SAND2023-12312C

On the Harmonic Balance Method Augmented with Non-Smooth Basis Functions for Contact/Impact Problems

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IMAC-XLI (41), submission #14374

February 13-16, 2023

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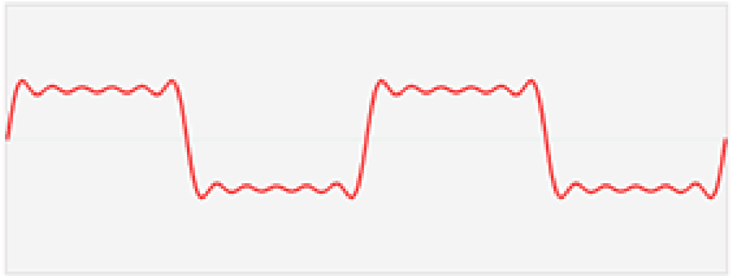
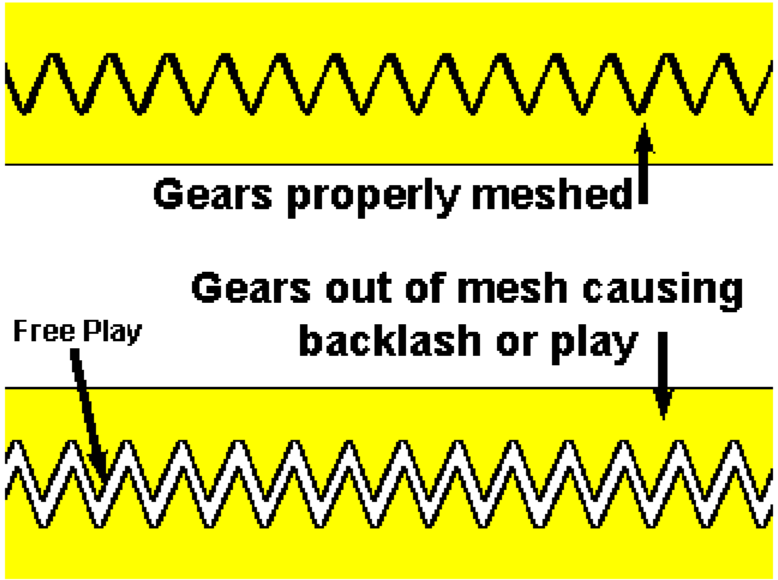
Presentation outline

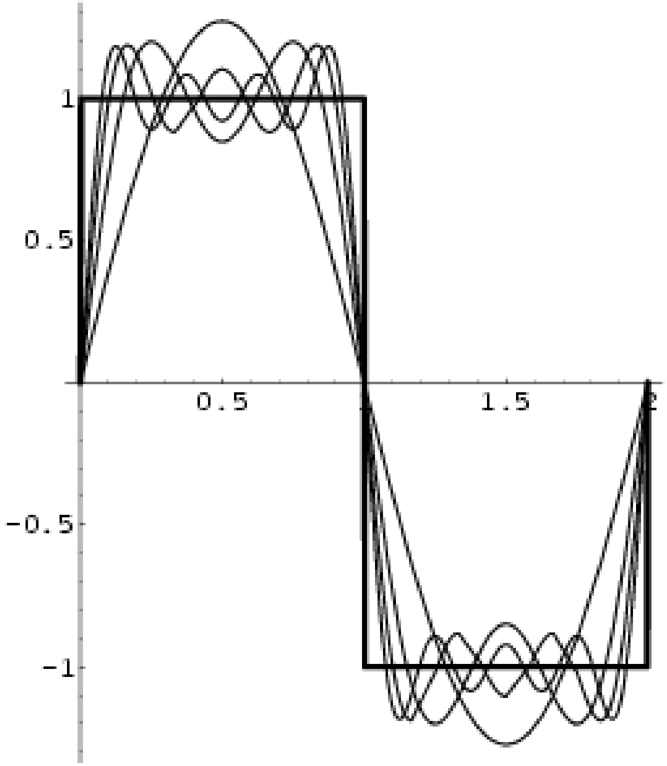
- **Introduction and motivation**
- **Non-smooth Galerkin formulation**
- **Least-squares regression analysis**
- **Additional studies**
- **Conclusions**

Introduction and motivation

- **Recent developments have efficiently applied the harmonic balance method to strongly nonlinear systems**
  - Systems include aircraft, spacecraft, gear drives, bladed disks, etc.
  - Applications include continuation procedures, bifurcation detection and tracking, nonlinear modal analysis, etc.
  - Advantages include reduced computational costs and capture of unstable solutions
  - Difficulties include chaotic/aperiodic responses and non-smooth or discontinuous solutions (Gibbs phenomenon)
  - How can we simulate "stiff" contact or friction systems efficiently with HBM?







Barbosa, L, 2013, "Fourier series and transform," Wikipedia. https://commons.wikimedia.org/wiki/File:Fourier_series_and_transform.gif
Weisstein, E, 1999 "Gibbs Phenomenon," https://archive.lib.msu.edu/crcmath/math/math/g/g164.htm
Astro-Physics, Inc., March 10, 2019, "400GTO - Adjustments to Remove Worm Gear Backlash". https://www.astro-physics.info/index.htm?products/mounts/400gto/400backlash

Introduction and motivation

- **Review of the literature–simulation approaches include:**
  - Lanczos filtering to improve the values of the Fourier coefficients
  - Append additional, non-smooth or discontinuous terms to a system's solution
  - Replace some or all the terms in classical Fourier series with non-smooth terms
  - Utilize event-driven schemes to find and integrate between the state transition times to compute nonlinear forces
  - Non-smooth temporal and spatial transformations

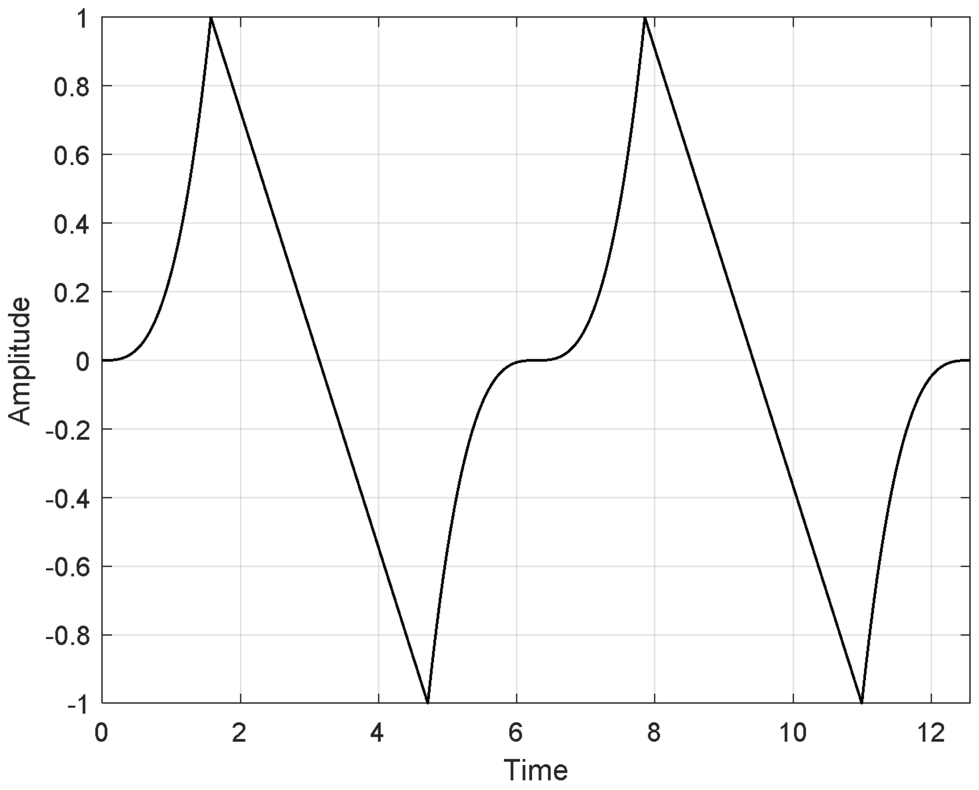
- **Difficulties:**
  - Gibbs phenomenon: slow convergence (polynomial) compared to smooth systems (exponential)
  - A priori knowledge of the state transition times may be required

Colaïtis, Y. and Batailly, A., 2021, "The harmonic balance method with arc-length continuation in blade-tip/casing contact problems," Journal of Sound and Vibration, **502**, p.116070. Doi: https://doi.org/10.1016/j.jsv.2021.116070
Brake, M.R. and Segalman, D.J., 2013, "Modelling localized nonlinearities in continuous systems via the method of augmentation by non-smooth basis functions," Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, **469**(2158), 20130260. Doi: https://doi.org/10.1098/rspa.2013.0260
Kim, W.J. and Perkins, N.C., 2003, "Harmonic balance/Galerkin method for non-smooth dynamic systems," Journal of Sound and Vibration, **261**(2), pp. 213-224. Doi: https://doi.org/10.1016/S0022-460X(02)00949-5
Krack, M., Panning-von Scheidt, L., and Wallaschek, J., 2013, "A high-order harmonic balance method for systems with distinct states," Journal of Sound and Vibration, **332**(21), pp. 5476-5488. Doi: https://doi.org/10.1016/j.jsv.2013.04.048
Pilipchuk, V.N., 2010. Nonlinear dynamics: between linear and impact limits (Vol. 52). Springer Science & Business Media.
Wang, W., Lu, Z.R., and Liu, J., 2021, "Convergence rates of harmonic balance method for periodic solution of smooth and non-smooth systems," Communications in Nonlinear Science and Numerical Simulation, **99**, 105826. Doi: https://doi.org/10.1016/j.cnsns.2021.105826

Introduction and motivation

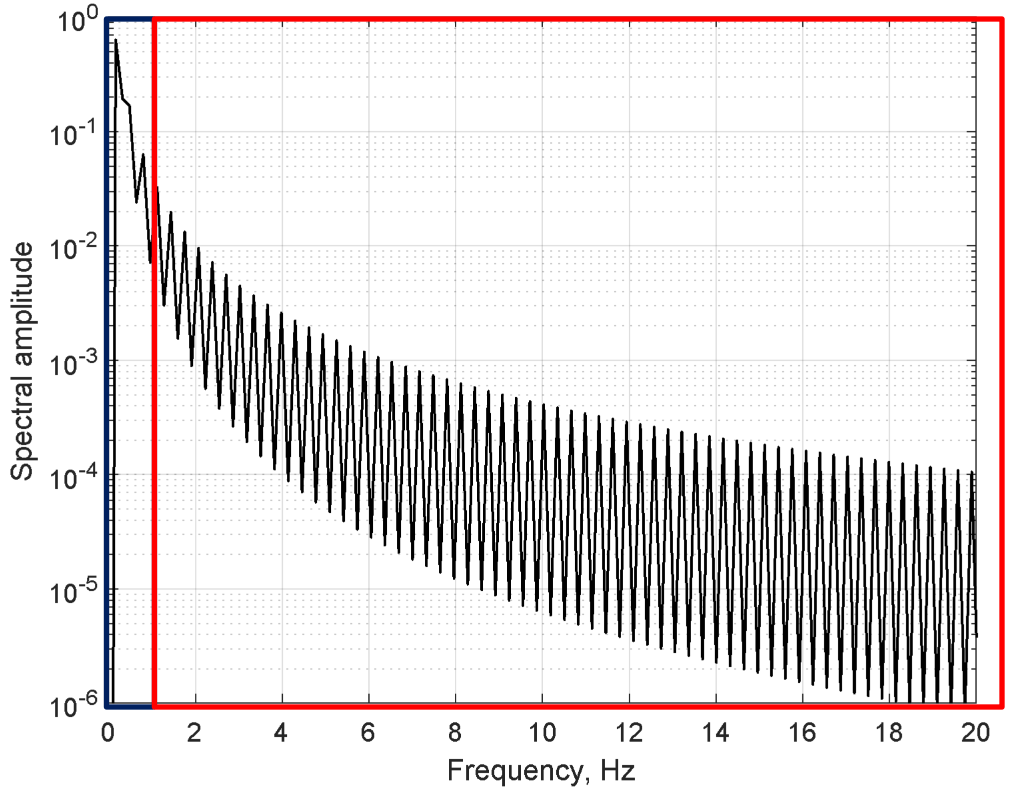
❑ Motivating idea:

