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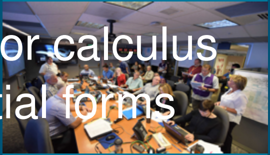
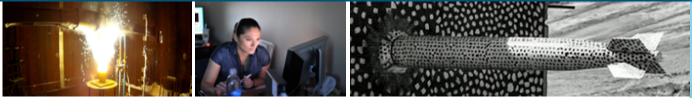
Progress towards a discrete exterior calculus for (vector) bundle-valued differential forms





Presented by:

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What is a (vector) bundle?

Vector Bundles E

- A vector space $V(x)$ attached to each point x of a $\mathscr{M}$
- Can define a *dual bundle* $E^\star$ to a vector bundle E
- Key spaces for continuum mechanics: *real bundle* $\mathbb{R}$, *pseudoscalar bundle* Ψ, *tangent bundle* T, *cotangent bundle* $T^\star$
- Can define *bundle metric* $\mathbf{g}_E$: gives an *inner product* on the bundle

Connections X_E

- Defines how to compare elements of a bundle at different x
- Connections are used to define derivatives, for example the *covariant derivative* ∇_{X}; also known as *parallel transport*
- Key connection for continuum mechanics: *Levi-Civita connection* for T on $\mathscr{M}$ (the unique torsion-free metric-compatible affine connection)
- Example: *affine connections* on T define the Christoffel symbols

What are (vector) bundle-valued differential forms?

(vector) bundle-valued differential form (BVDFs)

- $\mathbf{x}_E^k \in \Lambda^k(E)$ and $\tilde{\mathbf{x}}_E^k \in \tilde{\Lambda}^k(E)$: smooth section of the tensor product bundle of vector bundle E with the kth exterior power of the cotangent bundle $T^\star$
- Note $\tilde{\Lambda}^k(E) := \Lambda^k(E \otimes \Psi)$ (there are really only $\Lambda^k(E)$ forms)
- scalar-valued differential form (SVDFs) are just special cases of BVDFs with $E = \mathbb{R}$ or $E = \Psi$
  - $\Lambda^k := \Lambda^k(\mathbb{R})$
  - $\tilde{\Lambda}^k := \tilde{\Lambda}^k(\mathbb{R}) = \Lambda^k(\mathbb{R} \otimes \Psi) \approx \Lambda^k(\Psi)$

Why do SVDFs and BVDFs matter?

It turns out many physical quantities are best understood* as differential forms (see Tonti2013,Tonti2014,Gilbert2023,Eldred2023) ex. $\tilde{\rho}^n$, η^0, $\tilde{D}^2$, B^2, $\mathbf{u}^0_{\mathrm{T}}$, $\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}$

*under changes of coordinates and orientation, they transform as BVDFs

What is exterior calculus?

The calculus (integration, differentiation, products, etc.) of differential forms!

- SVDF exterior calculus is somewhat familiar:
  - exterior derivative d, Hodge star $\tilde{\star}$, wedge product $\wedge$, topological pairing $\langle\langle , \rangle\rangle$, inner product $\langle , \rangle$, Lie derivative L, flat $\flat$, sharp $\sharp$
- BVDF exterior calculus is less familiar, but same sort of operations with similar properties can be defined:
  - (covariant) exterior derivative d_X, Hodge star $\tilde{\star}$, topological pairing $\langle\langle , \rangle\rangle_\chi$, inner product $\langle , \rangle_\chi$, Trace $\mathbb{T}$, Inclusion $\mathbb{I}$, flat $\flat_1$, sharp $\sharp_1$
- BVDF exterior calculus reduces to SVDF exterior calculus when $E = \mathbb{R}$ or $E = \Psi$
- Exterior calculus is the natural language for developing *geometric mechanics formulations (variational, Hamiltonian, metriplectic, etc.)*
- It also underlies *mimetic* discretizations, such as *discrete exterior calculus* and *finite element exterior calculus*

What is (SVDF) discrete exterior calculus?

A discrete version of exterior calculus with the "same" properties!

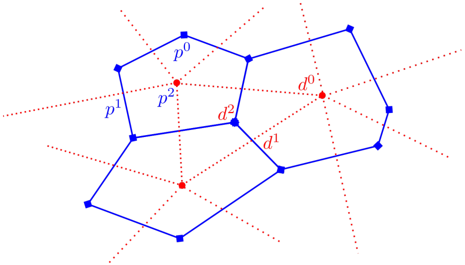
- See Hirani2003,Eldred2021,Eldred2022
- Specifically, a *Double deRham complex* method: a type of *mimetic* spatial discretization, ex. $\nabla \cdot \nabla \times = 0$, $\nabla \times \nabla = 0$, $\nabla^* = -\nabla \cdot$, etc.

$$
\begin{array}{ccccccc}
\mathbb{W}_0 & \xrightarrow[\;]{\vec{\nabla}} & \mathbb{W}_1 & \xrightarrow[\;]{\vec{\nabla}\times} & \mathbb{W}_2 & \xrightarrow[\;]{\vec{\nabla}\cdot} & \mathbb{W}_3 \\
 & d & & d & & d & \\
\downarrow * & & \downarrow * & & \downarrow * & & \downarrow * \\
\mathbb{W}^3 & \xleftarrow{\vec{\nabla}\cdot} & \mathbb{W}^2 & \xleftarrow{\vec{\nabla}\times} & \mathbb{W}^1 & \xleftarrow{\vec{\nabla}} & \mathbb{W}^0 \\
 & d & & d & & d &
\end{array}
$$

