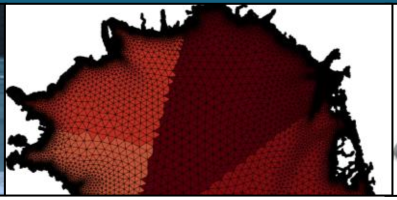
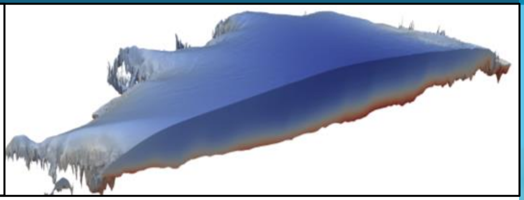
SAND2023-02452C

An Optimal Modal Finite-Element Discretization For Ice-Sheet Modeling

Center for Computing Research

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CFC, Cannes, April 27, 2023

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Talk Outline

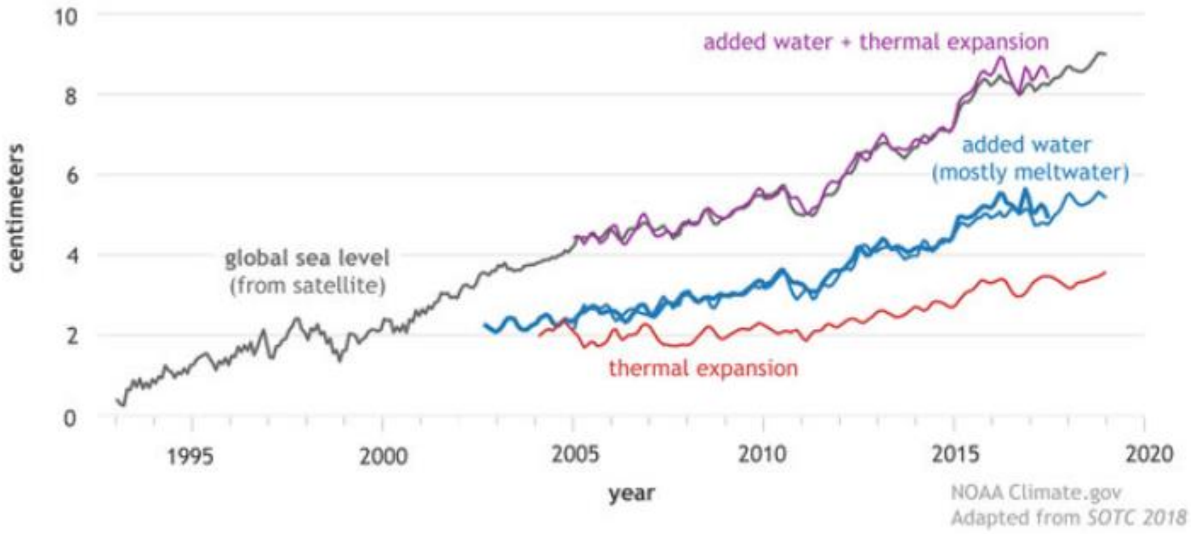
- Brief motivation and introduction to ice sheet equations
- MALI ice sheet model
- Reduced-order model based on prescribing the vertical profile from the Shallow Ice Approximation (SIA)
- Improvement of reduced-order model by optimally selecting the vertical modes

Supported by US DOE Office of Science projects:

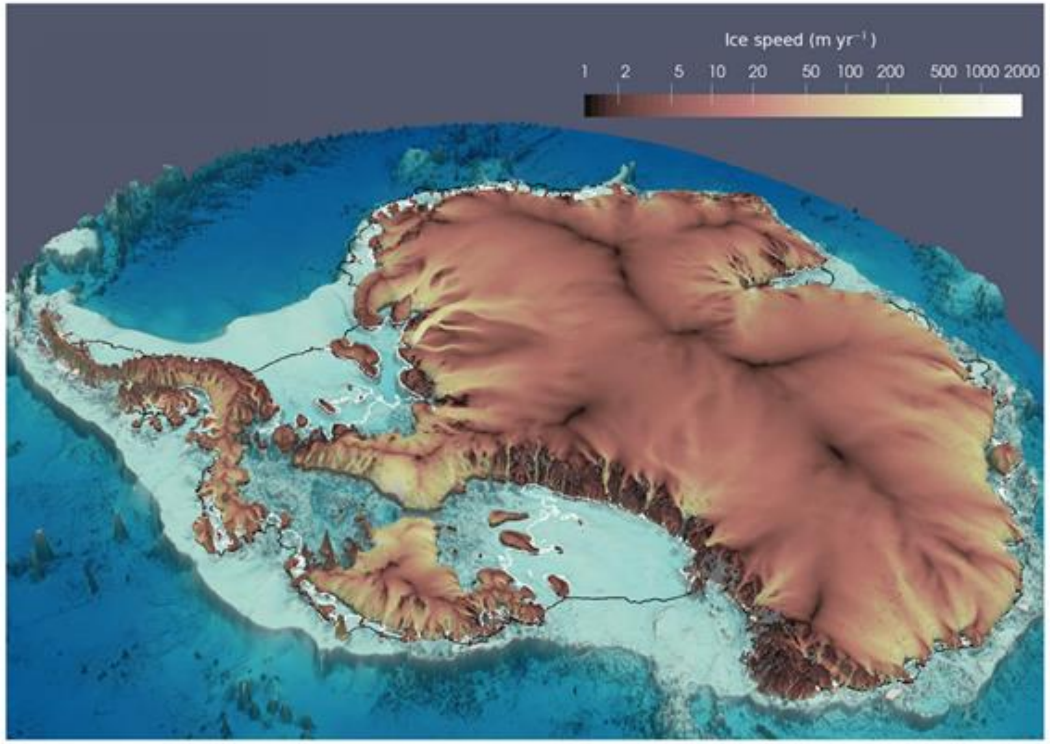
- ***FAnSSIE**: Framework for Antarctic System Science in E3SM*
- ***FASTMath**: Frameworks, Algorithms and Scalable Technologies for Mathematics*
- ***E3SM**: Energy Exascale Earth System Model*

Goal: probabilistic predictions of sea level rise

Contributors to global sea level rise (1993-2018):



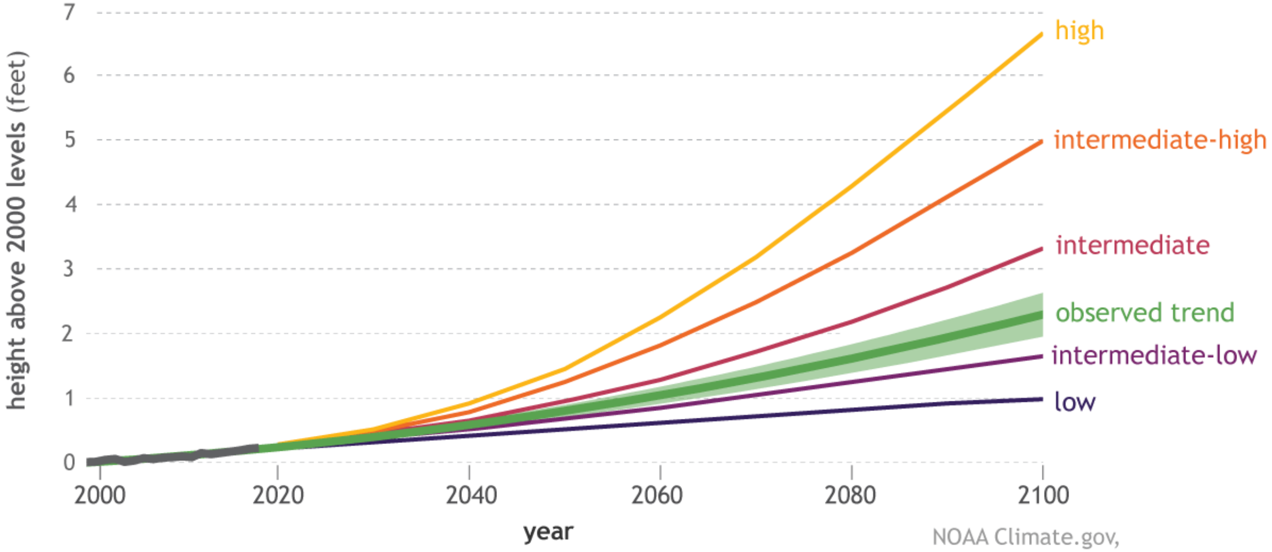
NOAA Climate.gov
Adapted from *SOTC 2018*



