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Gauss' Law Preserving Methods for the Multi-Fluid Plasma Model

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Multi Fluid Plasma Model

Euler's Equations + Maxwell's Equations

Continuity Equation

$$\int_K \partial_t \rho_\alpha \phi_\ddagger \, \mathrm{d}v = \int_K \rho_\alpha \mathbf{u}_\alpha \cdot \nabla \phi_\ddagger \, \mathrm{d}v - \oint_{\partial K} \hat{F}_\rho \cdot \mathbf{n} \phi_\ddagger \, \mathrm{d}a \quad \forall \phi_\ddagger \in V^h_{\ddagger\nabla}(\Omega)$$

Ampere's Equation

$$\int_\Omega (\varepsilon \partial_t \mathbf{E} + \mathbf{j}) \cdot \mathbf{\Psi} - \mu^{-1} \mathbf{B} \cdot \nabla \times \mathbf{\Psi} \, \mathrm{d}v = \mu^{-1} \oint_{\partial\Omega} (\mathbf{B} \times \mathbf{\Psi}) \cdot \mathbf{n} \, \mathrm{d}s \quad \forall \mathbf{\Psi} \in \mathbf{V}^h_{\nabla\times}(\Omega)$$

Gauss' Law

$$-\varepsilon \int_\Omega \mathbf{E} \cdot \nabla \phi \, \mathrm{d}v = \int_\Omega q \phi \, \mathrm{d}v \qquad \forall \phi \in V^h_\nabla(\Omega)$$

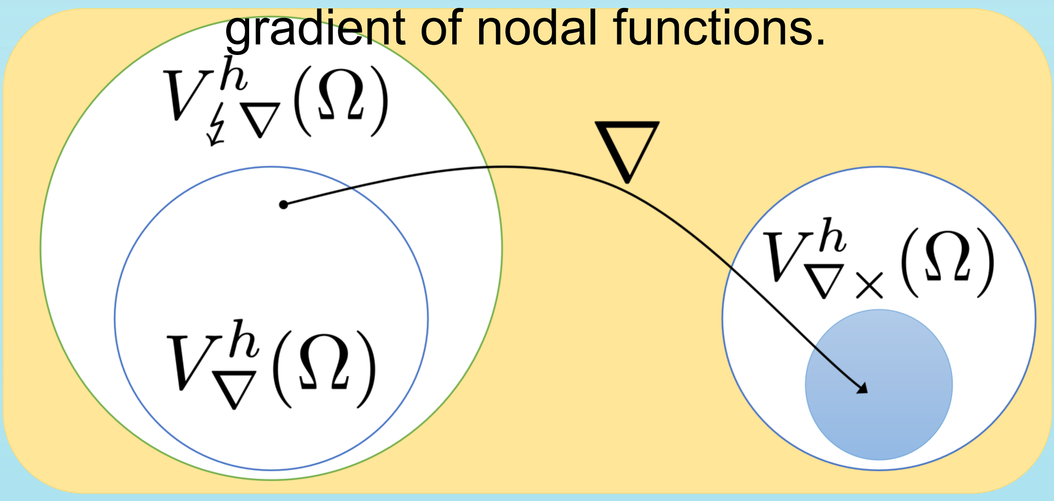
Current $\mathbf{j} = \sum_\alpha \left(\frac{Q_\alpha}{M_\alpha} \rho_\alpha \mathbf{u}_\alpha\right)$ Charge Density $q = \sum_\alpha \left(\frac{Q_\alpha}{M_\alpha} \rho_\alpha\right)$

Design Principles

- Conservative Fluid Discretization – DG ✓
- Shock Capturing – Slope Limiting ✓
- $\nabla \cdot B = 0$ – Compatible Discretization ✓
- Satisfies Gauss' Law ???