- ➢ Non-smooth periodic motions can be represented by infinite Fourier series
- ➢ Can we append the Fourier series representation with entire non-smooth basis functions?
- ➢ This approach may allow us to capture a large set of harmonics with a small number of non-smooth functions



⇒

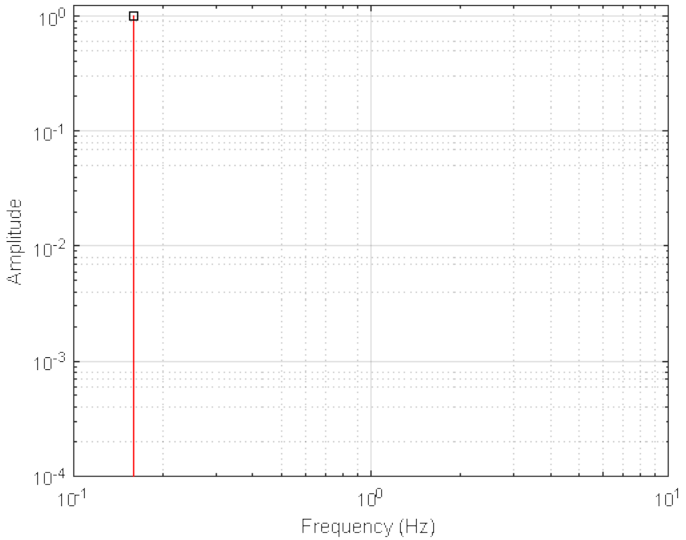
- Some sine/cosine terms
- Some non-smooth terms

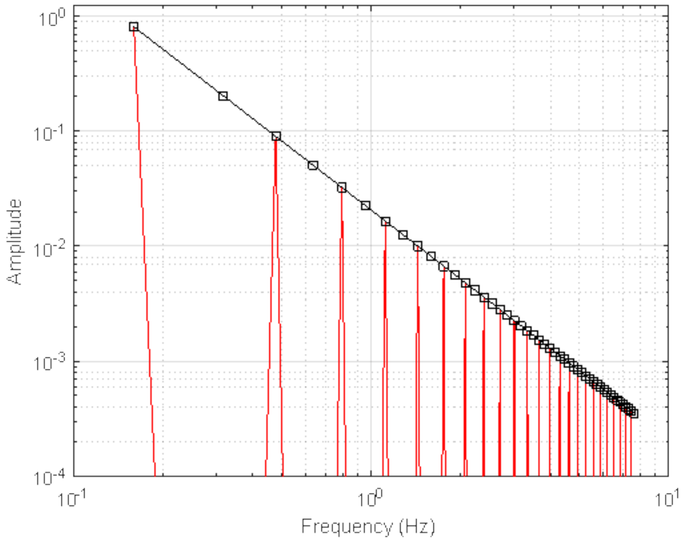


Introduction and motivation

- **Desirable basis function traits:**
  - Easy to mathematically represent
  - Convenient mathematical properties (Fourier series, derivatives, etc.)
  - Intuitive, non-smooth counterparts to sine and cosine functions

- **Goals of this work:**
  - Select functions with C^0 smoothness for capturing a non-smooth functional representation of a solution
  - Use goodness-of-fit metrics to evaluate the classical Fourier series and the non-smooth basis functions
  - Develop a framework that can later be implemented into harmonic balance formulations

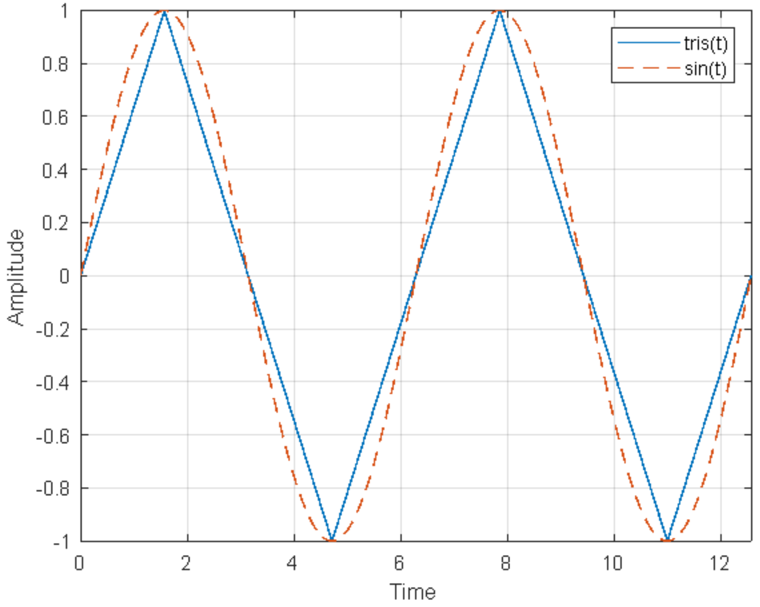


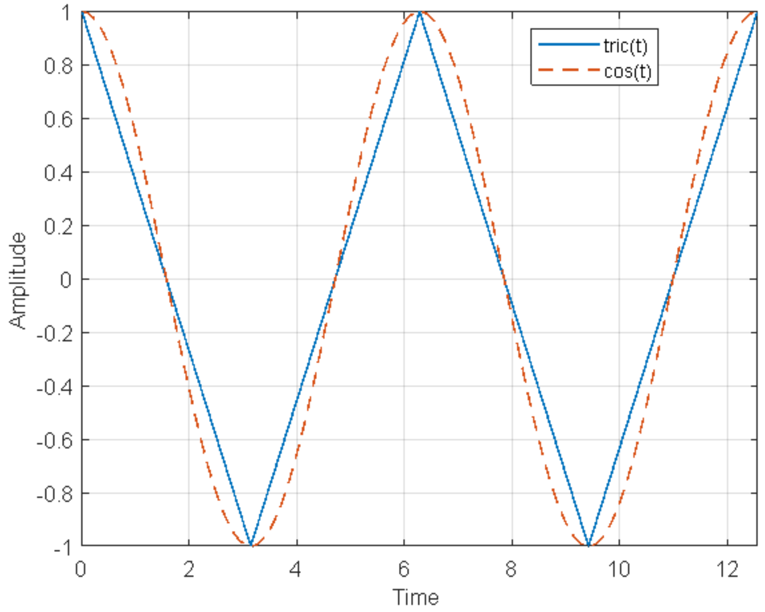


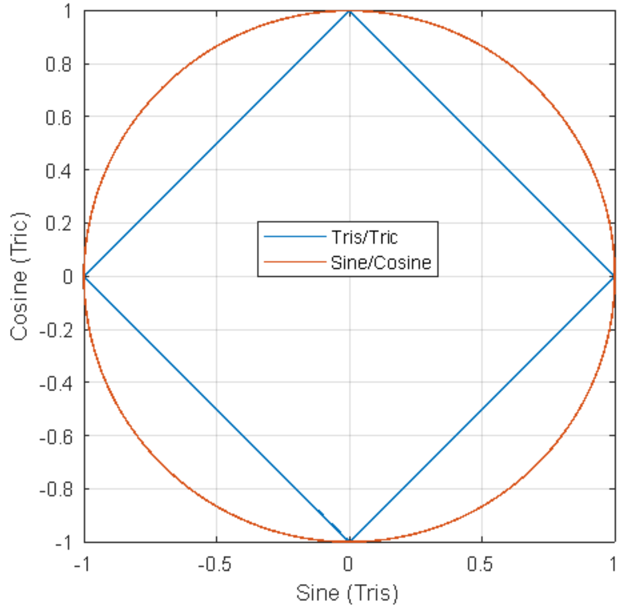
Non-smooth Galerkin formulation

- **Non-smooth triangle waves are chosen for study**
  - "Triangle sine" and "triangle cosine"
  - The same periodicity, maxima, minima, and roots as sine and cosine waves

$$tris(\omega t) = \begin{cases} \frac{4}{T} t_m, & t_m < \frac{T}{4} \\ \frac{-4}{T} t_m + 2, & \frac{T}{4} \le t_m \le \frac{3T}{4} \\ \frac{4}{T} t_m - 4, & t_m > \frac{3T}{4} \end{cases}, \qquad tric(\omega t) = \begin{cases} \frac{-4}{T} t_m + 1, & t_m \le \frac{T}{2} \\ \frac{4}{T} t_m - 3, & t_m > \frac{T}{2} \end{cases}, \qquad \begin{matrix} t_m = \omega t (mod T), \\ T = 2\pi \end{matrix}$$







Non-smooth Galerkin formulation

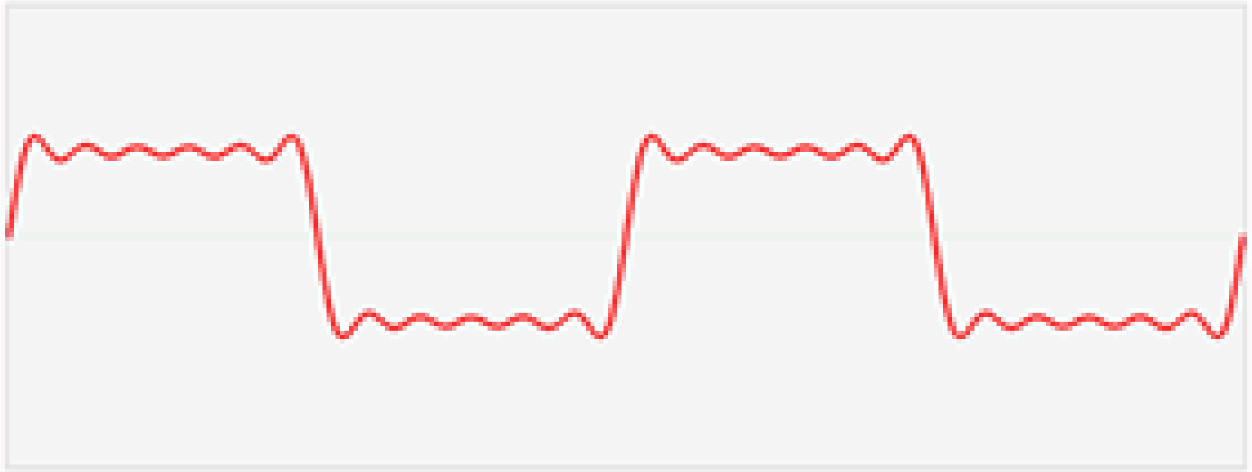
- **Infinite series representation:**

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n tric(n\omega t) + b_n tris(n\omega t)$$

- **A different form for numerical convenience**

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n tris(n\omega t + \theta_n)$$

- **Now we can study these functions for desirable properties and advantages**



Barbosa, L, 2013, "Fourier series and transform," Wikipedia. https://commons.wikimedia.org/wiki/File:Fourier_series_and_transform.gif

Mathematical properties

- **The non-smooth functions have similar properties to sine/cosine:**
  - Reflections:

θ reflected in $\alpha = 0$	θ reflected in $\alpha = \frac{\pi}{4}$	θ reflected in $\alpha = \frac{\pi}{2}$	θ reflected in $\alpha = \frac{3\pi}{4}$	θ reflected in $\alpha = \pi$
$tris(-\theta) = -tris(\theta)$	$tris\left(\frac{\pi}{2} - \theta\right) = tric(\theta)$	$tris(\pi - \theta) = +tris(\theta)$	$tris\left(\frac{3\pi}{2} - \theta\right) = -tric(\theta)$	$tris(2\pi - \theta) = -tris(\theta)$
$tric(-\theta) = +tric(\theta)$	$tric\left(\frac{\pi}{2} - \theta\right) = tris(\theta)$	$tric(\pi - \theta) = -tric(\theta)$	$tric\left(\frac{3\pi}{2} - \theta\right) = -tris(\theta)$	$tric(2\pi - \theta) = +tric(\theta)$