Simulation of Antarctic ice sheet as a consequence of extreme (unrealistic) climate forcing that induces collapse of ice shelves

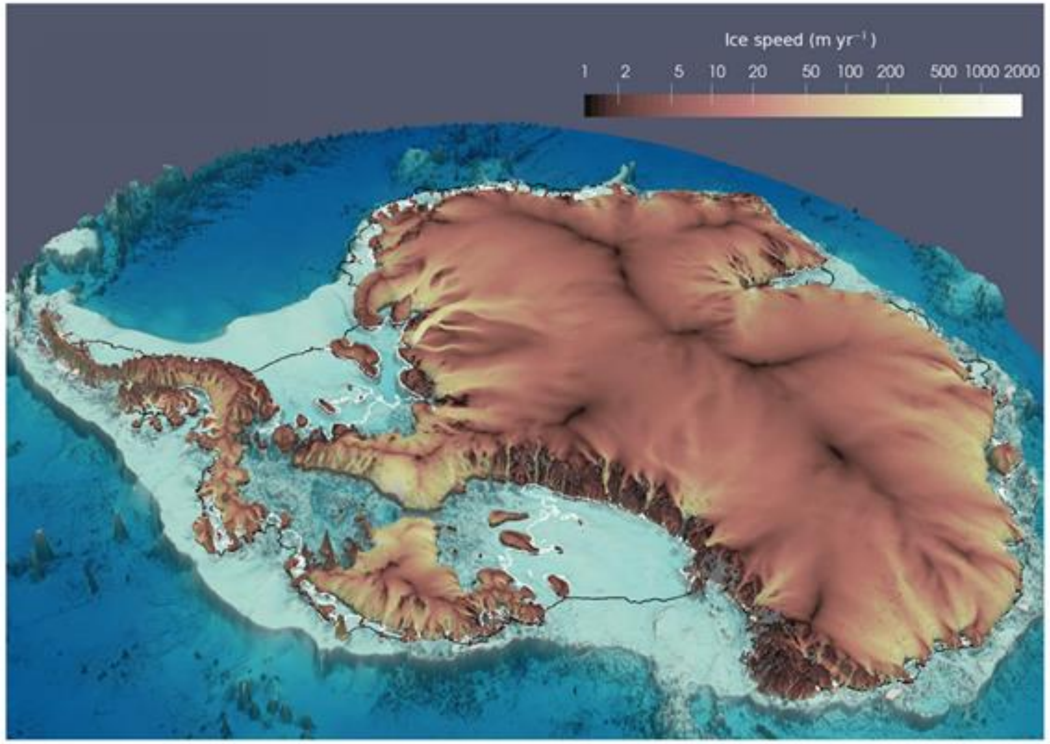
Goal: probabilistic projections of sea level rise

Possible pathways for future sea-level rise



NOAA Climate.gov,
adapted from Sweet et al., 2022

Note: regional sea-level rise can significantly exceed the global mean sea level rise in some areas (e.g. in the Gulf of Mexico).



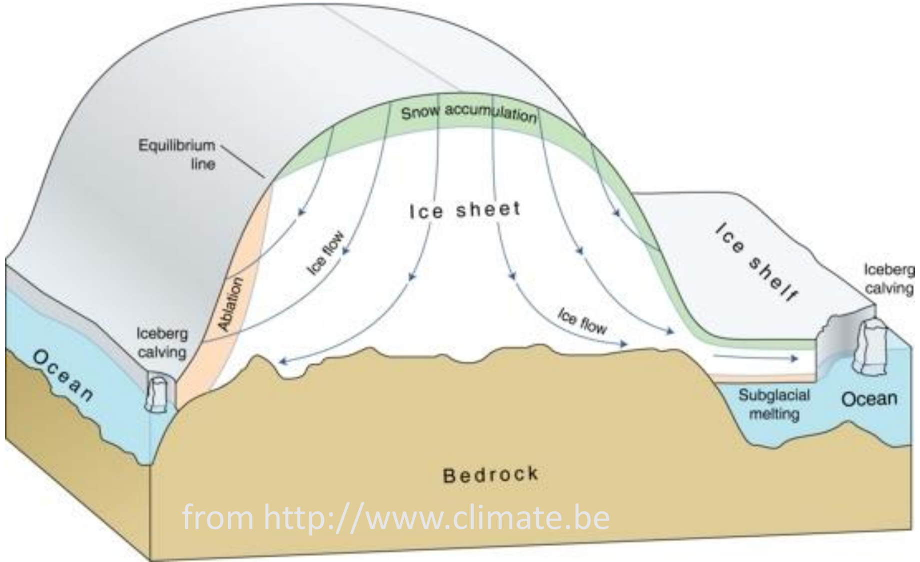
Simulation of Antarctic ice sheet as a consequence of extreme (unrealistic) climate forcing that induces collapse of ice shelves

Brief Motivation an basic physics

- Modeling ice sheets (Greenland and Antarctica) dynamics is essential to provide estimates for sea-level rise in next decades to centuries.

- Ice behaves like a very viscous shear-thinning fluid (similar to lava flow) driven by gravity.

- Inference problems to calibrate the model and uncertainty quantification (UQ) to determine how uncertainties in the data and the model affect projections of sea-level rise are major challenges and require several evaluation of the model. This motivates the need of reduced order models.



from http://www.climate.be

Model: Ice velocity equations

Stokes equations:

$$\begin{cases} -\nabla \cdot \sigma = \rho \mathbf{g} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

gravit. acceleration

ice velocity

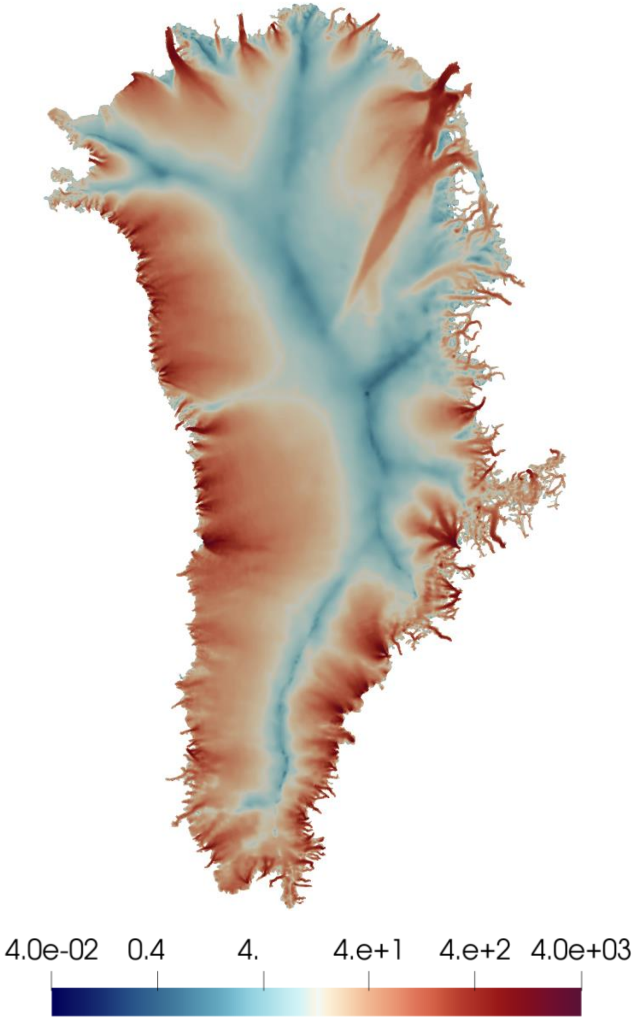
Stress tensor:

$$\sigma = 2\mu \mathbf{D} - pI, \qquad \mathbf{D}_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Ice viscosity (dependent on temperature):

$$\mu = \frac{1}{2} A(T)\, |\mathbf{D}(\mathbf{u})|^{\frac{1}{n}-1}, \quad n \geq 1, \quad (\text{tipically } n \simeq 3)$$

Modeled surface ice speed [m/yr]
(Greenland ice sheet)



Model: Ice velocity equations

Stokes equations:

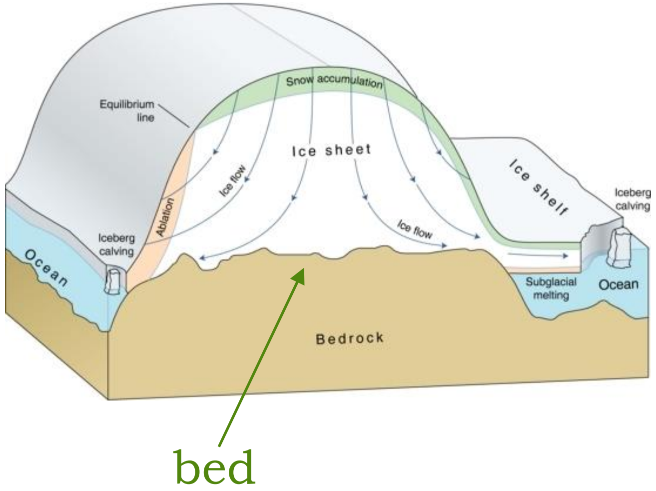
$$\begin{cases} -\nabla \cdot \sigma = \rho \mathbf{g} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

Sliding boundary condition at ice bed:

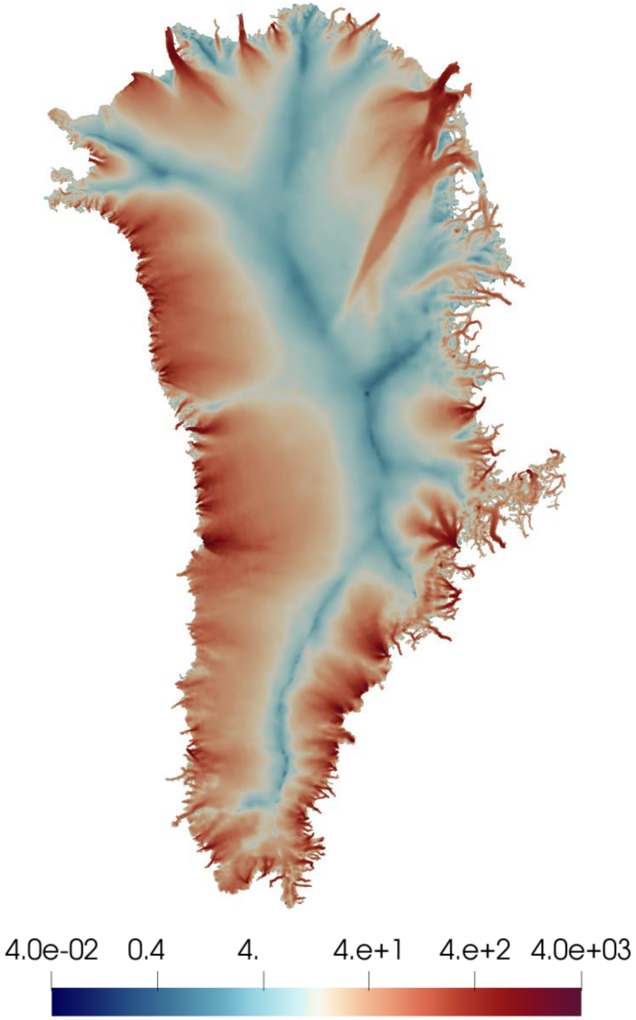
$$\begin{cases} \mathbf{u} \cdot \mathbf{n} = 0, \quad \text{(impenetrablity)} \\ (\sigma \mathbf{n})_{||} = \beta \mathbf{u} \end{cases}$$

Free slip: $\beta = 0$

No slip: $\beta = \infty$



Modeled surface ice speed [m/yr]
(Greenland ice sheet)



First Order of approximations of Stokes equations

Stokes equations are typically simplified exploiting the shallow nature of the ice sheets. Using scaling arguments it is possible to show that the pressure is almost hydrostatic (only depend on the elevation z) and simplify the model.

We implement a simplification of Stokes equations, called **First Order** (or Blatter-Pattyn) model.

First Order (FO) model

(3D elliptic PDE)

membrane stress tensor

$$-\nabla \cdot \left(2\mu\tilde{\mathbf{D}}\right) - \partial_z(\mu\,\partial_z\mathbf{u}) = -\rho g \nabla s$$

$$2\mu\tilde{\mathbf{D}}\mathbf{n} = \beta\mathbf{u}, \quad \text{on bed}$$

upper surface

Derived assuming:

$$\mathbf{D}(\mathbf{u}) = \begin{bmatrix} u_x & \frac{1}{2}(u_y + v_x) & \frac{1}{2}(u_z + \cancel{w_x}) \\ \frac{1}{2}(u_y + v_x) & v_y & \frac{1}{2}(v_z + \cancel{w_y}) \\ \frac{1}{2}(u_z + \cancel{w_x}) & \frac{1}{2}(v_z + \cancel{w_y}) & w_z \end{bmatrix} \qquad \mathrm{u} := \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

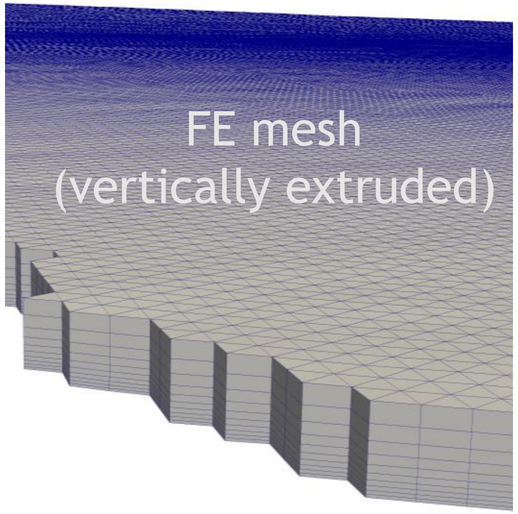
$$w_z = -(u_x + v_y)$$

which implies:

$$p = \rho g(s - z) - 2\mu(u_x + v_y)$$

Software: MPAS-Albany Land Ice model (MALI)

ALGORITHM	SOFTWARE TOOLS
Thickness evolution / Temperature Finite Volumes on Voronoi meshes	MPAS (Model for Prediction Across Scales)
Velocity/ SS Enthalpy solvers: Finite Elements on prisms	Albany
Optimization	ROL
Nonlinear solver (Newton method)	NOX
Krylov linear solvers/Prec	Belos/MueLu, Belos/FROSch
Automatic differentiation	Sacado



Typically 7-20 vertical layers.