- introduce a pair of grids (straight and twisted), one for each type of differential form, with a *deRham complex* on each grid
- Connect complexes through the *Hodge star* $\tilde{\star}$
- Explicit *codifferential* δ operator defined using *exterior derivative* d and Hodge star $\tilde{\star}$: $\delta = (-1)^k \tilde{\star}\,\mathrm{d}\,\tilde{\star}$
- Explicit *inner product* $\langle , \rangle$ operator defined using Hodge star $\tilde{\star}$ and wedge product $\wedge$: $\langle a, b \rangle = \int a \wedge \tilde{\star} b$

Main ideas behind (SVDF) DEC

- Two grids that are topologically dual* (primal and dual i.e straight and twisted): 1-1 relationship between k-cells on one grid (0=points, 1=lines, 2=faces, 3=volumes) and $n-k$-cells on the other grid
- Discrete k-forms are real numbers associated with a k-cell
- Key operator is the Hodge star $\tilde{\star}$; uses 1-1 relationship between k-cells and $n-k$-cells; highly grid geometry specific (ex. Voronoi, barycentric, etc.)
- Inner product $\langle , \rangle$ and codifferential δ defined using Hodge star $\tilde{\star}$



*with boundaries things get more complicated but can still be done consistently, see Eldred2021

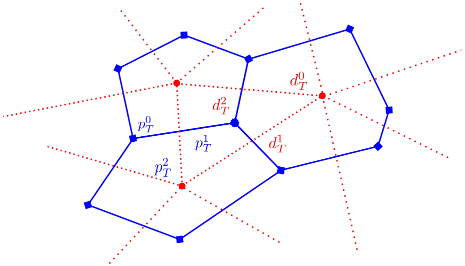
Approach to BVDF DEC

Approach: Retain main ideas from SVDF DEC

- Focus on $\mathbb{R}^n$, specifically tangent T and cotangent bundles $T^\star$
  - Metric **g** and connection X are trivial for T and $T^\star$ in $\mathbb{R}^n$; T and $T^\star$ have a position-independent, global basis
- Should reduce to SVDF DEC when $E = \mathbb{R}$ or $E = \Psi$
- Here we will show:
  - Discrete bundle-valued forms $\mathbf{x}_E^k$ and $\tilde{\mathbf{y}}_E^k$
  - (Covariant) exterior derivative d_{X}
  - Discrete wedge product $\dot{\wedge}$
  - Discrete Hodge star $\tilde{\star}$
  - Discrete pairings $\langle\langle , \rangle\rangle_\chi$ and $\langle , \rangle_\chi$
- Draws inspiration from Yavari2008, Angoshtari2013, Boom2022

Discrete bundle-valued forms $\mathbf{x}_E^k$ and $\tilde{\mathbf{y}}_E^k$

- Discrete scalar-valued forms x^k and $\tilde{y}^k$ are 1 real number attached to each k-cell (i.e. co-chains)
- Therefore, bundle-valued forms $\mathbf{x}_E^k$ and $\tilde{\mathbf{y}}_E^k$ are r real numbers attached to a k-cell, where r is the dimension of the vector space E, for T and $T^\star$ we have $r = n$
- Key here: "twistedness" is a property of the whole form, not the bundle or form part



Discrete covariant exterior derivative d_X

Covariant exterior derivative is

$$\mathrm{d}_\mathrm{X} : \Lambda^k(E) \to \Lambda^{k+1}(E) \tag{1}$$

The SVDF DEC discrete exterior derivatives $\mathbf{D}_k$ and $\bar{\mathbf{D}}_k$ are weighted $(-1, 1)$ sum of "nearest-neighbor" $k-1$ cells with weights given by orientations, ex. 1-forms in 2D:

$$(\mathbf{D}_2 x^1)_c = \sum_{e \in EC(c)} x_e^1 n_{ec} \tag{2}$$

Define BVDF DEC discrete exterior derivatives $\mathbf{D}_k^E$ and $\bar{\mathbf{D}}_k^E$ as *component-wise* versions of SVDF DEC operators, ex. T-valued 1-forms in 2D:

$$(\mathbf{D}_2^T \mathbf{x}_T^1)_{c,r} = \sum_{e \in EC(c)} (\mathbf{x}_T^1)_{e,r} n_{ec} \tag{3}$$

Relies on trivial connection $\mathrm{X}_T/\mathrm{X}_T^\star$ for T and $T^\star$ in $\mathbb{R}^3$

Discrete wedge products $\dot{\wedge}$ I

Continuous $\dot{\wedge}$ is

$$\dot{\wedge} : \Lambda^k(E_1), \Lambda^l(E_2) \rightarrow \Lambda^{k+l}(E_3) \tag{4}$$

where a *canonical trivialization* exists for $E_1 \otimes E_2 \rightarrow E_3$.

