 \times 10^{-4}$ [9] and $P(c \to h_c, \chi_c) \approx 8 \times 10^{-5}$ [11] obtained using the same inputs. With our inputs, the gluon fragmentation probability into the $^1D_2$ charmonium [8] is about

1.2×10^{-6}.

For the bottomonium, we use $\alpha_s(2m_b) \approx 0.174$ and $|R_D''(0)|^2 = 0.637\mathrm{GeV}^7$. The shape of the fragmentation functions for D-wave bottomonium is identical to those of the charmonium in Fig. 5, but the overall normalization is a factor of 200 smaller. The fragmentation probabilities are given by

$$\int_0^1 D_{b \to \delta^b}(z)dz = 1.6 \times 10^{-8} , \tag{76}$$

$$\int_0^1 D_{b \to \delta_1^b}(z)dz = 1.2 \times 10^{-8} , \tag{77}$$

$$\int_0^1 D_{b \to \delta_2^b}(z)dz = 1.8 \times 10^{-8} , \tag{78}$$

$$\int_0^1 D_{b \to \delta_3^b}(z)dz = 8.5 \times 10^{-9} . \tag{79}$$

The total probability for the D-wave bottomonium is very small $\sim 5.4 \times 10^{-8}$. This is to be compared with the probabilities of $P(b \to \Upsilon) \approx 2 \times 10^{-5}$ [9] and $P(b \to h_b, \chi_b) \approx 2.5 \times 10^{-6}$. We observe that the relative probabilities for the S-wave to P-wave quarkonium and for the P-wave to D-wave quarkonium are roughly suppressed by a similar factor of v^2 as in the $\bar{b}c$ case.

At the Tevatron the b-quark production cross section is of order $10\mu b$ with $p_T \gtrsim 6$ GeV, which implies about 10^9 b-quarks with an accumulated luminosity of 100 pb^{-1}. Using the fragmentation probabilities calculated above, we expect about 10^4 D-wave $(\bar{b}c)$ mesons, while there are only about 50 D-wave bottomonium. At large p_T, charm quark production is very similar to bottom quark production. Therefore, the c-quark production cross section is also of order 10 μb and there should be about 10^4 D-wave charmonium for the same luminosity. Although the production rates of D-wave $(\bar{b}c)$ mesons and charmonium are rather substantial, the small branching ratios and efficiencies of identification in subsequent levels of decays render their detections extremely difficult. For example, the combined branching ratio $\mathrm{Br}(^1D_2(c\bar{c}) \to {}^1P_1\gamma, {}^1P_1 \to J/\psi\pi^0, J/\psi \to \mu^+\mu^-)$ has been estimated to be about 10^{-4} [8]. Even after the installation of the Main Injector when the yearly luminosity can be boosted to $1' - 3\ fb^{-1}$, the chance of detecting the D-wave $(\bar{b}c)$ mesons or charmonium

is still quite slim. The future Large Hadron Collider (LHC) can produce $10^{12} - 10^{13}$ $b\bar{b}$ pairs and $c\bar{c}$ pairs per year running. It can then produce about $10^7 - 10^8$ D-wave $(\bar{b}c)$ mesons and charmonium. After taking into account the branching ratios and efficiencies, there should be a handful number of these D-wave states that can be identified. Nevertheless, detection of the D-wave bottomonium does not seem to be feasible even at the LHC! The production of $b\bar{b}$ and $c\bar{c}$ pairs at LEP, LEPII, and other higher energy e^+e^- or $\mu^+\mu^-$ colliders are much smaller than at the hadronic colliders. Therefore, it is much more difficult to identify the D-wave quarkonium or $(\bar{b}c)$ mesons at the e^+e^- or $\mu^+\mu^-$ facilities.

In conclusions, we have computed the heavy quark fragmentation functions into D-wave heavy-heavy mesons containing two heavy quarks to leading order in strong coupling constant and to leading order in the non-relativistic expansion within the framework of factorization model of Bodwin, Braaten, and Lepage. The color-singlet contributions can be expressed in terms of the NRQCD matrix elements which are related to the second derivative of the non-relativistic radial D-wave wave-function. The color-octet contributions are expected to be quite small in the heavy quark fragmentation and therefore we have ignored them in the present analysis. The fragmentation functions obtained in this work should be useful in the production of the D-wave charmonium, bottomonium, and $(\bar{b}c)$ mesons at the future Large Hadronic Collider.