MALI relies on **Trilinos** for achieving performance portability through ***Kokkos*** programming model.
And for providing large-scale PDE constrained optimization capabilities.

References:

1. *Watkins et al., arXiv 2022*
2. *Hoffman et al. GMD, 2018*
3. *Tuminaro, Perego, Tezaur, Salinger, Price, SISC, 2016.*
4. *Tezaur, Perego, Salinger, Tuminaro, Price, Hoffman, GMD*, 2015
5. *Perego, Price, Stadler,* JGR, 2014

Further simplifications of Stokes equations

Shallow Shelf Approx. (SSA)

(2d PDE, for floating fast-flowing ice)

$$-\nabla \cdot \left(2\mu H \tilde{\mathbf{D}}(\bar{\mathbf{u}})\right) + \beta \bar{\mathbf{u}} = -\rho g H \nabla s$$

Derived assuming:

$$\mathbf{D} = \begin{bmatrix} u_x & \frac{1}{2}(u_y + v_x) & 0 \\ \frac{1}{2}(u_y + v_x) & v_y & 0 \\ 0 & 0 & w_z \end{bmatrix}$$

Shallow Ice Approx. (SIA)

(for grounded slow-flowing ice)

$$\mathbf{u} = \left(\frac{2A(T)\rho^n g^n}{n+1}\left((s-z)^{n+1} - H^{n+1}\right)|\nabla s|^{n-1}\right)\nabla s$$

Derived assuming:

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & \frac{1}{2}u_z \\ 0 & 0 & \frac{1}{2}v_z \\ 0 & 0 & w_z \end{bmatrix}$$

Normalized
vertical coor.

$$\zeta \in (0,1), \quad z = s - H(1-\zeta).$$

An higher accuracy depth-integrated modal model

Normalized vertical coordinate

$$\zeta \in (0,1), \quad z = s - H(1-\zeta).$$

$\zeta = 0$: ice bed
$\zeta = 1$: ice surface

SIA and SSA Solutions:

$$u_{SIA} = u_{def}(x,y)\ (1-\zeta)^{n+1}$$

$$u_{SSA} = u_{SSA}(x,y)$$

Idea [1, 2], look for solution of FO with the ansatz:

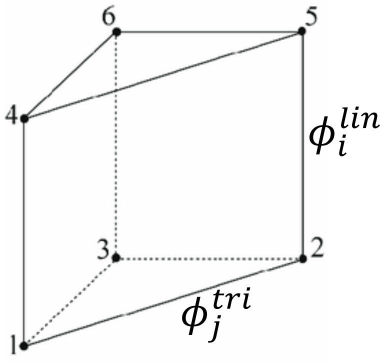
$$u_{MOD} = u_{bed}(x,y) + u_{def}(x,y)\ (1-\zeta)^{n+1}$$

- ➢ Leads to a system of two 2D PDEs

Our approach (allows to reuse most of the code base):
Solve a 3D problem with only one layer of wedges, and use as a wedge element basis:

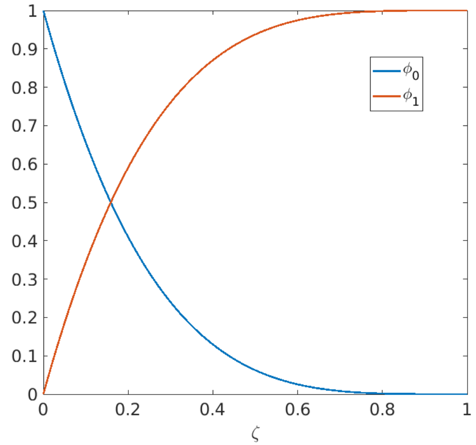
$$\phi_{ij}^{wedge}(\xi,\eta,\zeta) = \phi_j^{tri}(\xi,\eta)\ \phi_i^{lin}(\zeta)$$

piecewise linear
triangular basis



$$\phi_0^{lin}(\zeta) := (1-\zeta)^4$$

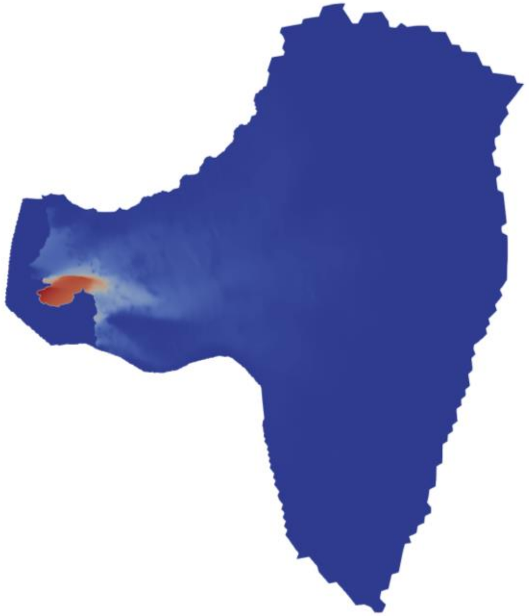
$$\phi_1^{lin}(\zeta) := 1-(1-\zeta)^4$$



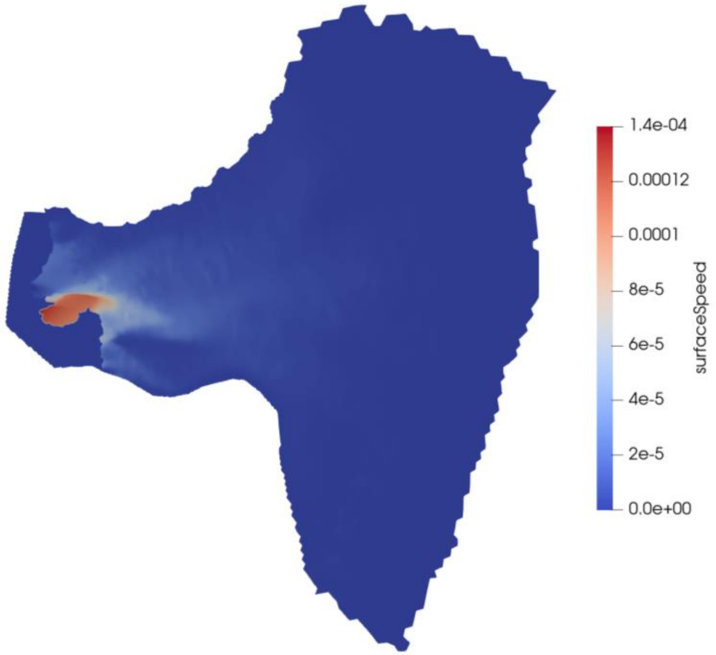
[1] J. Basis, J. Glaciology, 2017
[2] T. Dias dos Santos, M. Morlighem, D. Brinkerhoff, The Cryosphere, 2022
[3] Ern, Perotto, Veneziani, Multiscale Model. Simul., 2010

Model comparison for Thwaites glacier (Antarctica)