Important canonical trivializations

- $E \otimes E^\star \rightarrow \mathbb{R}$, $E^\star \otimes E \rightarrow \mathbb{R}$
- $E \otimes \mathbb{R} \rightarrow E$ (special case $\mathbb{R} \otimes \mathbb{R} \rightarrow \mathbb{R}$)
- $\Psi \otimes \Psi \rightarrow \mathbb{R}$
- $\mathbb{R} \otimes \Psi \rightarrow \Psi$, $\Psi \otimes \mathbb{R} \rightarrow \Psi$

These trivializations exist independent of the choice of a bundle metric $\mathbf{g}_E$.

Discrete wedge products $\dot{\wedge}$ II

- Here we focus on $\dot{\wedge}$ for E-valued straight k-forms and $E^\star$-valued twisted $n-k$-forms, which makes $E_3 = \mathbb{R}$-valued twisted n-forms
- The SVDF DEC $\wedge$ for this case is just scalar multiplication on each k-cell, relying on the duality between k-forms and $n-k$-forms of opposite type

$$x^k \wedge_h \tilde{y}^{n-k} := x^k \tilde{y}^{n-k} \tag{5}$$

- The BVDF DEC definition is just the natural extension of this: perform a scalar product between the r components of T and the r components of $T^\star$ at each k-cell

$$\mathbf{x}_T^k \dot{\wedge}_h \tilde{\mathbf{y}}_{T^\star}^{n-k} := \mathbf{x}_T^k \cdot \tilde{\mathbf{y}}_{T^\star}^{n-k} \tag{6}$$

- This definition of $\dot{\wedge}$ relies on trivial metric $\mathbf{g}_T/\mathbf{g}_T^\star$ for T and $T^\star$ in $\mathbb{R}^3$

Discrete Hodge star $\tilde{\star}$ I

The Hodge star is:

$$\tilde{\star} : \Lambda^k(E) \rightarrow \tilde{\Lambda}^{n-k}(E^\star) \tag{7}$$

The SVDF DEC Hodge stars $\mathbf{H}_k$ and $\bar{\mathbf{H}}_k$ are defined as weighted sum of $(n-k)$ cells for some *sparse* set of weights $H_{k,\widetilde{n-k}}$ or $H_{n-k,\tilde{k}}$, ex $\mathbf{H}_k$:

$$(\mathbf{H}_k x^k)_k = \sum_{\widetilde{n-k}} \tilde{x}^{n-k}_{\widetilde{n-k}} H_{k,\widetilde{n-k}} \tag{8}$$

- Uses duality between k-form and $(n-k)$-form on opposite grids.
- Many different Hodge stars in the literature ex. *Voronoi* (also known as *diagonal* or *circumcentric*), *Galerkin*, *barycentric*. Choice imposes restrictions on the grid geometry and/or topology.

Discrete Hodge star $\tilde{\star}$ II

Just as for covariant exterior derivative, define BVDF DEC Hodge star $\mathbf{H}_k^E$ and $\bar{\mathbf{H}}_k^E$ using SVDF DEC Hodge star component-wise, ex $\mathbf{H}_k^T$:

$$(\mathbf{H}_k^T \mathbf{x}_T^k)_{k,r} = \sum_{\widetilde{n-k}} (\tilde{\mathbf{x}}_{T^\star}^{n-k})_{\widetilde{n-k},r} H_{k,\widetilde{n-k}} \tag{9}$$

- Relies on trivial metric $\mathbf{g}_T/\mathbf{g}_T^\star$ for T and $T^\star$ in $\mathbb{R}^n$
- This will inherit all of the key properties of the SVDF DEC Hodge star, such as Symmetric Positive Definiteness

Often, define some Hodge stars implicitly (requires invertible Hodge stars) such that

$$\bar{\mathbf{H}}_k \mathbf{H}_{n-k} = (-1)^{k(n-k)} \mathbf{I} \tag{10}$$

which is discrete analogue of $\tilde{\star}\tilde{\star}(-1)^{k(n-k)}$.

Discrete topological pairing (Poincaré duality) $\langle\langle , \rangle\rangle_\chi$

Continuous definition is in terms of $\dot\wedge$ and $\int$:

$$\left\langle\left\langle \mathbf{a}^k_E, \tilde{\mathbf{b}}^{n-k}_{E^\star} \right\rangle\right\rangle_\chi = \int \mathbf{a}^k_E \,\dot\wedge\, \tilde{\mathbf{b}}^{n-k}_{E^\star} \tag{11}$$

The SVDF DEC definition (based on $\wedge$ from above) is:

$$\left\langle\left\langle a^k, \tilde{b}^{n-k} \right\rangle\right\rangle := \sum_k a^k_k \tilde{b}^{n-k}_{n-k} \tag{12}$$

$$\left\langle\left\langle \tilde{a}^k, b^{n-k} \right\rangle\right\rangle := \sum_k (-1)^{k(n-k)} \tilde{a}^k_k b^{n-k}_{n-k} \tag{13}$$