  - Shifts and periodicity:

Shift by one quarter period	Shift by one half period	Shift by three quarter periods	Shift by full periods
$\mathrm{tris}\left(\theta \pm \frac{\pi}{2}\right) = \pm \mathrm{tric}(\theta)$	$\mathrm{tris}(\theta \pm \pi) = -\mathrm{tris}(\theta)$	$\mathrm{tris}\left(\theta \pm \frac{3\pi}{2}\right) = \mp \mathrm{tric}(\theta)$	$\mathrm{tris}(\theta \pm 2\pi k) = +\mathrm{tris}(\theta)$
$\mathrm{tric}\left(\theta \pm \frac{\pi}{2}\right) = \mp \mathrm{tris}(\theta)$	$\mathrm{tric}(\theta \pm \pi) = -\mathrm{tric}(\theta)$	$\mathrm{tric}\left(\theta \pm \frac{3\pi}{2}\right) = \pm \mathrm{tris}(\theta)$	$\mathrm{tric}(\theta \pm 2\pi k) = +\mathrm{tric}(\theta)$

Miscellaneous:
- $|tris(\theta)| + |tric(\theta)| = 1$
- Angle sums/phase shifts: not straightforward

Mathematical properties

- **Fourier series representations (ω integer):**

$$tris(\omega t) = \sum_{n=1}^{\infty} b_n sin(nt), \qquad b_n = \frac{4\omega}{n^2\pi^2} \sum_{k=1}^{2\omega} (-1)^{k+1} \sin\left(\frac{(2k-1)n\pi}{2\omega}\right)$$

$$tric(\omega t) = a_0 + \sum_{n=1}^{\infty} a_n cos(nt), \qquad a_0 = 0, \qquad a_n = \frac{4\omega}{n^2\pi^2} \sum_{k=1}^{2\omega-1} (-1)^k cos\left(\frac{kn\pi}{\omega}\right)$$

- Examples:

$$tris(t) = \sum_{n=1}^{\infty} b_n sin(nt), \qquad b_n = \frac{8}{n^2\pi^2} sin\left(\frac{n\pi}{2}\right) = \frac{8}{\pi^2}, 0, \frac{-8}{9\pi^2}, 0, \frac{8}{25\pi^2}, 0, \frac{-8}{49\pi^2}, \ldots$$

$$tric(t) = \sum_{n=1}^{\infty} a_n cos(nt), \qquad a_n = \frac{4}{n^2\pi^2}[1 - cos(n\pi)] = \frac{8}{\pi^2}, 0, \frac{8}{9\pi^2}, 0, \frac{8}{25\pi^2}, 0, \frac{8}{49\pi^2} \ldots$$

- First derivatives: square waves with coefficients $\propto \frac{8\omega}{n\pi^2}$
- Second derivatives: Dirac combs with coefficients $\propto \frac{8\omega}{\pi^2}$

Mathematical properties

- **The functions numerically satisfy the following *orthogonality* relationships:**

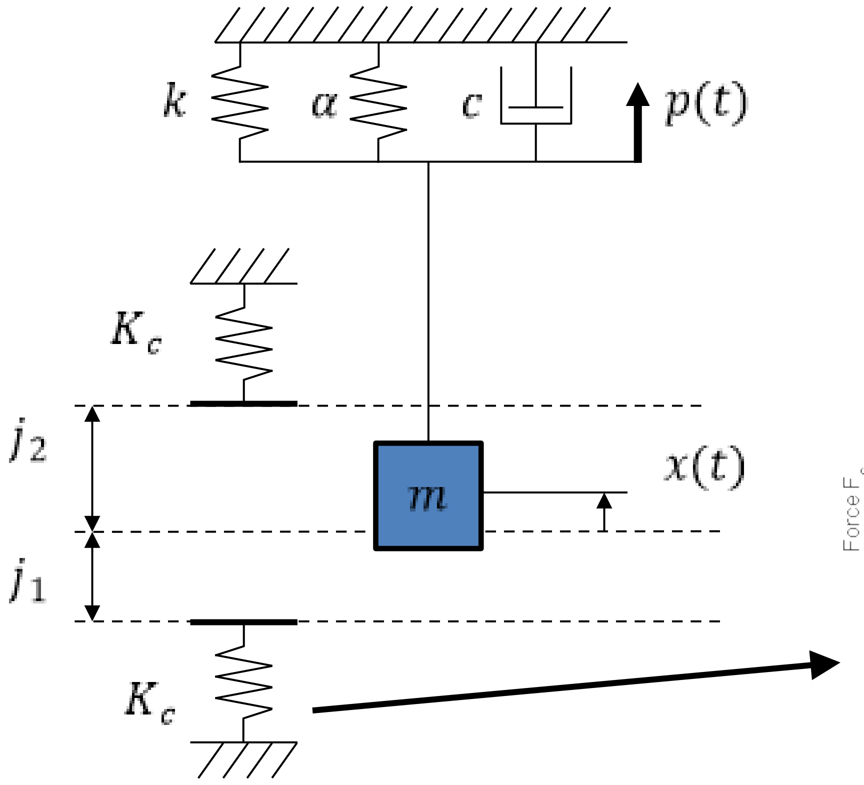
$$\int_0^{2\pi} tris(mx)dx = 0,$$
$$\int_0^{2\pi} tric(mx)dx = 0,$$
$$\int_0^{2\pi} tric(mx)tric(nx)dx = c_m\delta_{mn},$$
$$\int_0^{2\pi} tris(mx)tris(nx)dx = d_m\delta_{mn},$$
$$\int_0^{2\pi} tric(mx)tris(nx)dx = 0,$$

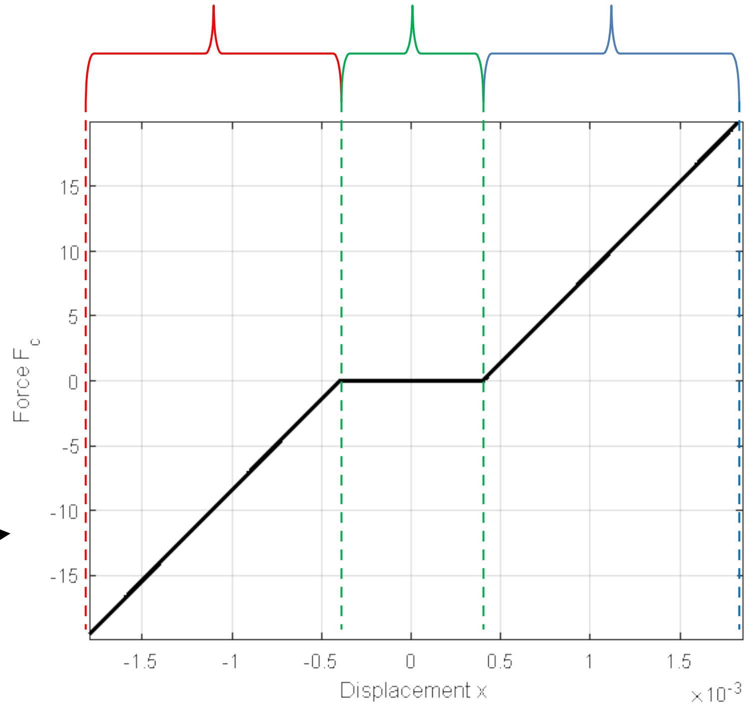
- For integers m, n and constants c_m, d_m
- δ_{mn} denotes Kronecker delta
- Result: biorthogonal system, just like sine and cosine
- A functional infinite series representation, similar to Fourier series, is possible
- Reiterate: no rigorous proof of this

Least-squares regression analysis

- **The system: a forced Duffing oscillator with freeplay**

$$\ddot{x} + 2\omega_n\zeta\dot{x} + \omega_n^2 x + \frac{\alpha}{m}x^3 + \frac{F_c(x)}{m} = \frac{p}{m}cos(\omega t),$$





- Two different contact laws—contact penalty stiffness, and elastic impact:

$$F_c = \begin{cases} K_c(x + j_1), & x < -j_1 \\ 0, & -j_1 \leq x \leq j_2 \\ K_c(x - j_2), & x > j_2 \end{cases}$$

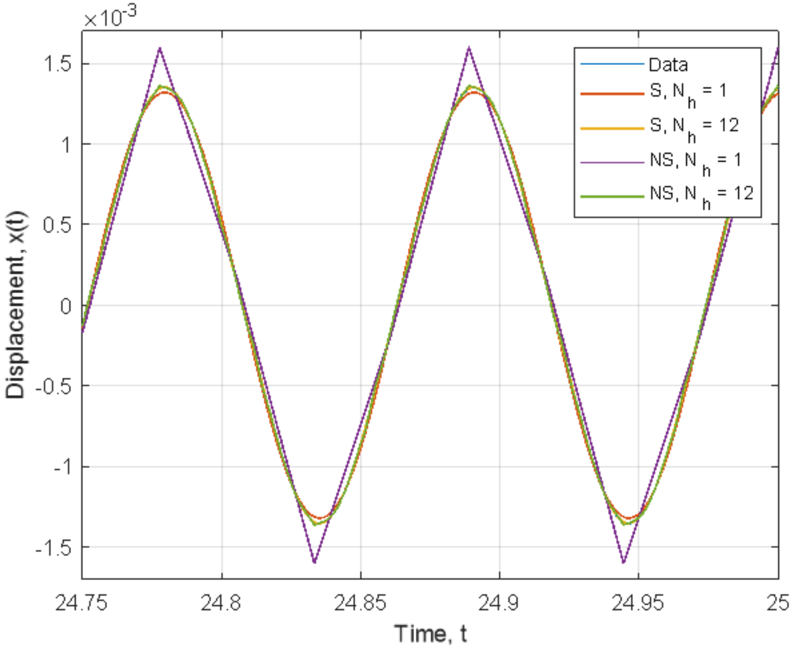
vs.

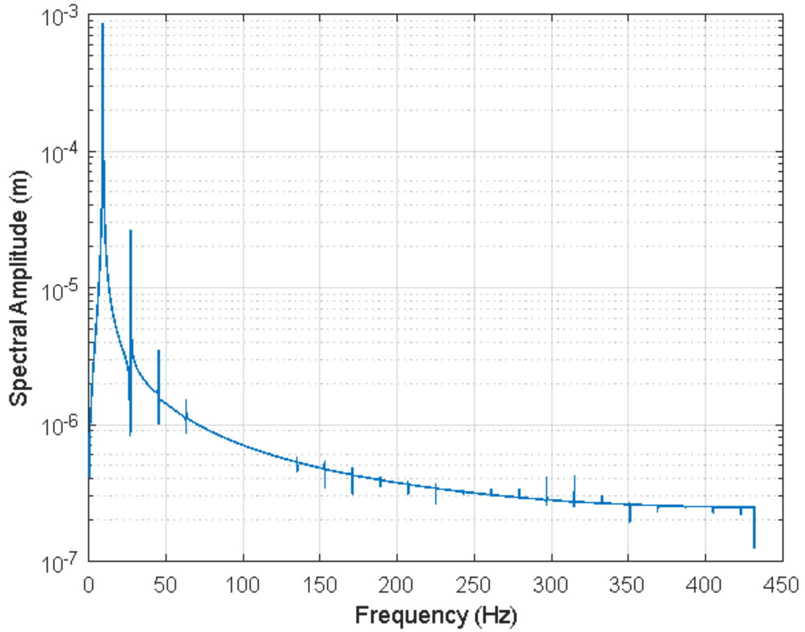
$$-j_1 \leq x(t) \leq j_2,$$
$$x^+ = -rx^-, r = 1$$

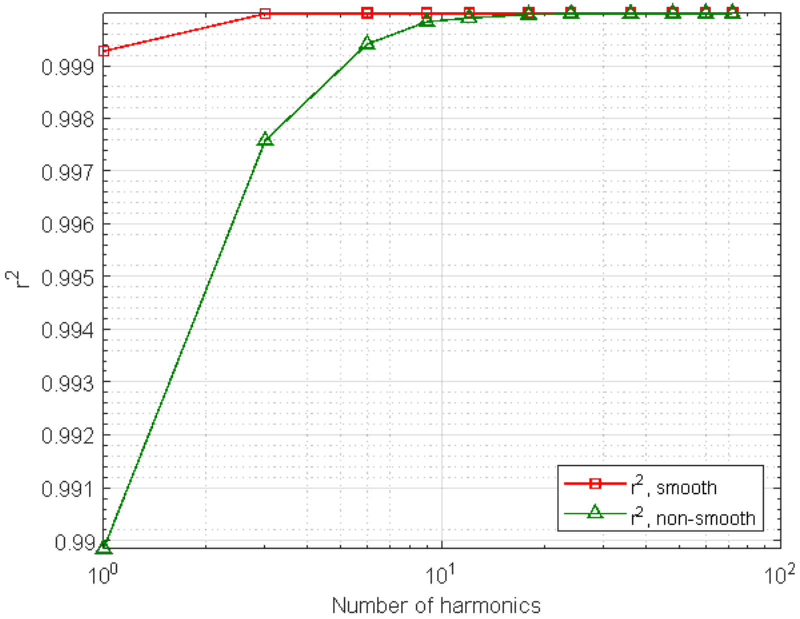
De Langre, E., Lebreton, G., 1996, "An Experimental and Numerical Analysis of Chaotic Motion in Vibration with Impact," ASME 8th International Conference on Pressure Vessel Technology, Montreal, Quebec, Canada, July 21 - 26, 1996.