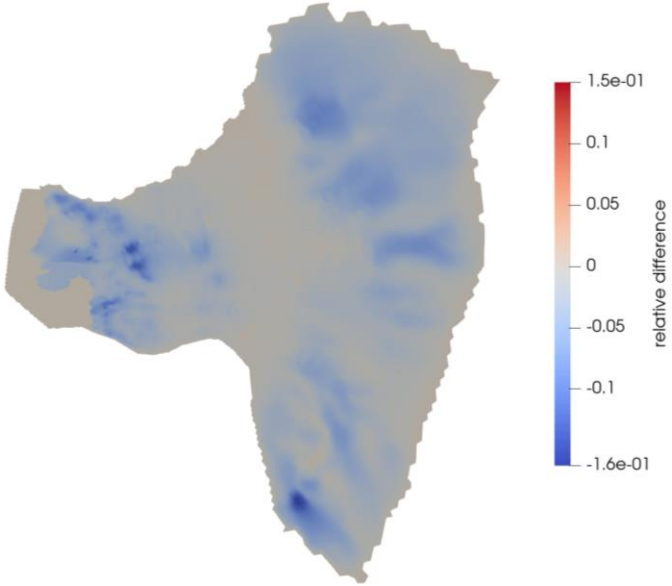
FO model
surface ice speed [m/s]



MOD model
surface ice speed [m/s]



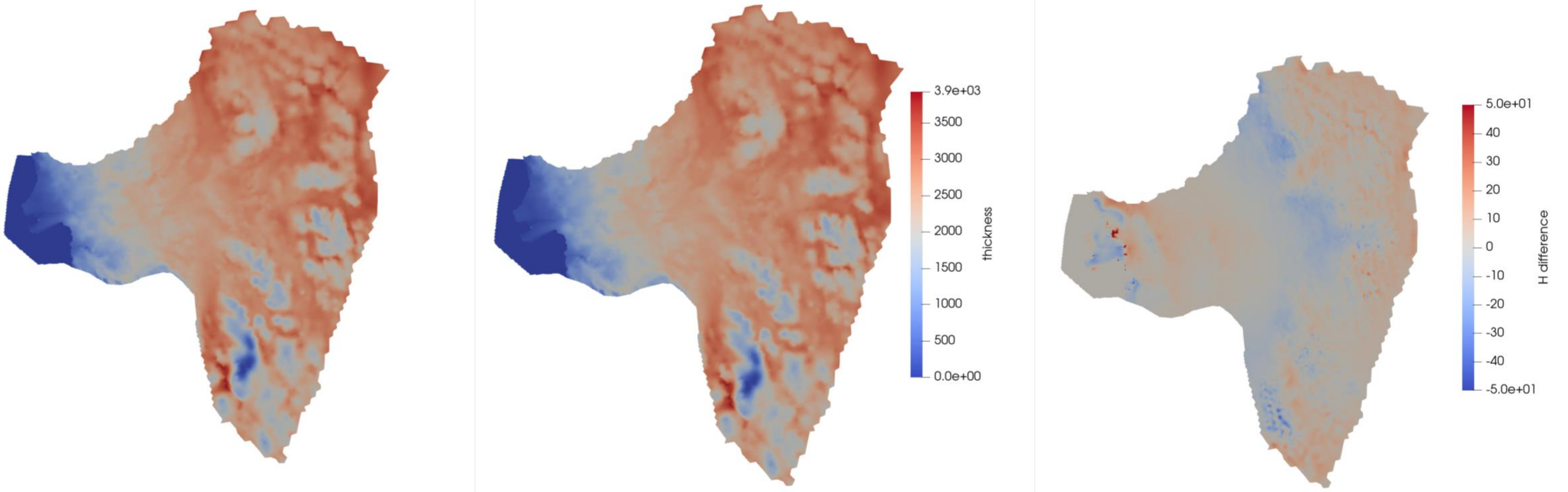
MOD - FO
relative speed difference



Overall, MOD is ~3 faster than FO with 10 layers, without a minor loss in accuracy.

Note: exploiting the tensor-product structure of the Wedge basis allows to minimize the additional cost of having a on-layer 3D model instead of two 2D models.
At the moment we have ~5x speedup in assembly phase but only ~2x in the solve phase.
This is likely due to the fact that our preconditioner already take advantage of the shallow structure.

Model comparison for Thwaites glacier (Antarctica)



FO model
ice thickness [m]

MOD model
ice thickness [m]

MOD - FO
thickness difference [m]

Ice thickness after 15 years of ice sheet evolution.

Optimal depth-integrated modal model

Can we consider polynomial modes other than the following?

$$\phi_0(\zeta) := (1-\zeta)^4, \qquad \phi_1(\zeta) := 1-(1-\zeta)^4$$

The desire of having Lagrangian basis functions with two degrees of freedom dictates

$$\phi_0(\zeta) := p(1-\zeta)\,, \qquad \phi_1(\zeta) := 1-p(1-\zeta), \qquad p(0)=0, \qquad p(1)=1$$

$$p(x) = x + x(x-1)(c_0 + c_1 x + c_2 x^2 + \ \ldots)$$

If $c_0 = c_1 = c_2 = 1$, we have the previous basis
If $c_0 = c_1 = c_2 = 0$, we have linear basis

Now we can optimize for these coefficients minimizing

$$\mathcal{J}(\boldsymbol{c}) = |u_{FO} - u_{mod}(\boldsymbol{c})|^2 \qquad \boldsymbol{c} = [c_0, c_1, c_2, \ldots]$$

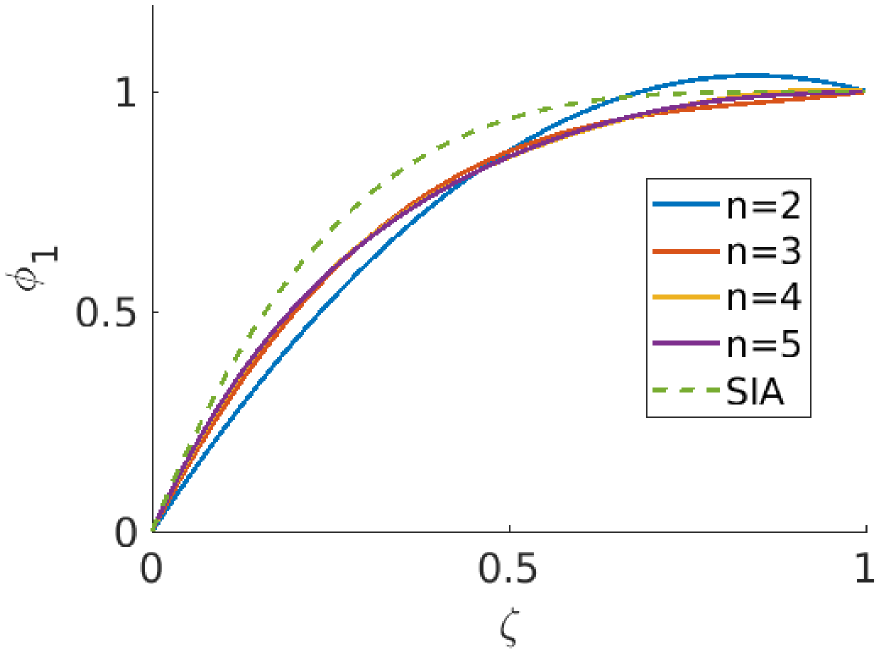
[1] J. Basis, J. Glaciology, 2017
[2] T. Dias dos Santos, M. Morlighem, D. Brinkerhoff, The Cryosphere, 2022
[3] Ern, Perotto, Veneziani, Multiscale Model. Simul., 2010