The BVDF DEC definition (based on $\dot\wedge$ from above) is:

$$\left\langle\left\langle \mathbf{a}^k_T, \tilde{\mathbf{b}}^{n-k}_{T^\star} \right\rangle\right\rangle_\chi := \sum_k (\mathbf{a}^k_T)_k \cdot (\tilde{\mathbf{b}}^{n-k}_{T^\star})_{n-k} \tag{14}$$

$$\left\langle\left\langle \tilde{\mathbf{a}}^k_T, \mathbf{b}^{n-k}_{T^\star} \right\rangle\right\rangle_\chi := \sum_k (-1)^{k(n-k)} (\tilde{\mathbf{a}}^k_T)_k \cdot (\mathbf{b}^{n-k}_{T^\star})_{n-k} \tag{15}$$

Discrete inner product $\langle , \rangle_\chi$

Continuous definition is:

$$\left\langle \mathbf{a}_E^k, \mathbf{b}_E^k \right\rangle_\chi = \int \mathbf{a}_E^k \dot{\wedge} \tilde{\star} \mathbf{b}_E^k \qquad \left\langle \tilde{\mathbf{a}}_E^k, \tilde{\mathbf{b}}_E^k \right\rangle_\chi = \int \tilde{\mathbf{a}}_E^k \dot{\wedge} \tilde{\star} \tilde{\mathbf{b}}_E^k \tag{16}$$

The SVDF DEC definition (based on $\wedge$ and $\mathbf{H}_k/\bar{\mathbf{H}}_k$ from above) is:

$$\left\langle x^k, y^k \right\rangle := (x^k)^T \mathbf{H}_k y^k, \qquad \left\langle \tilde{x}^k, \tilde{y}^k \right\rangle := (-1)^{k(n-k)} (\tilde{x}^k)^T \bar{\mathbf{H}}_k \tilde{y}^k \tag{17}$$

The BVDF DEC definition (based on $\dot{\wedge}$ and $\mathbf{H}_k^E/\bar{\mathbf{H}}_k^E$ from above) is:

$$\left\langle \mathbf{x}_T^k, \mathbf{y}_T^k \right\rangle_\chi := (\mathbf{x}_T^k)^T \mathbf{H}_k^T \mathbf{y}_T^k, \qquad \left\langle \tilde{\mathbf{x}}_T^k, \tilde{\mathbf{y}}_T^k \right\rangle_\chi := (-1)^{k(n-k)} (\tilde{\mathbf{x}}_T^k)^T \bar{\mathbf{H}}_k^T \tilde{\mathbf{y}}_T^k \tag{18}$$

with similar definitions for $T^\star$-valued forms.

Conclusions and Future Work

Summary

- Extended discrete exterior calculus to (vector) bundle-valued differential forms
  - Focused on fundamental exterior calculus operators: d_{X}, $\tilde{\star}$, $\dot{\wedge}$, $\langle , \rangle_{\chi}$, $\langle\langle , \rangle\rangle_{\chi}$
  - For $\mathbb{R}^3$, where tangent and cotangent bundles are flat and a global uniform basis exists

Future Work

- Transport operators for arbitrary BVDFs i.e. Lie derivatives $\mathrm{L}_{\mathbf{u}_{\mathrm{T}}^{0}}$, interior products i, wedge products $\wedge$; and associated raising/lowering operators: $\mathbb{T}$, $\mathbb{I}$, $\flat_1$, $\sharp_1$, $\flat$, $\sharp$
- Extension to arbitrary manifolds i.e. non-flat bundles: will require a discrete connection X
- Application to momentum-based formulations of fluids, especially charged fluid models

Questions?

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Extra Slides

Continuum mechanics models

Develop continuum mechanics models (CMMs) using geometric mechanics formulations (GM, ex. variational Hamiltonian, metriplectic, etc.):

$$\delta \int \mathscr{L}[\mathbf{x}] = 0 \qquad \frac{\partial \mathbf{x}}{\partial t} = \mathbb{J}(\mathbf{x}) \frac{\delta \mathscr{H}}{\delta \mathbf{x}} + \mathbb{M}(\mathbf{x}) \frac{\delta \mathscr{S}}{\delta \mathbf{x}}$$

Predicted quantities $\mathbf{x}$ used to build CMMs: momentum, velocity, stress, (mass) density, entropy, electromagnetic fields, etc.

Fundamental questions:

(1) What types of mathematical objects should be used to represent $\mathbf{x}$?

(2) What mathematical language should be used to build CMMs?

Traditional answers: (1) scalars, vectors and tensors (2) vector/tensor calculus

Limitations of vector/tensor calculus

Use of vector/tensor calculus starts to break down when considering:

- arbitrary manifolds and dimensions
- coordinate system independent expressions

Additionally, transport behaviour is tricky and unintuitive:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad \frac{\partial \eta}{\partial t} + \mathbf{u} \cdot \nabla \eta = 0$$

$$\frac{\partial \mathbf{m}}{\partial t} + \nabla \mathbf{m} \cdot \mathbf{u} + \mathbf{m} \cdot \nabla \mathbf{u} + \mathbf{m} \nabla \cdot \mathbf{u} = 0 \qquad \frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{u}) + \mathbf{u} \nabla \cdot \mathbf{B} = 0$$

How do we resolve these issues?

Use exterior calculus instead but keep GM formulations:
(1) physical quantities = differential forms (2) language = exterior calculus

What is exterior calculus?

The calculus of differential forms!

Integration, differentiation, etc.

Why use differential forms?