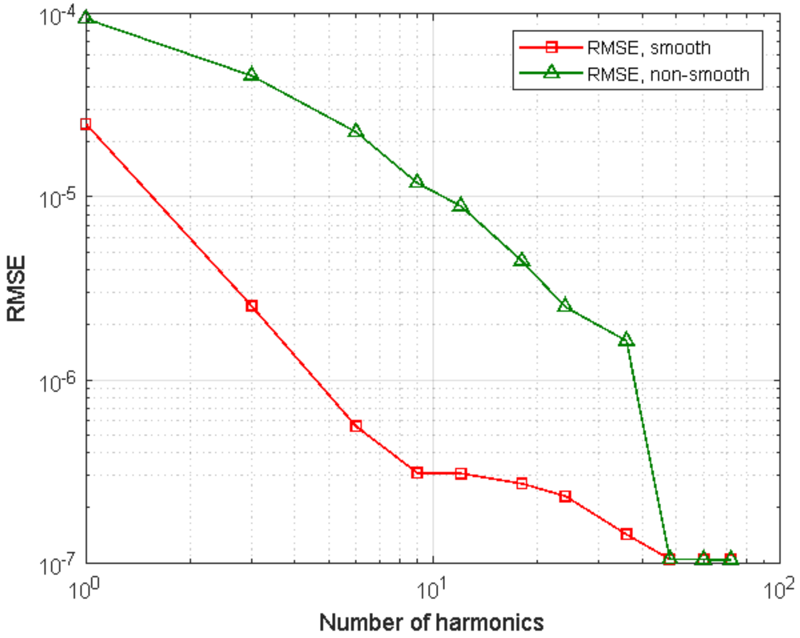
Least-squares regression analysis

Medium stiffness, $K_c = 1.4 * 10^4 \frac{N}{m}, \omega = 9\ Hz$



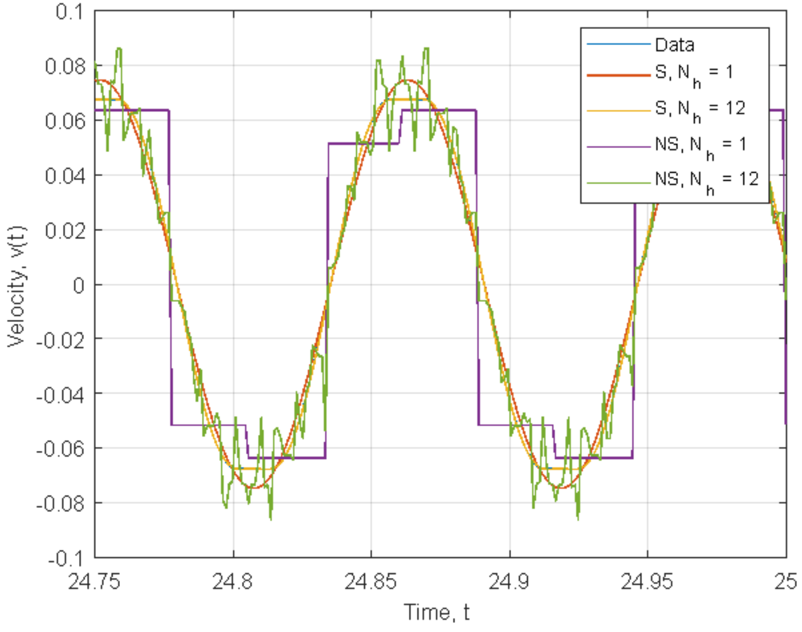


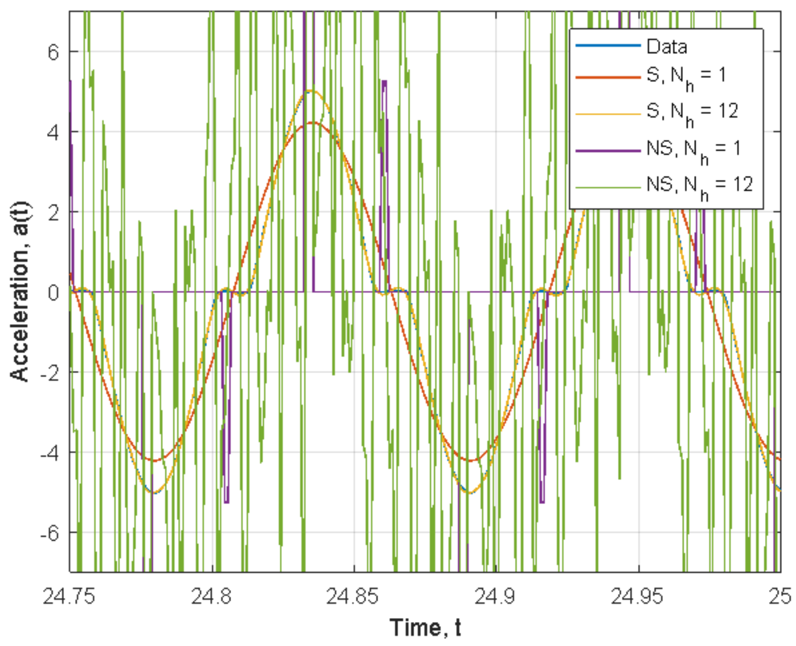


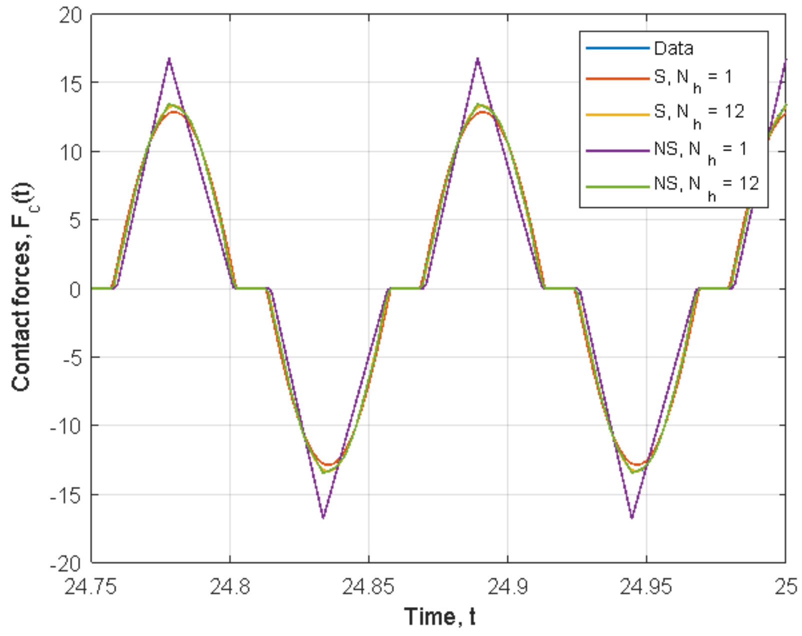


- ➢ Smooth system response
- ➢ Classical Fourier series converges much faster than non-smooth Fourier series

Least-squares regression analysis







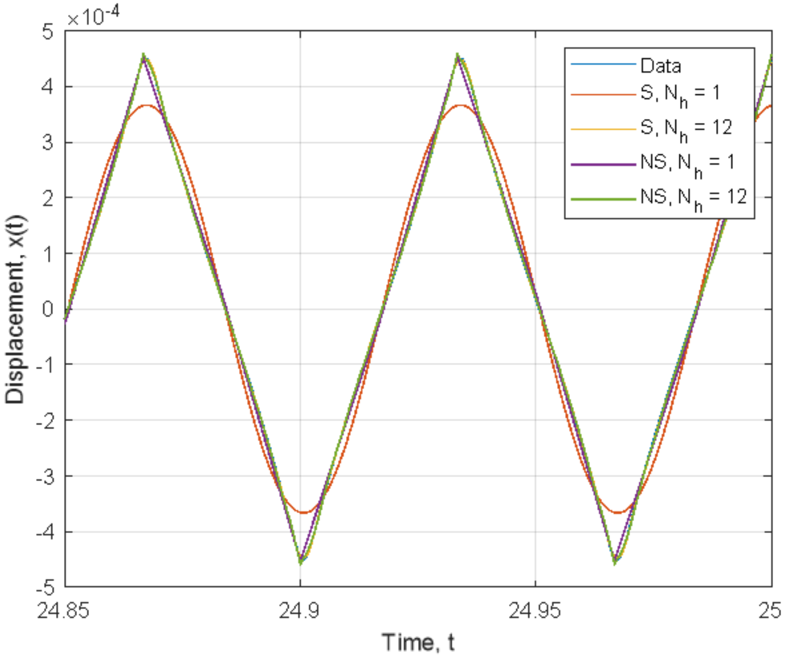
- Now let's look at the velocity, acceleration, and contact forces obtained using low- and high-quality fits
- Both smooth and non-smooth velocities use the coefficients obtained from curve-fitting the displacement:

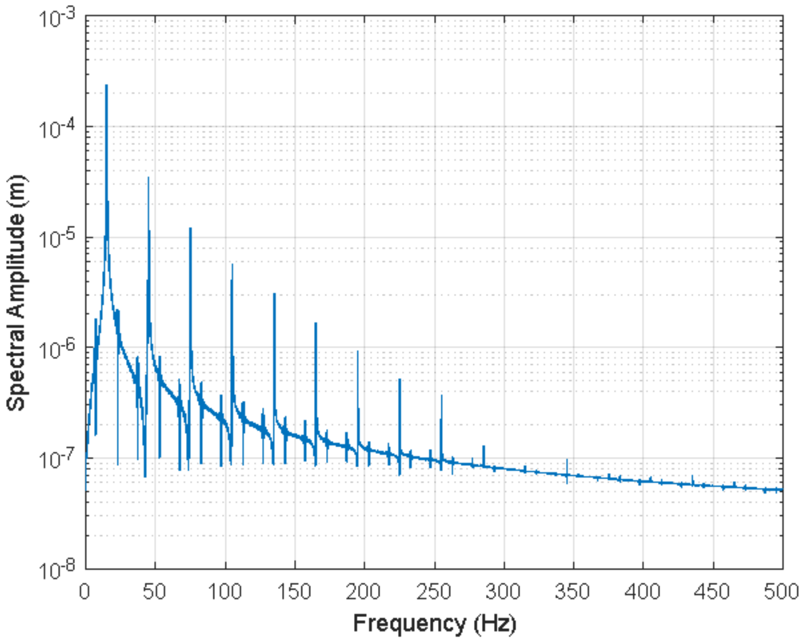
$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n tric(n\omega t) + b_n tris(n\omega t) \Rightarrow v(t) = \sum_{n=1}^{\infty} a_n n\omega squc(n\omega t) + b_n n\omega squs(n\omega t)$$