Optimal depth-integrated modal model

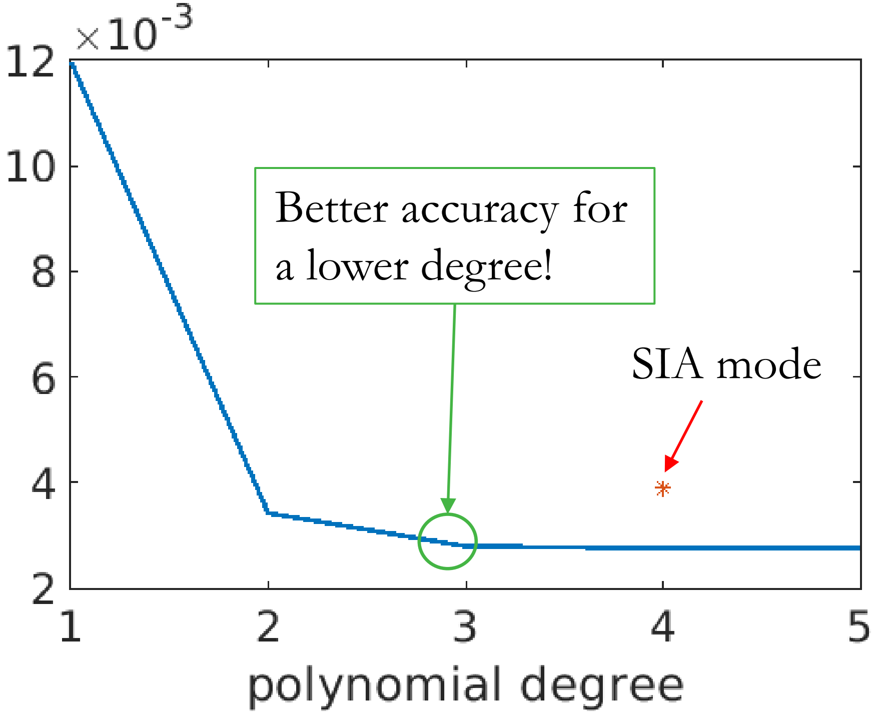
$$\min_{\boldsymbol{c}} \; \mathcal{J}(\boldsymbol{c}) = |u_{FO} - u_{MOD}(\boldsymbol{c})|^2$$

$\boldsymbol{u}_{FO}$ from FO simulation on Thwaites glacier, with vertically averaged temperature

Optimal modes



Relative L^2 error



Better accuracy for a lower degree!

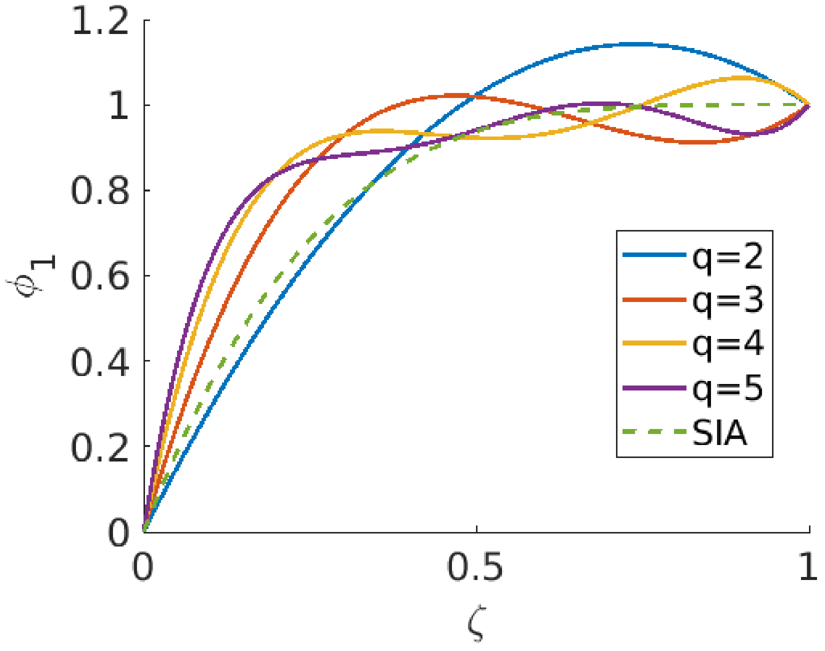
SIA mode

Optimal depth-integrated modal model

$$\min_{\boldsymbol{c}} \ \mathcal{J}(\boldsymbol{c}) = |\boldsymbol{u}_{FO} - \boldsymbol{u}_{MOD}(\boldsymbol{c})|^2$$

$\boldsymbol{u}_{FO}$ from FO simulation on Thwaites glacier, with vertically varying temperature

Optimal modes



Velocity less smooth in the vertical direction when temperature is not constant

SIA derived with assumption of vertically averaged temperature.

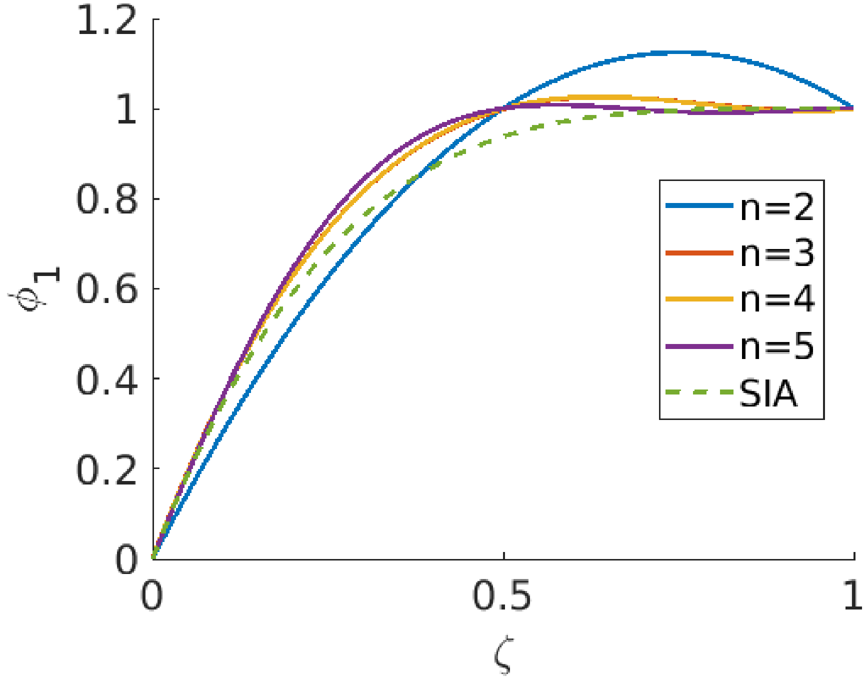
Optimal depth-integrated modal model

$$\min_{\boldsymbol{c}} \ \mathcal{J}(\boldsymbol{c}) = |\boldsymbol{u}_{FO} - \boldsymbol{u}_{MOD}(\boldsymbol{c})|^2 + \alpha \left|\partial_{\zeta\zeta}\phi_1(\zeta, c)\right|^2$$