Strong argument (see Tonti2013, Tonti2014) that physical quantities should be associated with (oriented) geometric entities (=differential forms). See also Frankel2011, Kanso2007, Gilbert2023.

What is a differential form?

- **scalar-valued differential form (SVDFs)** $x^k \in \Lambda^k$ and $\tilde{x}^k \in \tilde{\Lambda}^k$: smooth section of the kth exterior power of the cotangent bundle $T^\star$; ex. $\tilde{\rho}^n$, η^0, $\tilde{D}^2$, B^2, ...
- **(vector) bundle-valued differential form (BVDFs)** $\mathbf{x}_E^k \in \Lambda^k(E)$ and $\tilde{\mathbf{x}}_E^k \in \tilde{\Lambda}^k(E)$: smooth section of the tensor product bundle of vector bundle E with the kth exterior power of the cotangent bundle $T^\star$, ex. $\mathbf{u}_\mathrm{T}^0$, $\tilde{\mathbf{m}}_{\mathrm{T}^\star}^\mathrm{n}$, ...
  - Typical vector bundles: $\mathbb{R}$ (=Λ^k), Ψ (=$\tilde{\Lambda}^k$), T, $T^\star$

How are "vectors" represented in exterior calculus?

- There are four "vector proxies" in exterior calculus: $\mathbf{x}^0_T$, $\tilde{\mathbf{x}}^n_{T\star}$, x^1 and $\tilde{x}^{n-1}$
- Related through various operations such as $i_{\mathbf{u}^0_T}$, $\tilde{\star}$, $\flat/\flat_1$, $\sharp/\sharp_1$, etc. using volume form $\tilde{\mu}^n$
- Look the "same" in vector calculus in $\mathbb{R}^3$, distinct in exterior calculus: source of much confusion
- All play a key role in geometric mechanics formulations, as various measures of fluid flow:
  - $\mathbf{u}^0_T$ = convective velocity (Euler-Poincaré and Lie-Poisson)
  - $\tilde{\mathbf{m}}^n_{T\star} = \frac{\delta \mathscr{L}}{\delta \mathbf{u}^0_T}$ = momentum (Euler-Poincaré and Lie-Poisson)
  - v^1 = absolute velocity? circulation velocity? etc. (Curl-Form)
  - $\mathbf{F}^0_T = \frac{\delta \mathscr{H}}{\delta v^1}$ = mass flux (Curl-Form)
- This fits with the discuss in Tonti2013/Tonti2014 about the dual nature of velocity, see for example FLU3 (=SVDFs) vs. FLU6 (=BVDFs) in Tonti2014
  - Connects with question of what $\rho\mathbf{u}$ is? Mass flux or momentum density?

Also have "pseudovector proxies": $\tilde{\mathbf{x}}^0_T$, $\mathbf{x}^n_{T\star}$, $\tilde{x}^1$ and x^{n-1}, not discussed here

Operators on SVDFs and BVDFs

- Covariant exterior derivative d_{X}, reduces to d for SVDFs and ∇_{X} for vector fields
- Covariant wedge product $\dot{\wedge}$, reduces to $\wedge$ for SVDFs
- Covariant Hodge star $\tilde{\star}$, reduces to $\tilde{\star}$ for SVDFs
- Inner product $\langle , \rangle_{\chi}$, reduces to $\langle , \rangle$ for SVDFs
- Topological pairing $\langle\langle , \rangle\rangle_{\chi}$, reduces to $\langle\langle , \rangle\rangle$ for SVDFs
- Lie derivative $\mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}$, reduces to Lie bracket for vector fields
- Diamond operator $\diamond$, (formal) adjoint of Lie derivative

$$\left\langle\left\langle \tilde{a}^{n-k}, \mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}} b^k \right\rangle\right\rangle = -\left\langle \mathbf{u}^0_{\mathrm{T}}, \tilde{a}^{n-k} \diamond b^k \right\rangle_{\chi} \quad (19)$$

$$\left\langle\left\langle \tilde{\mathbf{a}}^{n-k}_{E^\star}, \mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}} \mathbf{b}^k_E \right\rangle\right\rangle_{\chi} = -\left\langle \mathbf{u}^0_{\mathrm{T}}, \tilde{\mathbf{a}}^{n-k}_{E^\star} \diamond \mathbf{b}^k_E \right\rangle_{\chi} \quad (20)$$

$\diamond : \tilde{a}^{n-k}, b^k \rightarrow \tilde{\mathbf{x}}^n_{T^\star}$ and $\tilde{\mathbf{a}}^{n-k}_{E^\star}, \mathbf{b}^k_E \rightarrow \tilde{\mathbf{x}}^n_{T^\star}$

Geometric Mechanics Formulations

Assume fluid can be characterized by:

- velocity $\mathbf{u}^0_\mathrm{T} \in \Lambda^0(T)$
- an arbitrary number of *simple* advected SVDF's $a \in \{\Lambda^k, \tilde{\Lambda}^k\}$ and BVDF's $\mathbf{b} \in \{\Lambda^k(T), \Lambda^k(T^\star), \tilde{\Lambda}^k(T), \tilde{\Lambda}^k(T^\star)\}$:

$$\frac{\partial a}{\partial t} + \mathrm{L}_{\mathbf{u}^0_\mathrm{T}} a = 0 \qquad \frac{\partial \mathbf{b}}{\partial t} + \mathrm{L}_{\mathbf{u}^0_\mathrm{T}} \mathbf{b} = 0 \tag{21}$$