Mathematical Observation

Weak solutions tested against DG functions are also weak solutions when tested against CG functions. Weak solutions tested against Nédélec edge functions are also weak solutions when tested against the gradient of nodal functions.



$$\int_\Omega \partial_t q \phi \, \mathrm{d}v - \int_\Omega \mathbf{j} \cdot \nabla \phi \, \mathrm{d}v + \oint_{\partial\Omega} (\hat{\mathbf{j}} \cdot n) \phi \, \mathrm{d}s$$

$$\int_\Omega (\varepsilon \partial_t \mathbf{E} + \mathbf{j}) \cdot \nabla \phi \, \mathrm{d}v = \mu^{-1} \oint_{\partial\Omega} (B \times \nabla \phi) \cdot n \, \mathrm{d}s \qquad \forall \phi \in V^h_\nabla(\Omega)$$

Algorithm

Step Fluid with Forward Euler

$$\int_K \rho_\alpha^{n+1} \phi_\ddagger \, \mathrm{d}v = \int_K \rho_\alpha^n \phi_\ddagger \, \mathrm{d}v + \Delta t \left(\int_K \rho_\alpha^n \mathbf{u}_\alpha^n \cdot \nabla \phi_\ddagger \, \mathrm{d}v - \oint_{\partial K} \hat{F}_\rho^n \cdot \mathbf{n} \phi_\ddagger \, \mathrm{d}a \right)$$

Compute Current

$$\int_\Omega \mathbf{j}^n \cdot \mathbf{\Psi} \, \mathrm{d}v = \int_\Omega \sum_\alpha \left(\frac{Q_\alpha}{M_\alpha} \rho_\alpha^n \mathbf{u}_\alpha^n \right) \cdot \mathbf{\Psi} \, \mathrm{d}v$$

Limit Fluid Solution

$$\rho_\alpha^{n+1} \to \tilde{\rho}_\alpha^{n+1} \Rightarrow q^{n+1} \to \tilde{q}^{n+1}$$

Step Maxwell's Equations with Backward Euler

$$\int_\Omega \left(\frac{\varepsilon}{\Delta t} (\mathbf{E}^{n+1} - \mathbf{E}^n) + \mathbf{j}^n \right) \cdot \mathbf{\Psi} \, \mathrm{d}v = \mu^{-1} \int_\Omega \mathbf{B}^{n+1} \cdot \nabla \times \mathbf{\Psi} \, \mathrm{d}v$$

Check Gauss' Law

$$\Delta t \int_\Omega \mathbf{j}^n \cdot \nabla \phi \, \mathrm{d}v = \int_\Omega (q^{n+1} - q^n) \phi \, \mathrm{d}v$$

$$\Downarrow$$

$$-\varepsilon \int_\Omega \mathbf{E}^{n+1} \cdot \nabla \phi \, \mathrm{d}v = \int_\Omega q^{n+1} \phi \, \mathrm{d}v \neq \int_\Omega \tilde{q}^{n+1} \phi \, \mathrm{d}v$$

Error!

Gauss Correction

Find Correction, $w \in L^2(\Omega)$

$$-\int_\Omega \mathbf{w} \cdot \nabla \phi \, \mathrm{d}v = \int_\Omega (\tilde{q}^{n+1} - q^{n+1}) \phi \, \mathrm{d}v \quad \forall \phi \in V^h_\nabla(\Omega)$$