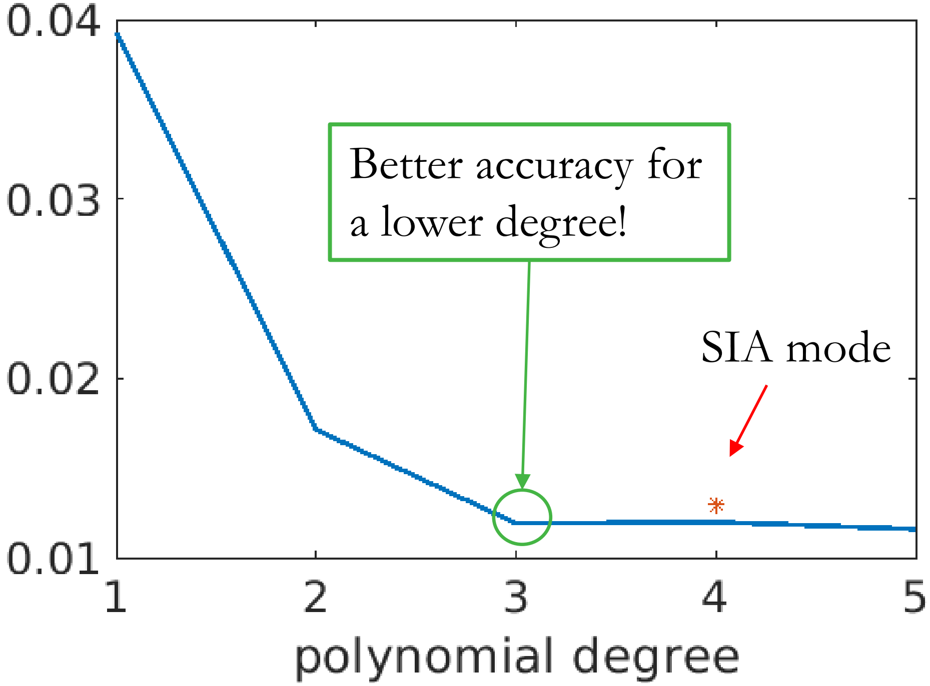
Penalize second derivative

$\boldsymbol{u}_{FO}$ from FO simulation on Thwaites glacier, with vertically varying temperature

Optimal modes



Relative L^2 error



Better accuracy for a lower degree!

SIA mode

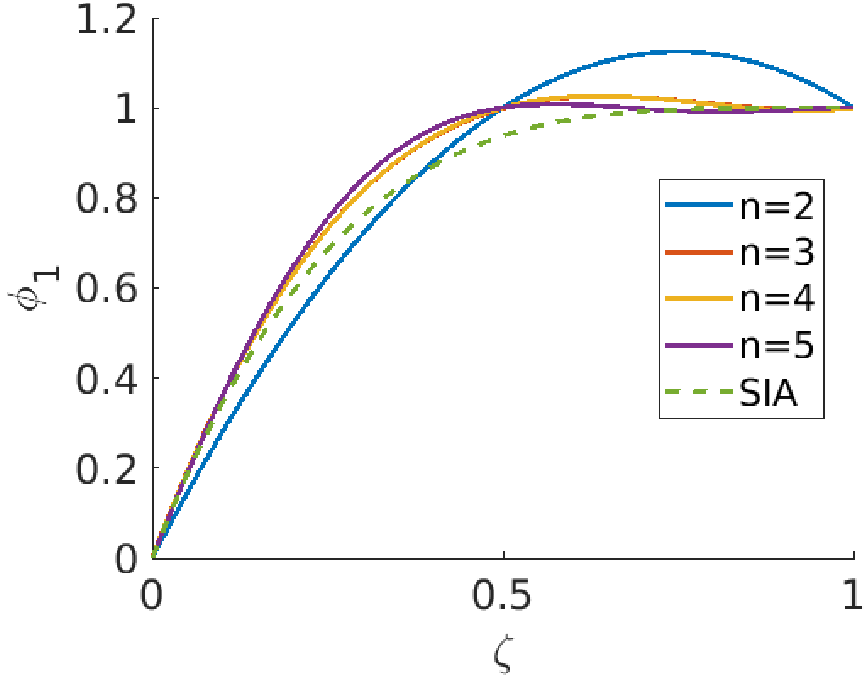
Optimal depth-integrated modal model

$$\min_{\boldsymbol{c}} \ \mathcal{J}(\boldsymbol{c}) = |\boldsymbol{u}_{FO} - \boldsymbol{u}_{MOD}(\boldsymbol{c})|^2 + \alpha \left|\partial_{\zeta\zeta}\phi_1(\zeta, \boldsymbol{c})\right|^2$$

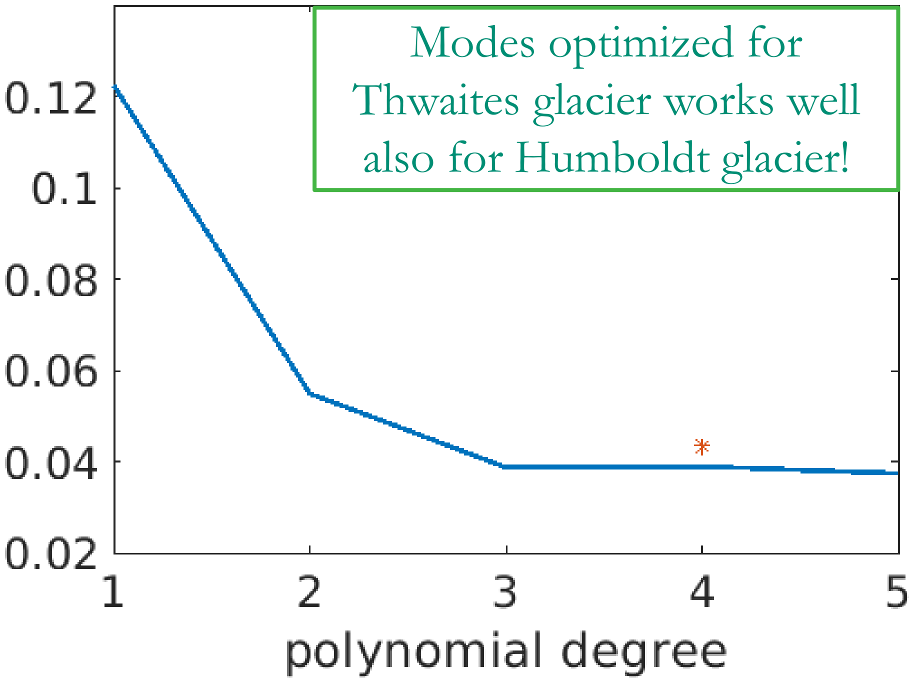
Penalize second derivative

$\boldsymbol{u}_{FO}$ from FO simulation on Thwaites glacier, with vertically varying temperature

Optimal modes



Relative L^2 error, Humboldt



Modes optimized for Thwaites glacier works well also for Humboldt glacier!

Can we use the same discretization for modeling temperature?

Heat equation (for cold ice):

$$\rho c\,\partial_t T + \nabla \cdot (\underset{\uparrow}{k}\nabla T) + \underset{\uparrow}{\rho c}\,\mathbf{u}\cdot\nabla T = 4\mu|D(\mathbf{u})|^2$$

conductivity

heat capacity

dissipation heating

Boundary condition at the ice bed (includes melting and refreezing):