Dynamics are given by *semi-direct product theory* (special case of *matched pair dynamics* for simple advected quantities):

- Euler-Poincaré (Variational) Formulation
- Lie-Poisson (Hamiltonian) Formulation
- Curl-Form (Hamiltonian) Formulation (not shown)

Euler-Poincaré (Variational) Formulations

Lagrangian $\mathscr{L}[\mathbf{u}^0_\mathrm{T}, a, \mathbf{b}]$ and Action $\mathscr{S}[\mathbf{u}^0_\mathrm{T}, a, \mathbf{b}] = \int_{t_1}^{t_2} \mathscr{L}$

$$\delta \mathscr{S} = \delta \int_{t_1}^{t_2} \mathscr{L} = 0 \tag{22}$$

subject to the constraints

$$\delta \mathbf{u}^0_\mathrm{T} = \partial_t \zeta^0_\mathrm{T} + \mathrm{L}_{\zeta^0_\mathrm{T}} \mathbf{u}^0_\mathrm{T} \tag{23}$$

$$\delta a = -\mathrm{L}_{\zeta^0_\mathrm{T}} a \tag{24}$$

$$\delta \mathbf{b} = -\mathrm{L}_{\zeta^0_\mathrm{T}} \mathbf{b} \tag{25}$$

Introducing the momentum $\tilde{\mathbf{m}}^\mathrm{n}_{\mathrm{T}^\star} = \frac{\delta \mathscr{L}}{\delta \mathbf{u}^0_\mathrm{T}} \in \tilde{\Lambda}^n(T^\star)$, the (constrained) variational principle (22) gives

$$\frac{\partial}{\partial t} \tilde{\mathbf{m}}^\mathrm{n}_{\mathrm{T}^\star} + \mathrm{L}_{\mathbf{u}^0_\mathrm{T}} \tilde{\mathbf{m}}^\mathrm{n}_{\mathrm{T}^\star} - \sum \frac{\delta \mathscr{L}}{\delta a} \diamond a - \sum \frac{\delta \mathscr{L}}{\delta \mathbf{b}} \diamond \mathbf{b} = 0 \tag{26}$$

This is basically Holm1998, written in the language of BVDFs. 1-form density = covector-valued twisted volume form.

Lie-Poisson (Hamiltonian) Formulations

Use Legendre transform (assume invertible) to go from $\mathbf{u}_{\mathrm{T}}^{0}$ to $\tilde{\mathbf{m}}_{\mathrm{T}\star}^{\mathrm{n}}$

$$\mathrm{H}[\tilde{\mathbf{m}}_{\mathrm{T}\star}^{\mathrm{n}}, a, \mathbf{b}] = \langle\langle \tilde{\mathbf{m}}_{\mathrm{T}\star}^{\mathrm{n}}, \mathbf{u}_{\mathrm{T}}^{0} \rangle\rangle_{\chi} - \mathscr{L}[\mathbf{u}_{\mathrm{T}}^{0}, a, \mathbf{b}] \tag{27}$$

The functional derivatives of H are

$$\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}_{\mathrm{T}\star}^{\mathrm{n}}} = \mathbf{u}_{\mathrm{T}}^{0} \qquad \frac{\delta \mathrm{H}}{\delta a} = -\frac{\delta \mathscr{L}}{\delta a} \qquad \frac{\delta \mathrm{H}}{\delta \mathbf{b}} = -\frac{\delta \mathscr{L}}{\delta \mathbf{b}} \tag{28}$$

Thus we can write the Euler-Poincaré (26) and transport (21) equations as

$$\frac{\partial}{\partial t} \tilde{\mathbf{m}}_{\mathrm{T}\star}^{\mathrm{n}} + \mathrm{L}_{\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}_{\mathrm{T}\star}^{\mathrm{n}}}} \tilde{\mathbf{m}}_{\mathrm{T}\star}^{\mathrm{n}} + \sum \frac{\delta \mathrm{H}}{\delta a} \diamond a + \sum \frac{\delta \mathrm{H}}{\delta \mathbf{b}} \diamond \mathbf{b} = 0 \tag{29}$$

$$\frac{\partial a}{\partial t} + \mathrm{L}_{\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}_{\mathrm{T}\star}^{\mathrm{n}}}} a = 0 \tag{30}$$

$$\frac{\partial \mathbf{b}}{\partial t} + \mathrm{L}_{\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}_{\mathrm{T}\star}^{\mathrm{n}}}} \mathbf{b} = 0 \tag{31}$$