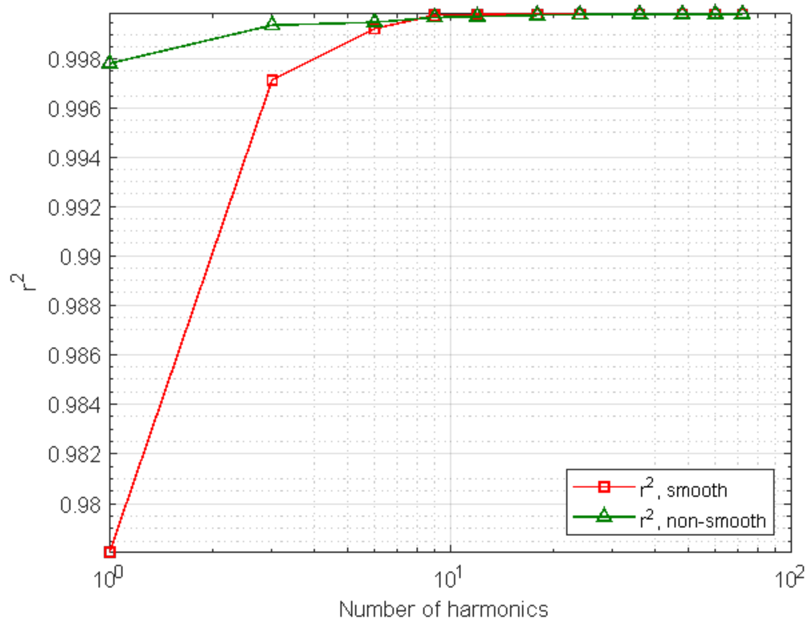
- Smooth acceleration is done similarly; non-smooth is computed using Matlab *gradient*
- Velocities and contact forces are captured well
- Accelerations show a limitation in the non-smooth formulation
- Dirac combs everywhere!

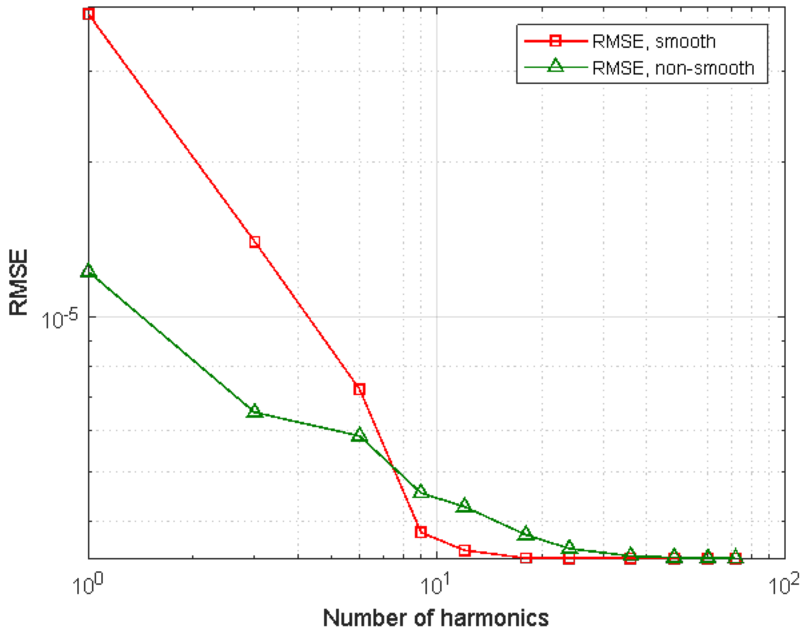
Least-squares regression analysis

$$\text{Hard stiffness, } K_c = 1.4 * 10^6 \frac{\text{N}}{\text{m}}, \omega = 15\ Hz$$



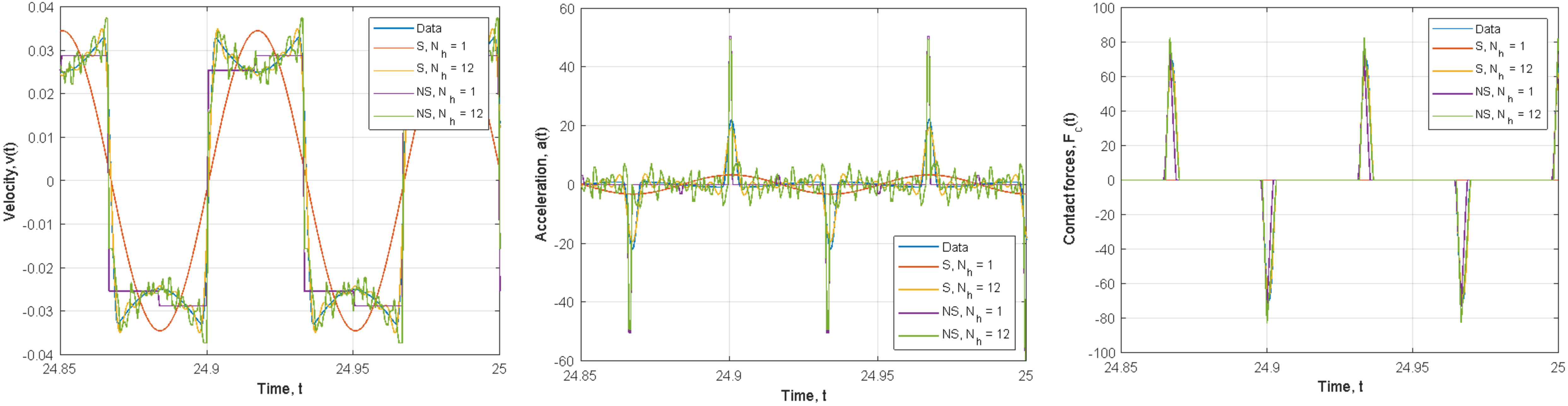






- Less-smooth of a system response
- The non-smooth series converges faster until $N_h = 8$
- For more harmonics, the smooth curve-fit becomes better

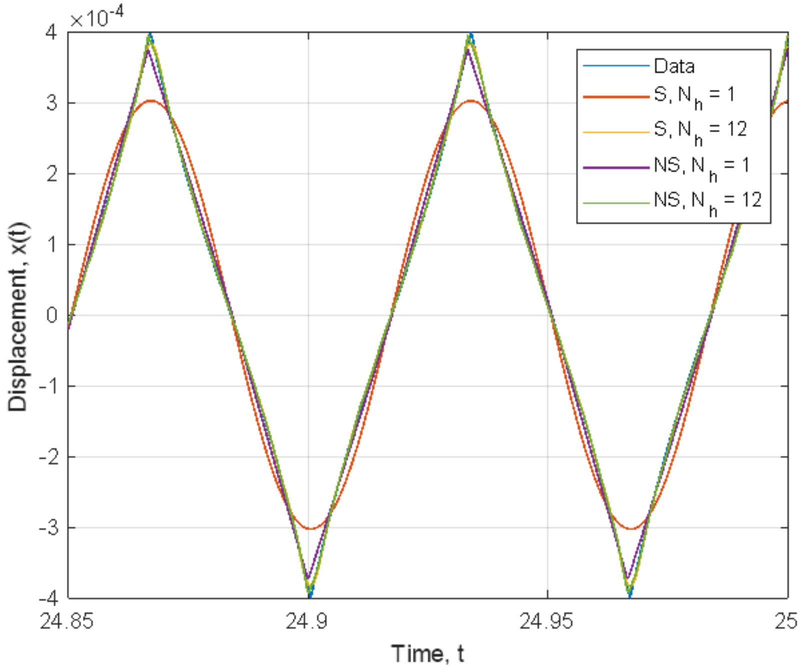
Least-squares regression analysis

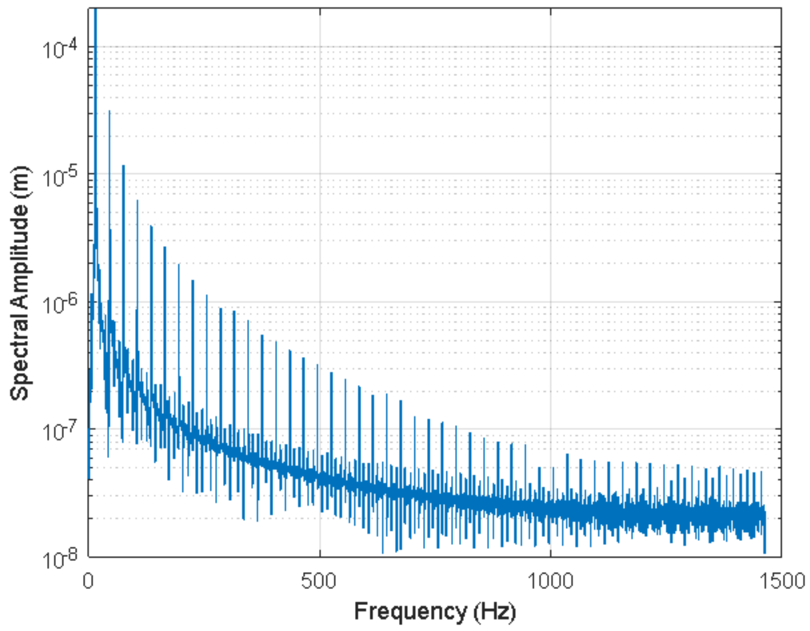


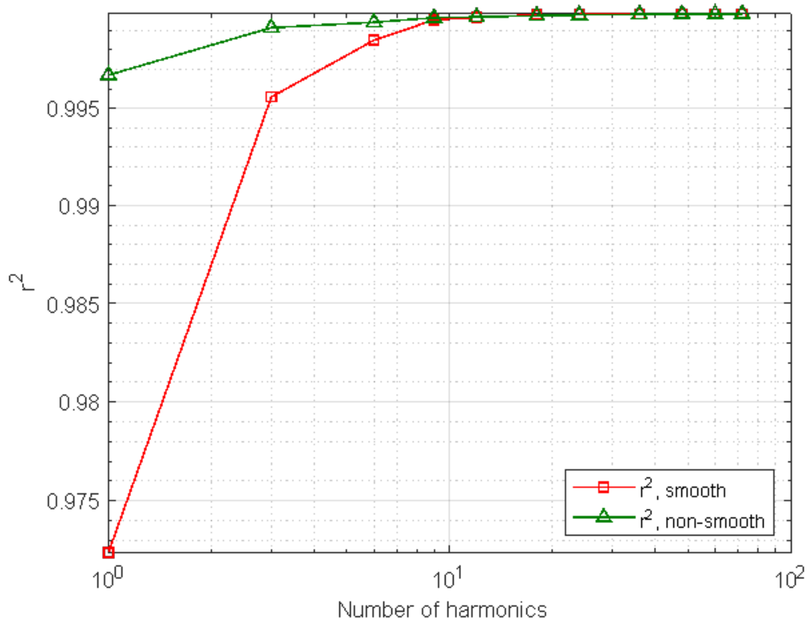
- Both smooth and non-smooth velocities agree well and match closely to the data
- Smooth and non-smooth accelerations agree well away from the points of contact
- Non-smooth acceleration overshoots significantly
- Contact forces agree very well