Specifically, Find Local DG Correction, $p \in V^h_{\ddagger\nabla}(\Omega)$

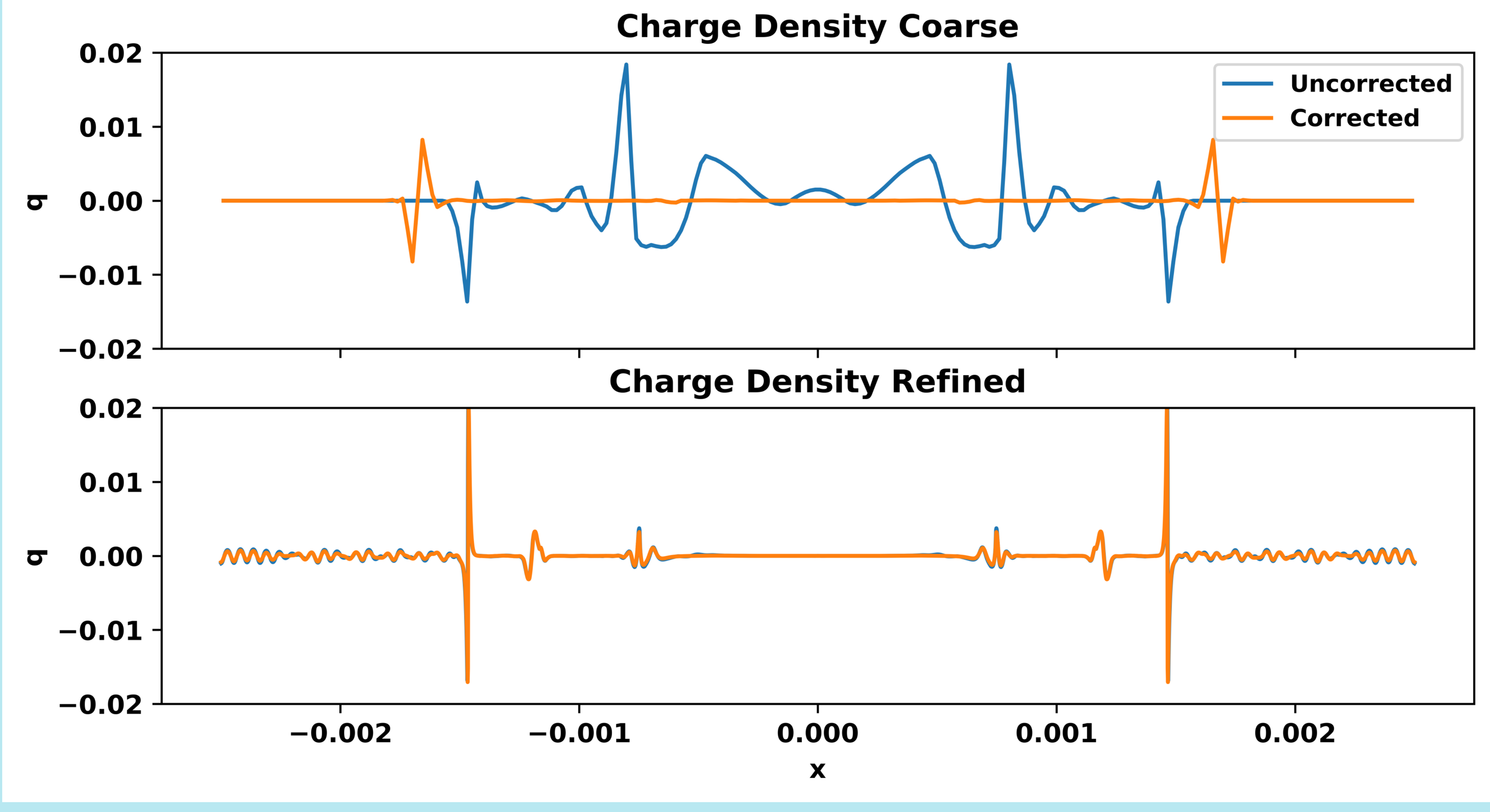
$$-\int_K \nabla p \cdot \nabla \phi_\ddagger \, \mathrm{d}v = \int_K (\tilde{q}^{n+1} - q^{n+1}) \phi_\ddagger \, \mathrm{d}v \quad \forall \phi_\ddagger \in V^h_{\ddagger\nabla}(\Omega)$$

Correction!