Velocity v^1 and Kelvin-Noether Form

Consider $\mathrm{v}^1 =$ instead of $\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}\star}$ Define a momentum straight 1-form m^1 as

$$\mathrm{m}^1 = (\tilde{\star}\,\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}\star})^{\flat} = \mathbb{T}\,\mathbf{m}^0_{\mathrm{T}\star} \tag{32}$$

where $\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}\star} = \mathbf{m}^0_{\mathrm{T}\star} \dot{\wedge} \tilde{\mu}^n$. Note that $\mathbf{m}^0_{\mathrm{T}\star} = \sharp_1 \tilde{\star}\,\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}\star}$.
Then assume the existence of a total mass density twisted n-form $\tilde{D}^n$ that is a linear combination of advected densities $\tilde{a}^n$:

$$\tilde{D}^n = \sum c_i \tilde{a}^n_i \tag{33}$$

with associated straight 0-form $D^0 = \tilde{\star}\,\tilde{D}^n$.
Using m^1 and D^0, the velocity straight 1-form v^1 is

$$\mathrm{v}^1 = \frac{1}{D^0} \wedge (\tilde{\star}\,\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}\star})^{\flat} = \frac{1}{D^0} \wedge \mathrm{m}^1 \tag{34}$$

Algebra yields *Kelvin-Noether form* of the Euler-Poincaré equations (26)

$$D^0 \wedge \left(\frac{\partial}{\partial t} \mathrm{v}^1 + \mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}\, \mathrm{v}^1 \right) - \sum \frac{\delta \mathscr{L}}{\delta a} \diamond_{\mathrm{kn}} a - \sum \frac{\delta \mathscr{L}}{\delta \mathbf{b}} \diamond_{\mathrm{kn}} \mathbf{b} = 0 \tag{35}$$

Curl-Form[1] (Hamiltonian) Formulation I

Let $\mathscr{H}[\mathrm{v}^1, a, \mathbf{b}] = \mathrm{H}[\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}\star}, a, \mathbf{b}]$, use chain rule along with (34) to get

$$\frac{\delta \mathscr{H}}{\delta \mathrm{v}^1} = \tilde{\star}\left[\frac{1}{D^0} \wedge \left(\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}\star}}\right)^{\flat}\right] = \frac{1}{D^0} \wedge \tilde{\star}\left(\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}\star}}\right)^{\flat} = \mathrm{i}_{\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}\star}}} \tilde{D}^n \tag{36}$$

$$\frac{\delta \mathscr{H}}{\delta \tilde{a}^n} = \frac{\delta \mathrm{H}}{\delta \tilde{a}^n} + \frac{\partial \tilde{D}^n}{\partial \tilde{a}^n} \wedge \mathrm{i}_{\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}\star}}} \mathrm{v}^1 \qquad \frac{\delta \mathscr{H}}{\delta a} = \frac{\delta \mathrm{H}}{\delta a} \qquad \frac{\delta \mathscr{H}}{\delta \mathbf{b}} = \frac{\delta \mathrm{H}}{\delta \mathbf{b}} \tag{37}$$

The equations of motion for advective velocity $\mathbf{U}^0_{\mathrm{T}} = \left[\tilde{\star}\left(\frac{1}{D^0} \wedge \frac{\delta \mathscr{H}}{\delta \mathrm{v}^1}\right)\right]^{\sharp} = \frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}\star}}$ are:

$$\frac{\partial \mathrm{v}^1}{\partial t} + \mathrm{i}_{\mathbf{U}^0_{\mathrm{T}}} \mathrm{d}\,\mathrm{v}^1 + \sum \frac{1}{D^0} \wedge \left[\frac{\delta \mathscr{H}}{\delta a} \diamond_{\mathrm{kn}} a\right] - \sum \frac{1}{D^0} \wedge \left[\frac{\delta \mathscr{H}}{\delta \mathbf{b}} \diamond_{\mathrm{kn}} \mathbf{b}\right] = 0 \tag{38}$$

$$\frac{\partial a}{\partial t} + \mathrm{L}_{\mathbf{U}^0_{\mathrm{T}}} a = 0 \tag{39}$$

$$\frac{\partial \mathbf{b}}{\partial t} + \mathrm{L}_{\mathbf{U}^0_{\mathrm{T}}} \mathbf{b} = 0 \tag{40}$$

[1] Also known as *vector-invariant* or *Carter-Licnerowicz*

Curl Form (Hamiltonian) Formulation II

Alternatively, for mass flux $\mathbf{F}^0_\mathrm{T} = \left[\tilde{\star}\frac{\delta\mathscr{H}}{\delta \mathrm{v}^1})\right]^\sharp$ the equations are:

$$\frac{\partial \mathrm{v}^1}{\partial t} + \mathrm{i}_{\mathbf{F}^0_\mathrm{T}} \frac{\mathrm{d}\,\mathrm{v}^1}{D^0} + \sum \frac{1}{D^0} \wedge \left[\frac{\delta\mathscr{H}}{\delta a} \diamond_{\mathrm{kn}} a\right] - \sum \frac{1}{D^0} \wedge \left[\frac{\delta\mathscr{H}}{\delta \mathbf{b}} \diamond_{\mathrm{kn}} \mathbf{b}\right] = 0 \tag{41}$$

$$\frac{\partial a}{\partial t} + \mathrm{L}_{\mathbf{F}^0_\mathrm{T}} \frac{a}{D^0} = 0 \tag{42}$$

$$\frac{\partial \mathbf{b}}{\partial t} + \mathrm{L}_{\mathbf{F}^0_\mathrm{T}} \frac{\mathbf{b}}{D^0} = 0 \tag{43}$$

Scalar/Vector Proxies- SVDFs I

Let's count the number of degrees of freedom for various scalar-valued differential forms and group them, for $n = 3$ (similar results for other n)

Scalar-Valued Forms (Straight or Twisted)

Form	Tensor Equivalent	Number of Dofs	Proxy Type	Examples
x^0	$(0, 0_{AS})$	1	Scalar	T, η
x^1	$(0, 1_{AS})$	3	Vector	$\mathbf{v}$, $\mathbf{H}$, $\mathbf{E}$
x^2	$(0, 2_{AS})$	3	Vector	$\mathbf{F}$, $\mathbf{B}$, $\mathbf{D}$
x^3	$(0, 3_{AS})$	1	Scalar	ρ

How do we convert between scalars/vectors and various proxies?