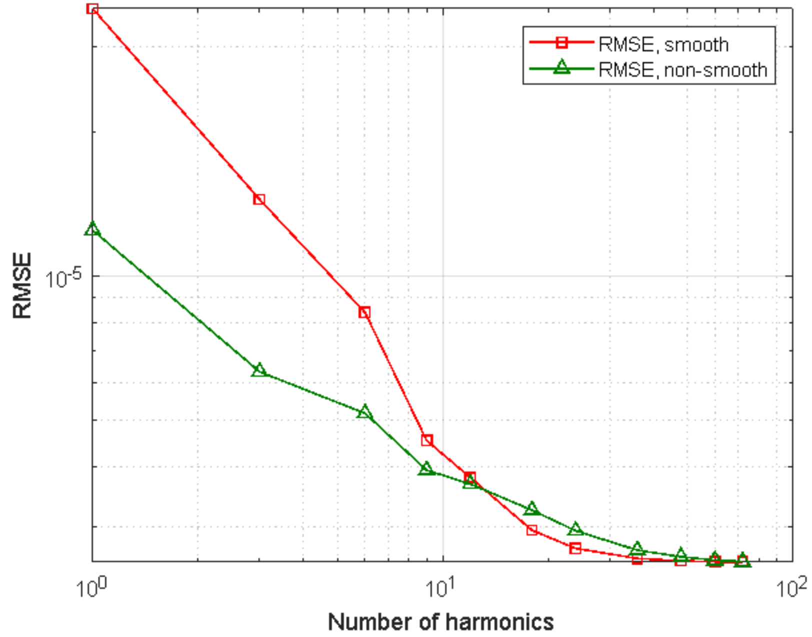
Least-squares regression analysis

Very hard stiffness, $K_c = 1.4 * 10^{10}\,\frac{N}{m}, \omega = 15\,Hz$



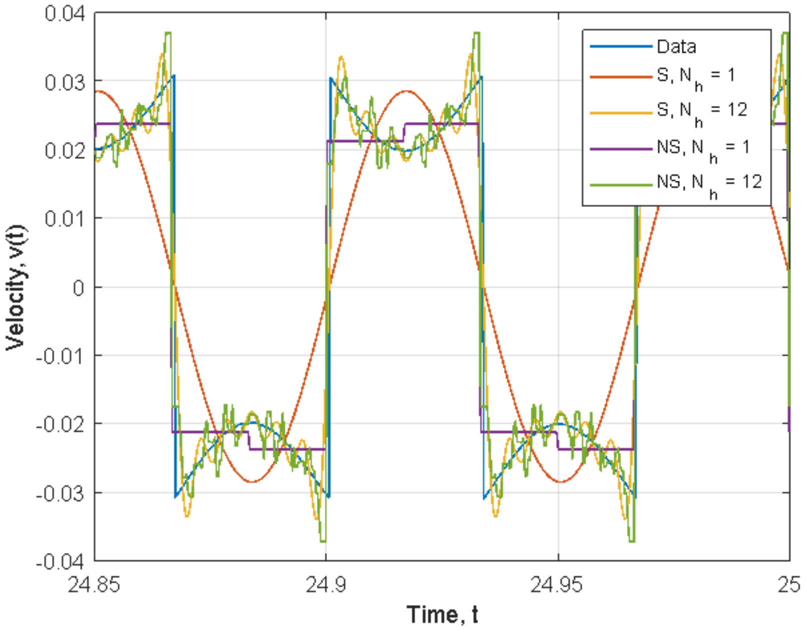


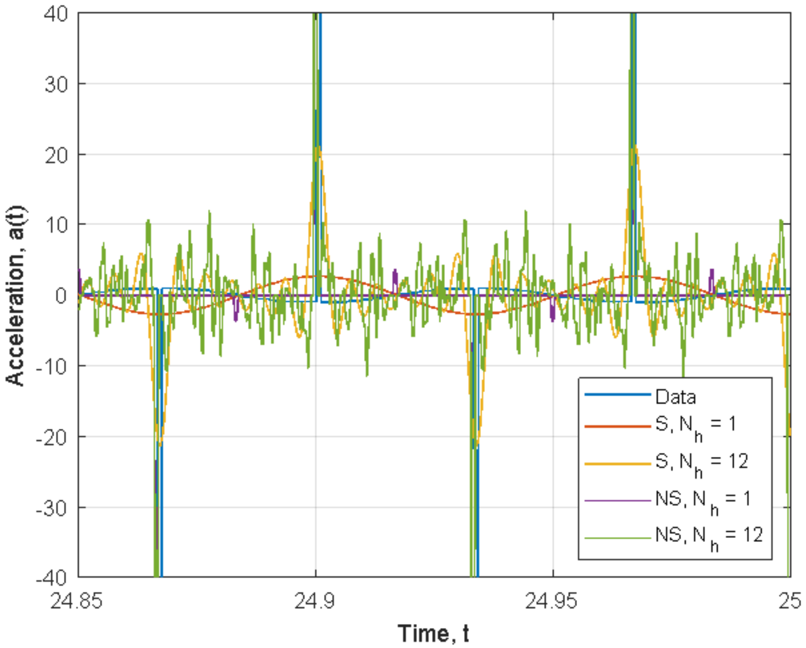


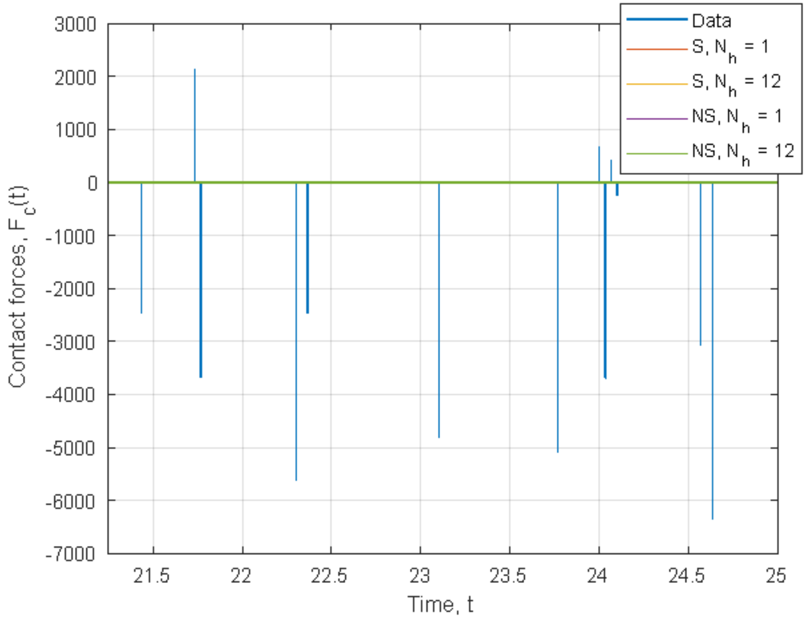


- Strongly non-smooth response
- The non-smooth series converges faster until $N_h = 12$
- For more harmonics, the smooth curve-fit becomes better

Least-squares regression analysis



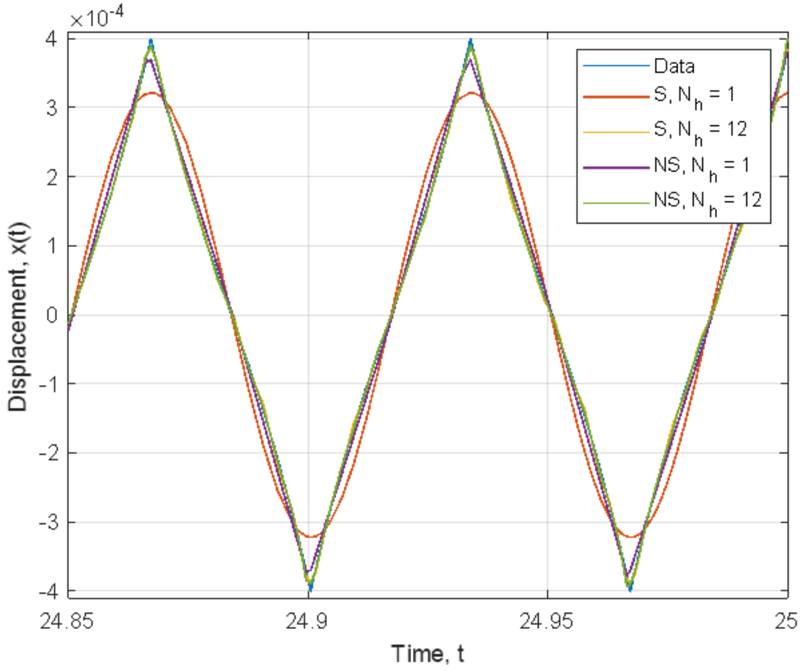


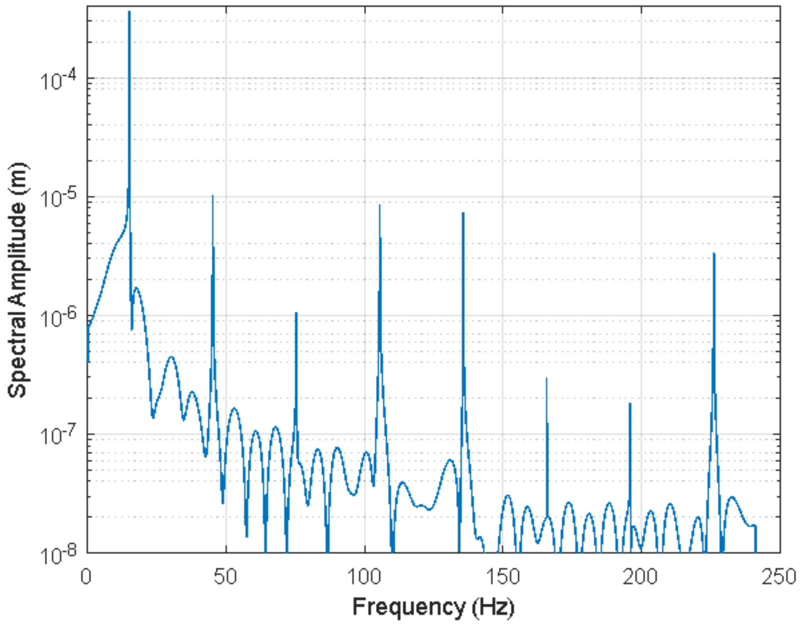


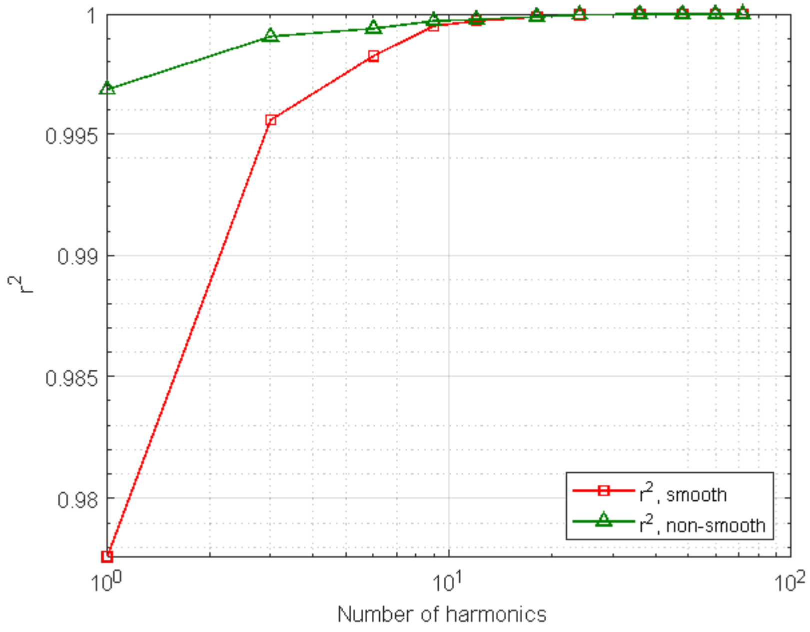
- Velocities are again in agreement with each other and the data
- The non-smooth response best captures the peaks in acceleration
- Poor agreement everywhere else, however, for both smooth and non-smooth
- Contact forces are essentially Dirac impulses by now
- *None* of the curve-fits capture the contact force
- Why? Because the penalty stiffness force definition