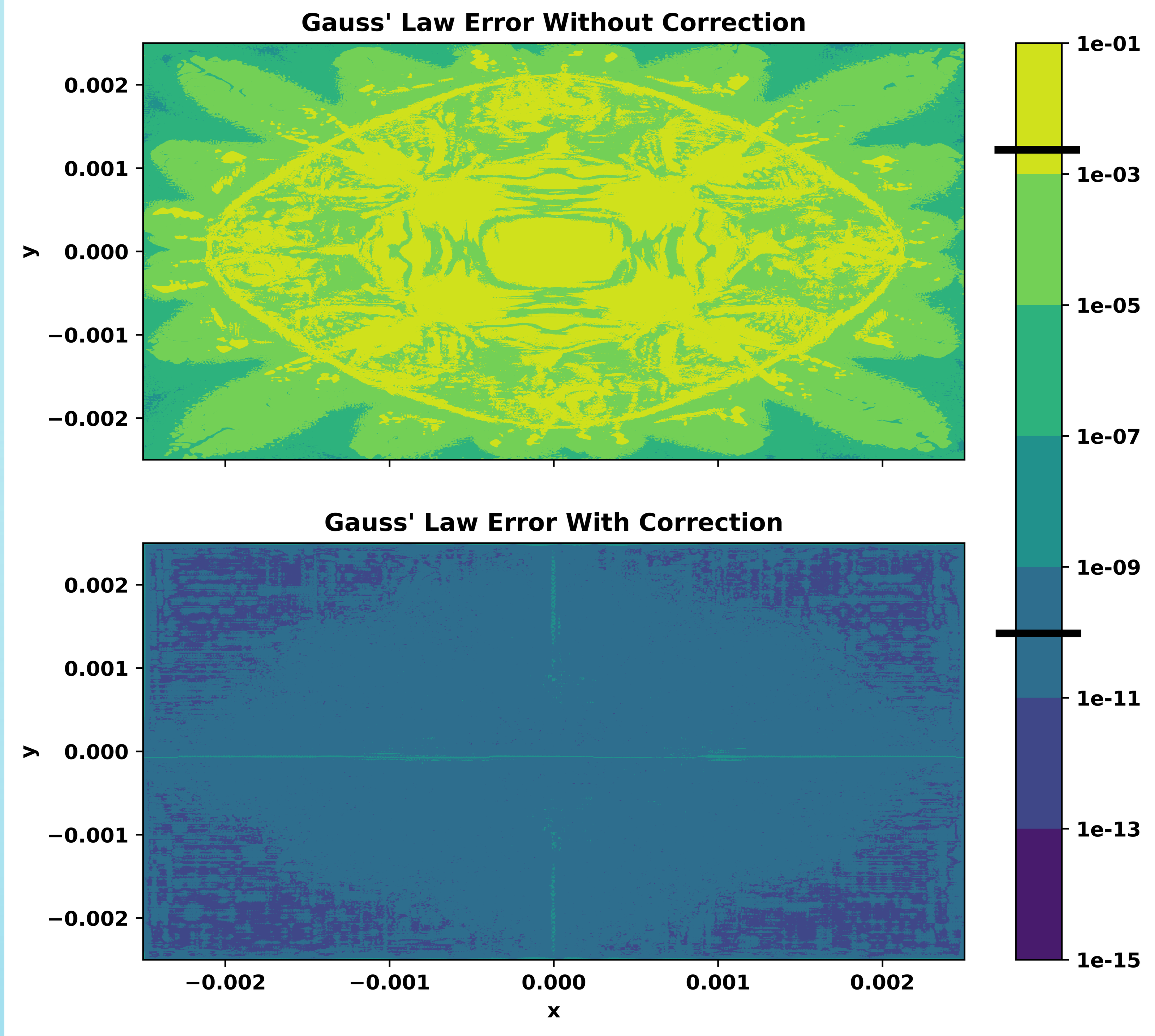
Corrected Electric Field Satisfies

$$\int_\Omega \tilde{\mathbf{E}}^{n+1} \cdot \mathbf{\Psi} \, \mathrm{d}v = \int_\Omega \left(\mathbf{E}^{n+1} + \frac{1}{\varepsilon} \nabla p \right) \cdot \mathbf{\Psi} \, \mathrm{d}v \quad \forall \mathbf{\Psi} \in V^h_{\nabla\times}(\Omega)$$

1D Riemann



2D Riemann



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