Use wedge products $\wedge$, interior products i, Hodge stars $\tilde{\star}$, flat $\flat$ and sharp $\sharp$ operators

Scalar/Vector Proxies- SVDFs II

Given vector field $\mathbf{x}^1_T$ (i.e. straight vector-valued 0-form), can get straight 1-form x^1 (circulation) and twisted $n-1$-form $\tilde{x}^{n-1}$ (flux):

$$x^1 = (\mathbf{x}^0_T)^\flat \tag{44}$$

$$\tilde{x}^{n-1} = \tilde{\star} x^1 = \mathrm{i}_{\mathbf{x}^0_T} \tilde{\mu}^n \tag{45}$$

Given pseudovector field $\tilde{\mathbf{x}}^1_T$ (i.e. twisted vector-valued 0-form), can get twisted 1-form $\tilde{x}^1$ (circulation) and straight $n-1$-form x^{n-1} (flux):

$$\tilde{x}^1 = (\tilde{\mathbf{x}}^0_T)^\flat \tag{46}$$

$$x^{n-1} = \tilde{\star} \tilde{x}^1 = \mathrm{i}_{\tilde{\mathbf{x}}^0_T} \tilde{\mu}^n \tag{47}$$

Given scalar x^0 (i.e. straight 0-form), can get twisted volume form $\tilde{x}^n$ (density):

$$\tilde{x}^n = \tilde{\star} x^0 = x^0 \wedge \tilde{\mu}^n \tag{48}$$

Given pseudoscalar $\tilde{x}^0$ (i.e. twisted 0-form), can get straight volume form x^n (density):

$$x^n = \tilde{\star} \tilde{x}^0 = \tilde{x}^0 \wedge \tilde{\mu}^n \tag{49}$$

Vector/Tensor Proxies- BVDFs I

Let's count the number of degrees of freedom for various bundle-valued differential forms and group them, for $n = 3$ (similar results for other n)

Bundle-Valued Forms (Straight or Twisted, T or $T^\star$)

Form	Tensor Equivalent	Number of Dofs	Proxy Type	Examples
$\mathbf{x}^0_T$	$(1, 0_{AS})$	3	Vector	$\mathbf{u}$
$\mathbf{x}^1_T$	$(1, 1_{AS})$	9	2-Tensor	
$\mathbf{x}^2_T$	$(1, 2_{AS})$	9	2-Tensor	τ
$\mathbf{x}^3_T$	$(1, 3_{AS})$	3	Vector	$\mathbf{m}$

How do we convert between vectors/tensors and various proxies?

Use covariant wedge products $\dot\wedge$, covariant Hodge stars $\tilde\star$, covariant flat $\flat_1$, covariant sharp $\sharp_1$, trace $\mathbb{T}$ and inclusion $\mathbb{I}$ operators

Vector/Tensor Proxies- BVDFs II

Given a straight vector-valued 0-form $\mathbf{x}_T^1$, can get straight 1-form x^1 (circulation), the twisted $n-1$-form $\tilde{x}^{n-1}$ (flux) and the twisted covector-valued n-form $\tilde{\mathbf{x}}_{T^\star}^n$:

$$
\begin{aligned}
x^1 &= (\mathbf{x}_T^0)^\flat && (50)\\
\tilde{x}^{n-1} &= \tilde{\star} x^1 = \mathrm{i}_{\mathbf{x}_T^0} \tilde{\mu}^n && (51)\\
\tilde{\mathbf{x}}_{T^\star}^n &= \tilde{\star} \mathbf{x}_T^1 = \tilde{\star}(x^1)^\sharp && (52)
\end{aligned}
$$

The twisted covector-valued n-form $\tilde{\mathbf{x}}_{T^\star}^n$ can be decomposed into a straight covector-valued 0-form $\mathbf{x}_{T^\star}^0$ and the volume form $\tilde{\mu}^n$ as

$$\tilde{\mathbf{x}}_{T^\star}^n = \mathbf{x}_{T^\star}^0 \dot{\wedge} \tilde{\mu}^n \quad (53)$$

Using the trace/inclusion operators, the following relationships hold

$$x^1 = \mathbb{T} \mathbf{x}_{T^\star}^0 \leftrightarrow \mathbf{x}_{T^\star}^0 = \mathbb{I} x^1 \quad (54)$$

$$\tilde{x}^{n-1} = \mathbb{T}^\star \tilde{\mathbf{x}}_{T^\star}^n \leftrightarrow \tilde{\mathbf{x}}_{T^\star}^n = \mathbb{I}^\star \tilde{x}^{n-1} \quad (55)$$