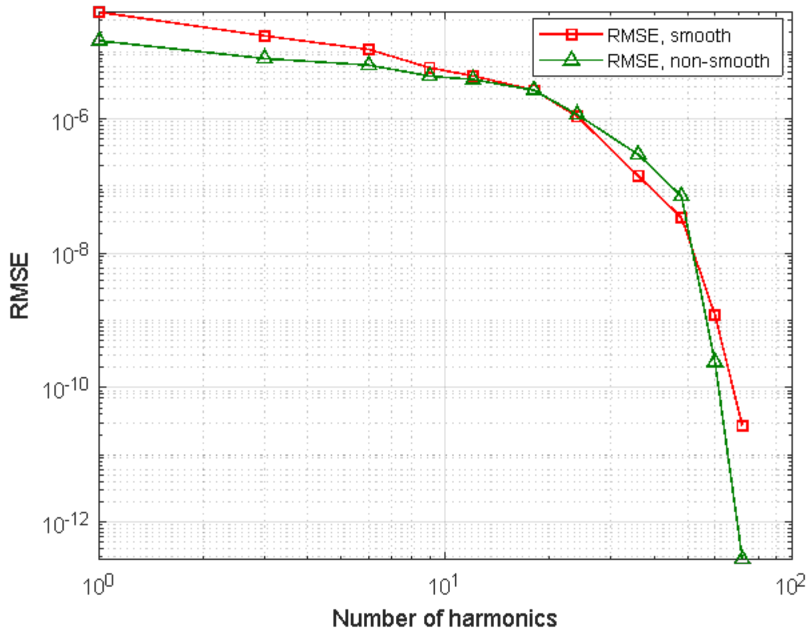
Least-squares regression analysis

Elastic impact, $COR = 1, \omega = 15\ Hz$



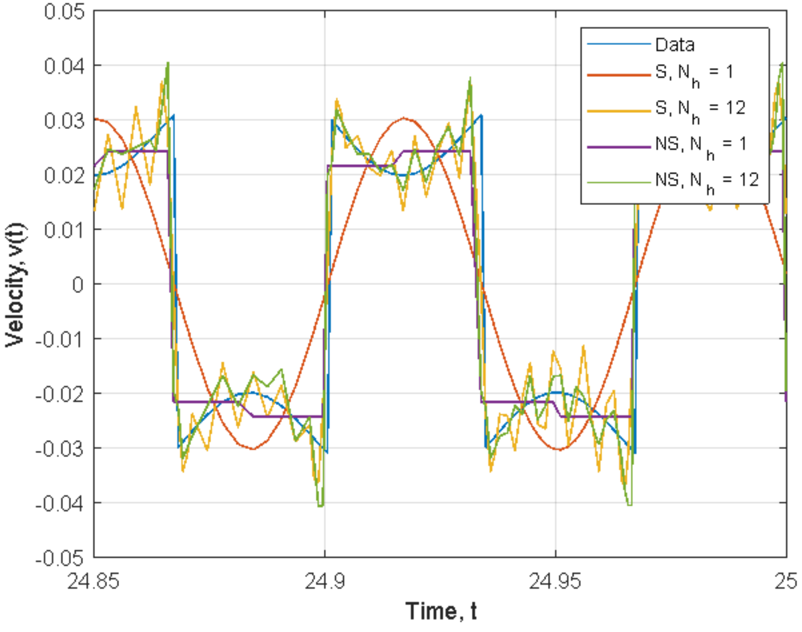


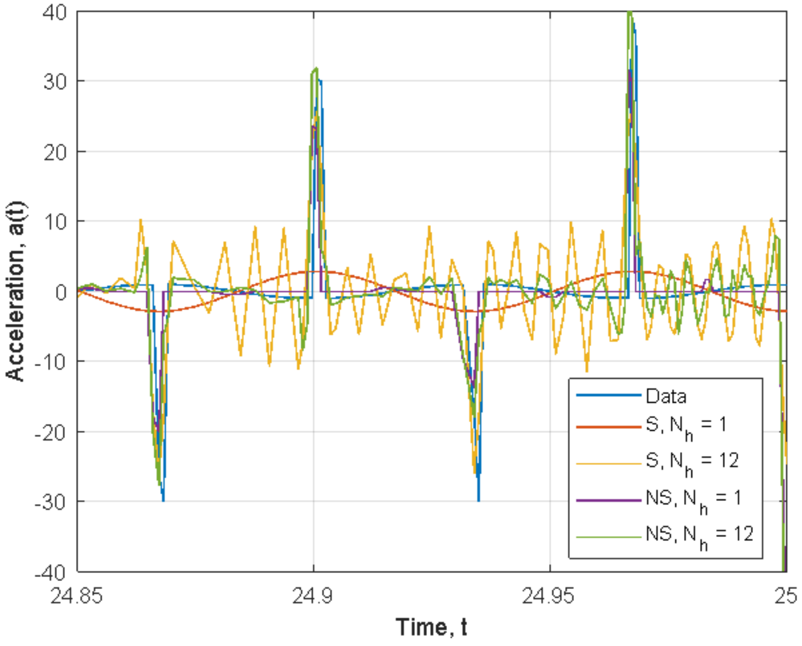


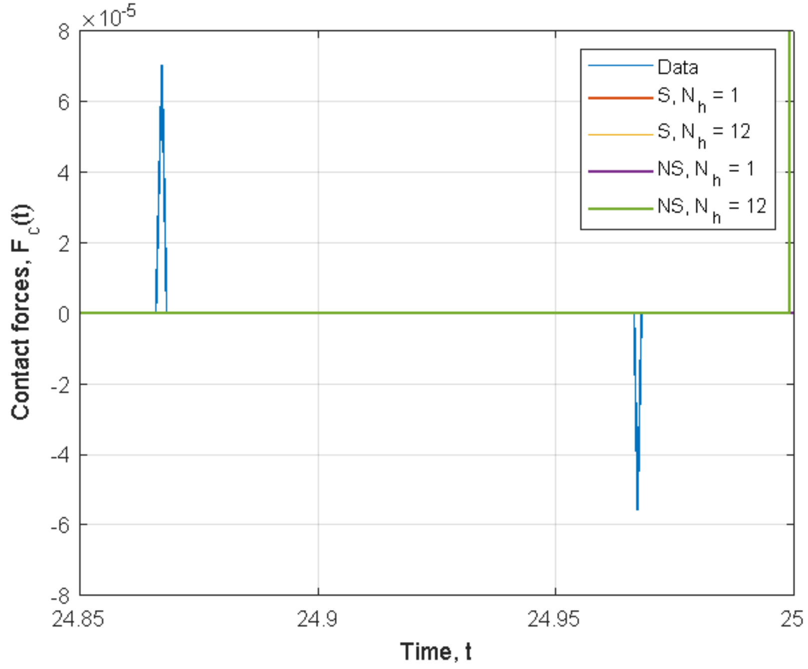


- Explicit hard impact using coefficient of restitution instead of penalty stiffness, aka the limit as $K_c \rightarrow \infty$
- The non-smooth fit converges faster until $N_h = 18$
- Then the smooth fit until $N_h \approx 54$
- Then the non-smooth fit again

Least-squares regression analysis







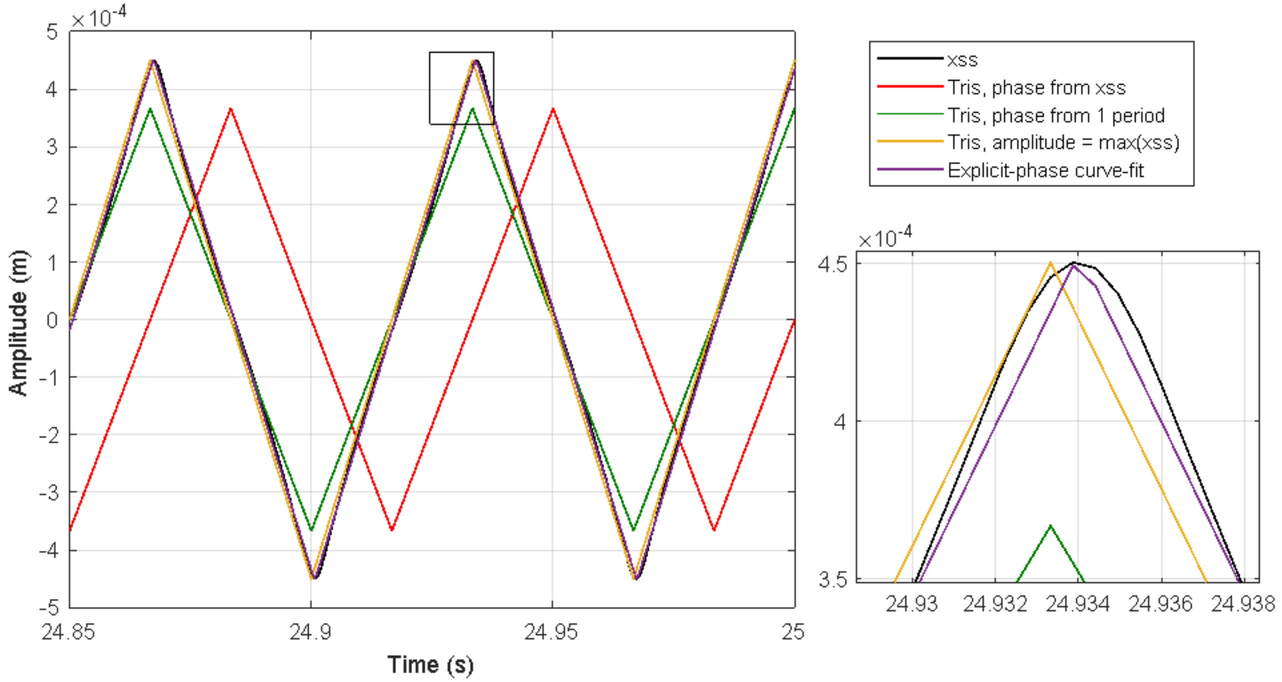
- Velocities are still in good agreement
- Both smooth and non-smooth capture the peaks in acceleration
- Poor agreement everywhere else, still
- Contact forces are Dirac impulses

Additional studies

- **Consider a different curve fit of the following form:**

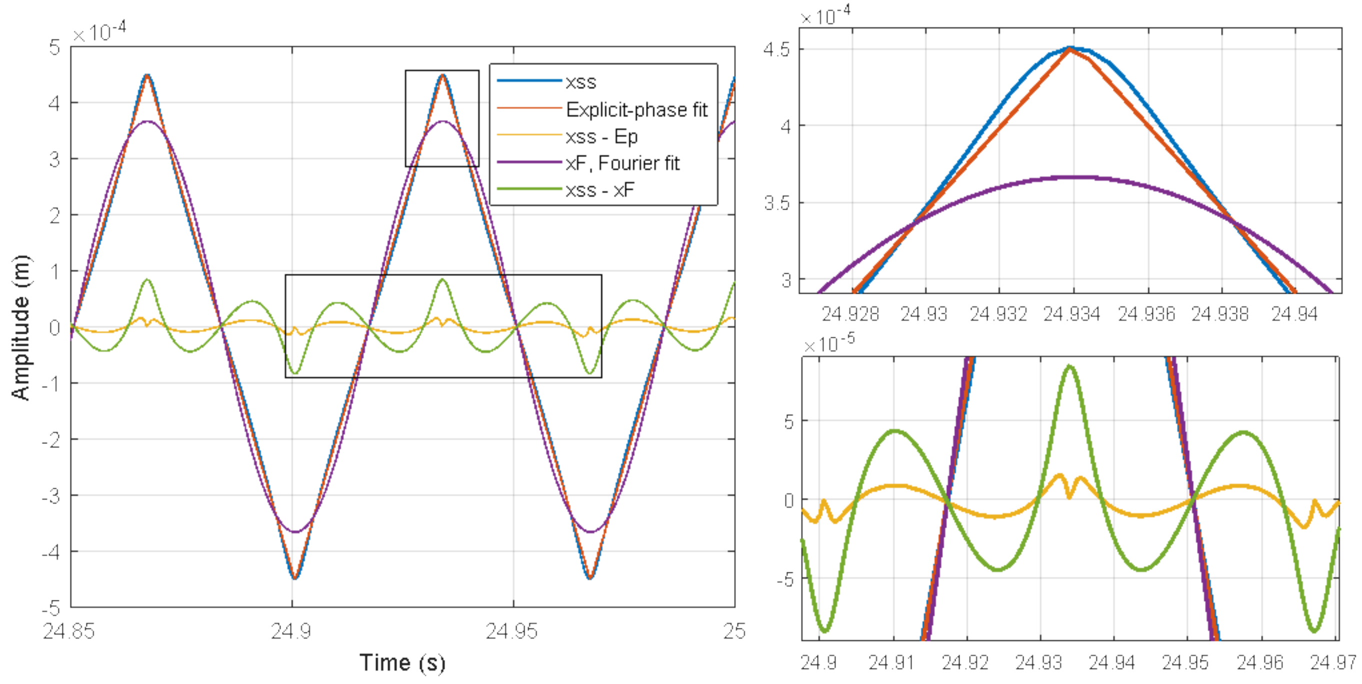
$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n tris(n\omega t + \theta_n)$$