frictional heating

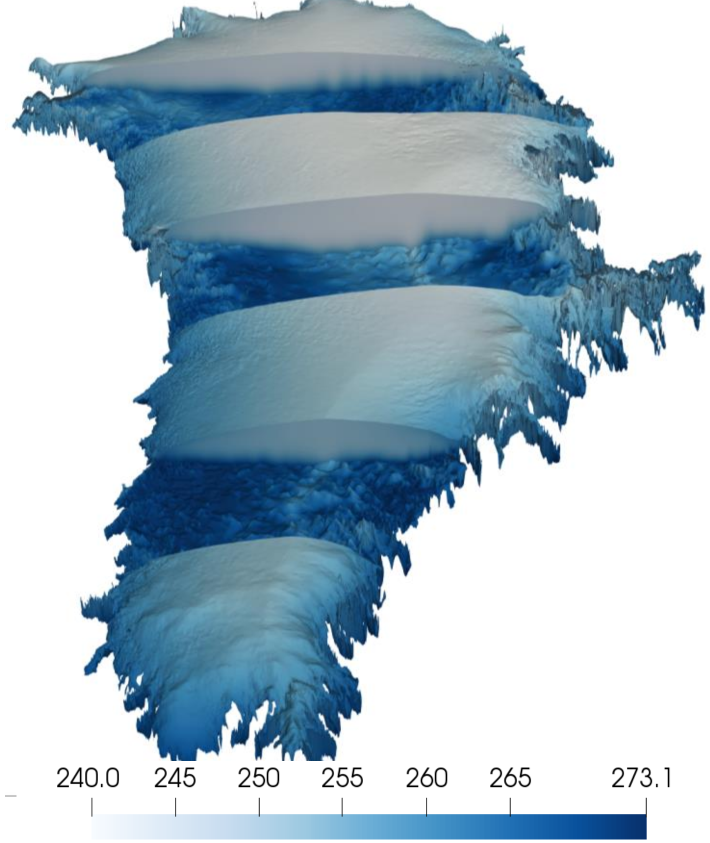
$$\mathrm{m} = \mathrm{G} + \beta|\mathbf{u}|^2 - k\nabla T\cdot\mathbf{n}$$

melting rate

geothermal heat flux

temperature flux

modeled temperature [K] (Greenland ice sheet)



MALI implements a enthalpy formulation that accounts for temperate ice as well.

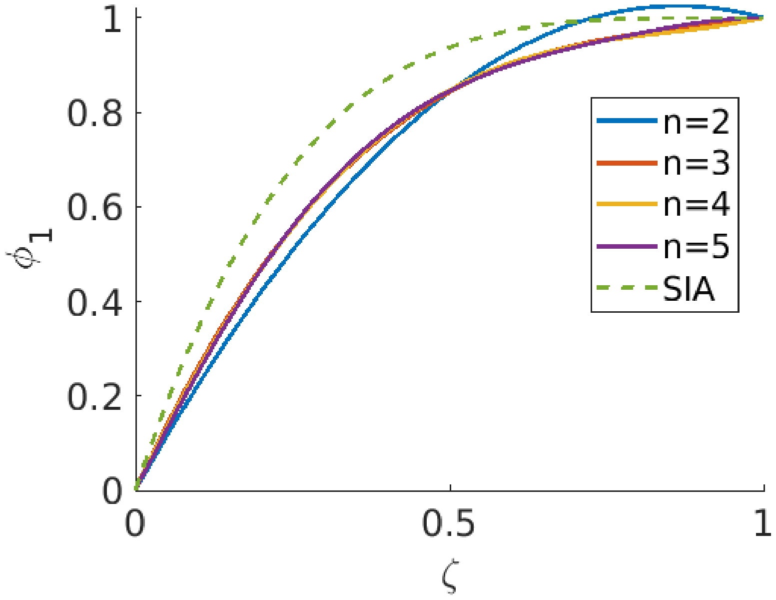
[1] A. Aschwanden, E. Bueler, C. Khroulev, and H. Blatter, Journal of Glaciology, 2012
[2] J. Hewitt and C. Schoof, The Cryosphere, 2017

Optimal depth-integrated modal model

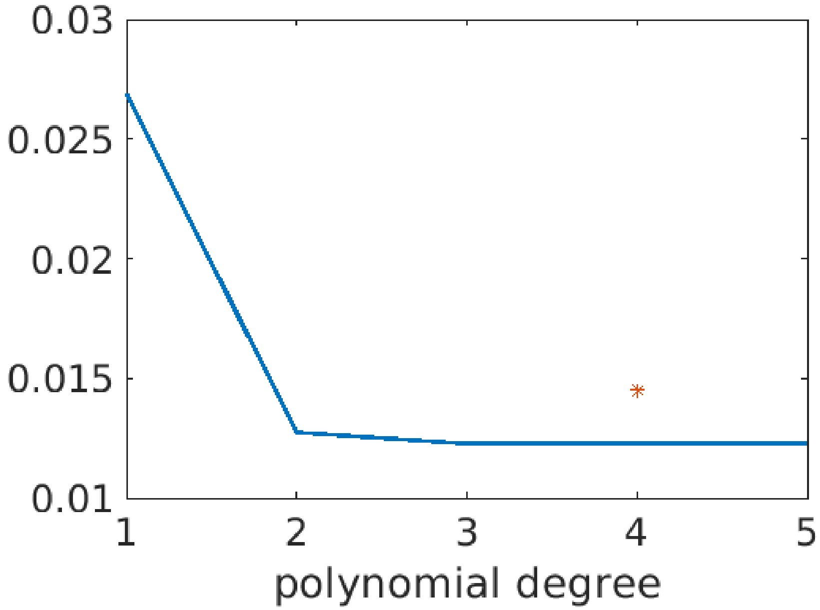
$$\min_{\boldsymbol{c}} \ \mathcal{J}(\boldsymbol{c}) = |T - T_{MOD}(\boldsymbol{c})|^2$$

Penalize second derivative

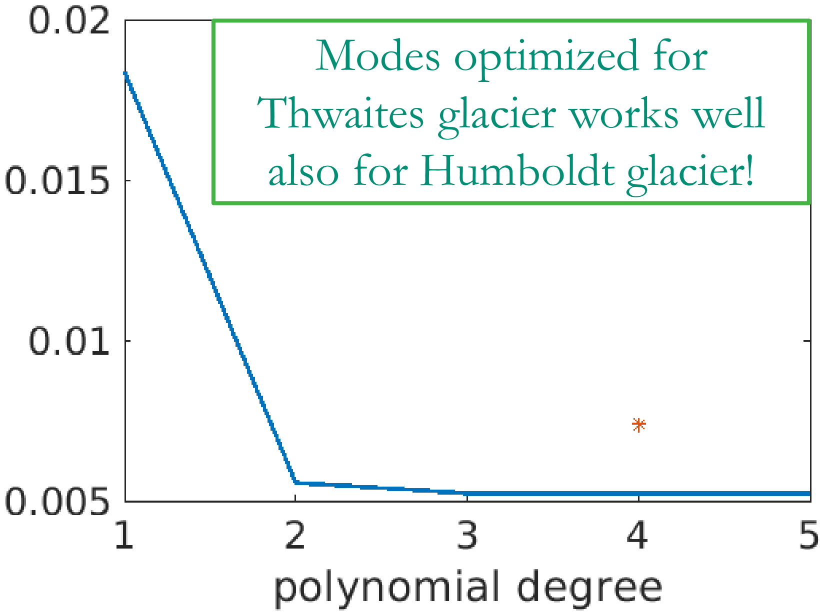
T from 3D simulation on Thwaites glacier, with <u>vertically varying temperature</u>



Optimal modes, Thwaites



Relative L^2 error, Thwaites



Relative L^2 error, Humboldt

Modes optimized for Thwaites glacier works well also for Humboldt glacier!

Final Remarks

- Implemented fast reduced-order model based on a well-known modal discretization based on the SIA approximation.

- Simulations of Thwaites and Humboldt glaciers show good agreement with high fidelity model

- Derived optimal vertical mode that has slightly better accuracy than SIA-based model at a lower degree, which allow using lower-order quadrature rule.

- TODO: Validate the model over several glaciers, possibly adjusting the optimal coefficients

- TODO: use reduced-order model for accelerating inference and UQ problems

- TODO: go high-order (e.g. P3, P4) in the vertical direction.