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DISCLAIMER

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FIGURES CAPTION

1. Feynman diagram for $Q^*(q) \rightarrow Q\overline{Q}(P, \underline{1} \text{ or } \underline{8}) + Q(p')$.

2. Another Feynman diagram for $Q^*(q) \rightarrow Q\overline{Q}(P, \underline{8}) + Q(p')$.

3. Feynman diagram for $Q^*(q) \rightarrow Q\bar{q}(P, \underline{1}) + q(p')$.

4. The fragmentation functions for $\bar{b}$ to fragment into D-wave $(\bar{b}c)$ mesons at the heavy quark mass scale.

5. The fragmentation functions for c to fragment into D-wave charmonium at the heavy quark mass scale.
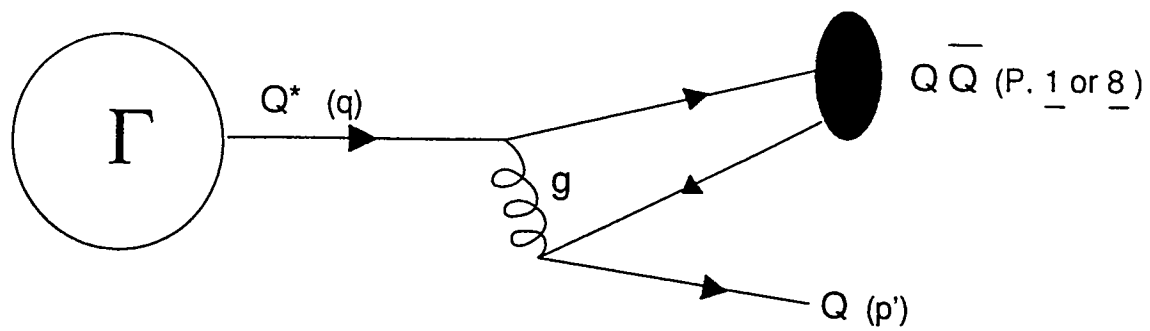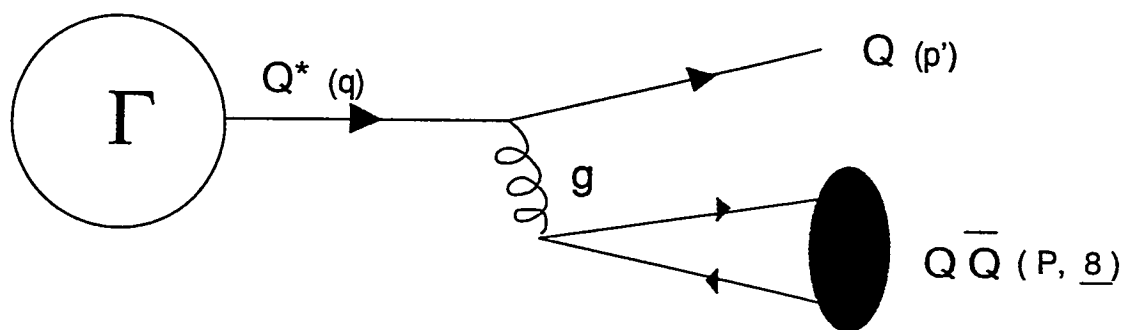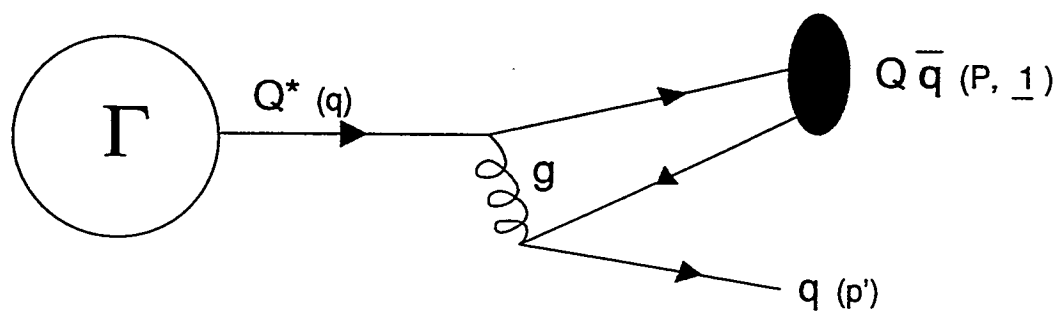
Fig. 1

Fig. 2

Fig. 3