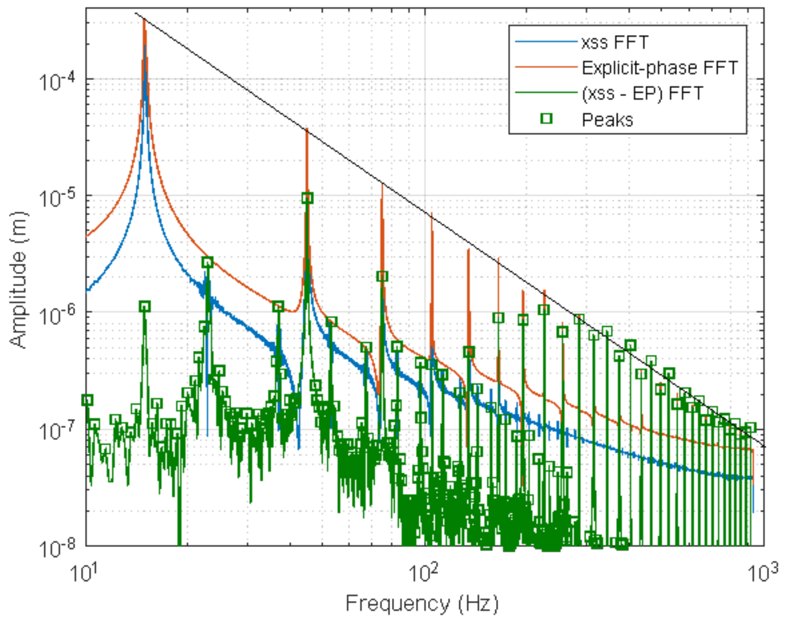
- Motivation: to better match up the locations of contacts/impacts with the phase-shifts that would produce them, hopefully reducing the number of terms needed
- Every additional non-smooth term means additional contacts and impulses/Dirac combs in the acceleration, which becomes unrealistic

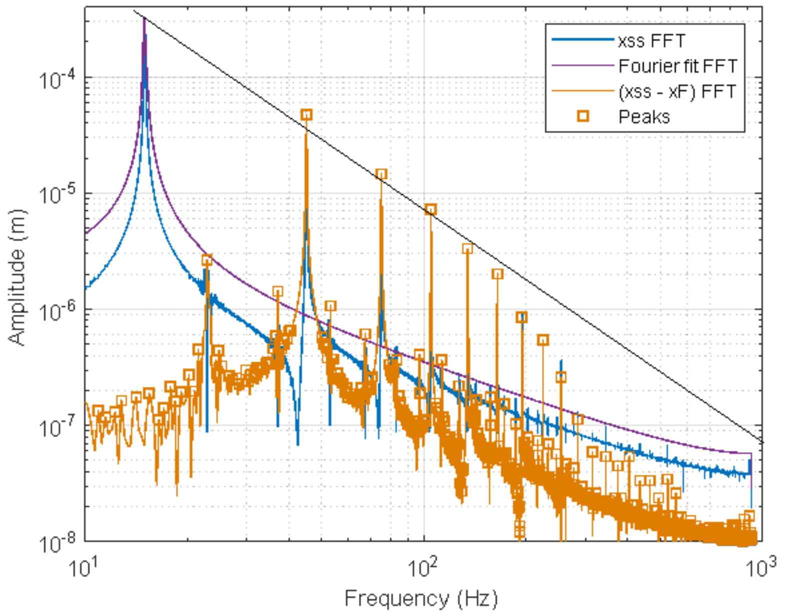


- Left: hard-contact case with different curve fits of a single triangle function
- 3 different manual curve fits
- 4th curve fit using Matlab nonlinear least squares
- First 3 tend to capture only amplitude or phase accurately, not both
- 4th one captures both well

Additional studies

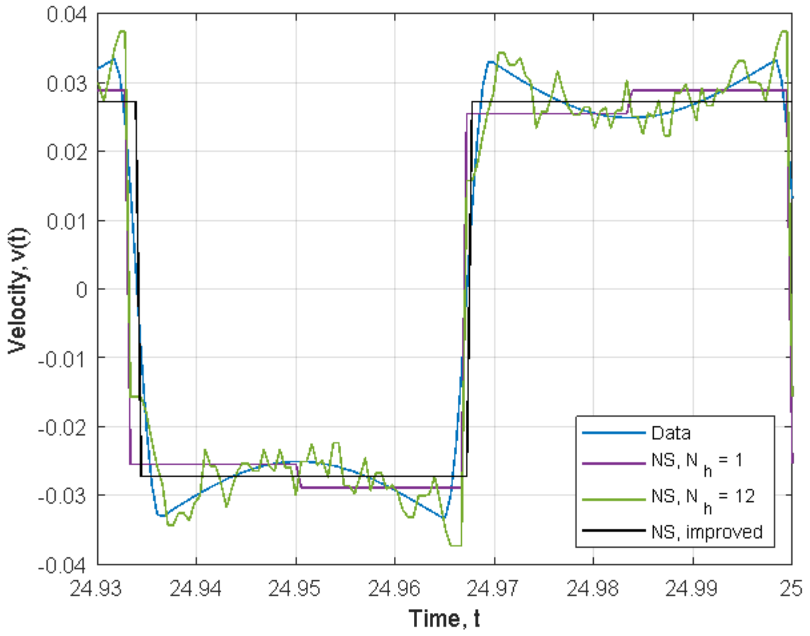


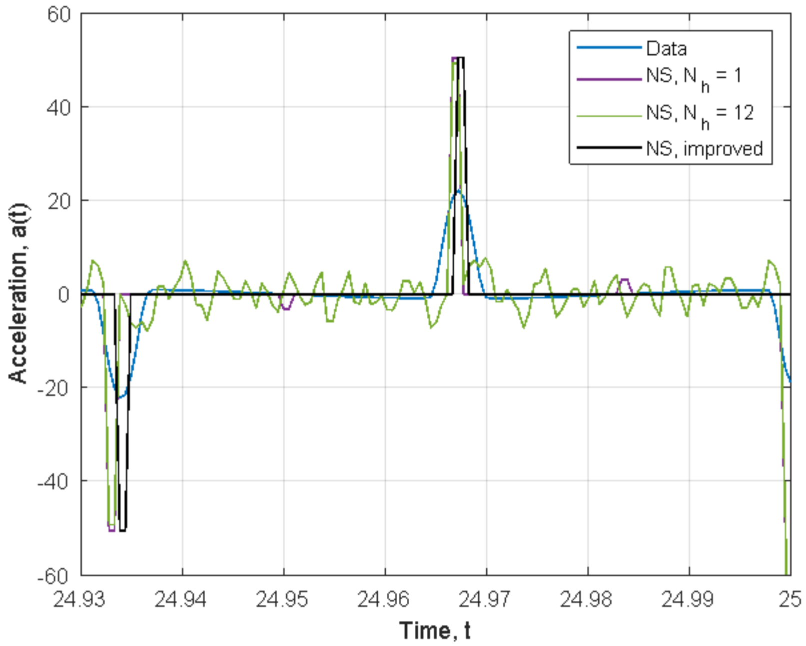


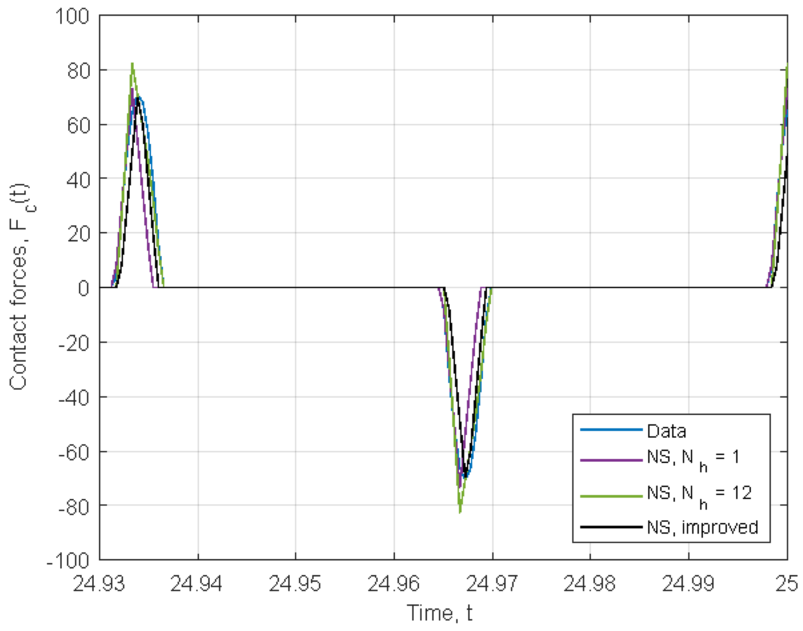


- Compare the improved curve fit to the Fourier fit for $N_h = 1$
- Original tris: RMSE $= 1.22 * 10^{-5}$
- Fourier: RMSE $= 3.88 * 10^{-5}$
- New tris: RMSE $= 8.55 * 10^{-6}$
- 1.5 times better now!

Additional studies







- The improved non-smooth fit keeps good agreement with the velocity, acceleration, and contact force
- Further evidence that more terms does not necessarily improve the fit if performed naively

Conclusions

- Evaluated the usefulness of non-smooth basis functions for obtaining the response of a contact/impact system
- Non-smooth, triangular sine and cosine functions were defined
- Mathematical properties were highlighted
- Applied curve fits to time histories of a contact/impact system and studied for quality
- Results show Fourier series is superior for smooth responses, as expected
- The non-smooth series becomes superior for increasingly non-smooth responses
- Fourier series tends to become more accurate again when many harmonics are used
- A modified series form showed better results than the original naïve series form

Future work

- Continue studying non-smooth series representation and how to improve accuracy
- Optimal combinations of smooth and non-smooth terms based on when Fourier series regains highest accuracy
- Mathematical properties amenable to addition in a harmonic balance code

Acknowledgements

New Mexico State University

Sandia National Laboratories—Laboratory-Directed Research and Development (LDRD)

São Paulo State University (UNESP)

Thank you for your attention!

Please ask any questions

This study describes objective technical results and analysis. Any subjective views or opinions that might be expressed in the paper do not necessarily represent the views of the U.S. Department of Energy or the United States Government. This article has been authored by an employee of National Technology & Engineering Solutions of Sandia, LLC under Contract No. DE-NA0003525 with the U.S. Department of Energy (DOE). The employee owns all right, title and interest in and to the article and is solely responsible for its contents. The United States Government retains and the publisher, by accepting the article for publication, acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this article or allow others to do so, for United States Government purposes. The DOE will provide public access to these results of federally sponsored research in accordance with the DOE Public Access Plan https://www.energy.gov/downloads/doe-public-access-plan. SAND2023-12312C. R. Vasconcellos acknowledges the financial support of the Brazilian agency CAPES (grant 88881.302889/2018-01).