National Laboratories

# An Inexact Trust-Region Algorithm for Nonsmooth Nonconvex Optimization

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# Problem Formulation



**Goal:** Develop an efficient algorithm to solve the **nonsmooth optimization problem**,

$$\min_{x \in H} \; f(x) + \phi(x).$$

- $H$ is a Hilbert space with inner product $(\cdot, \cdot)$ and associated norm $\|\cdot\|$;
- $f : H \to \mathbb{R}$ has Lipschitz continuous gradients on an open set containing dom $\phi$;
- $\phi : H \to [-\infty, +\infty]$ is proper, closed and convex, but may be **nonsmooth**;
- $F := f + \phi$ is bounded below on dom$\phi$.

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**Key Requirements of Algorithm**

1. **Large-Scale Problems:** Rapid convergence, mesh independence, and matrix free.
2. **Leverage Inexactness:** Converges even when $f$ and $\nabla f$ are computed inexactly via adaptive discretization, reduced-order modelling, compression, etc.

# Inexact Computations

Motivation

In infinite-dimensional optimization, the objective function and its gradient are often **impossible** to compute without discretization, iteration, etc., leading to **inexactness**.

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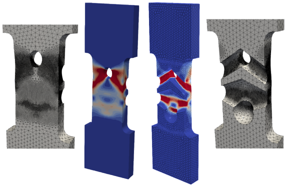
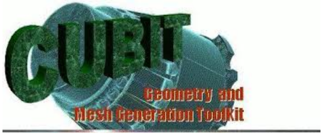
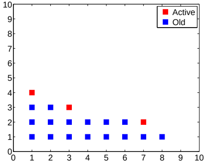
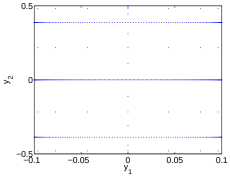
### Adaptive Finite Elements



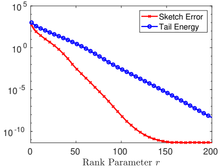



### Adaptive Quadrature




### Adaptive Compression




# Motivating Application



**Goal:** Determine a control $z$ that produces a state close to $w$ and that has ***small support***.

Given a domain $\Omega \subset \mathbb{R}^d$, a target state $w \in L^2(\Omega)$, bounds $a \leq 0 \leq b$ a.e., and penalty parameters $\alpha, \beta \geq 0$,

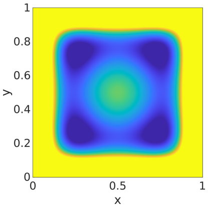
$$\min_{z \in L^2(\Omega)} \int_\Omega |S(z) - w|^2(x)\,\mathrm{d}x + \frac{\alpha}{2}\int_\Omega |z|^2(x)\,\mathrm{d}x + \beta \int_\Omega |z|(x)\,\mathrm{d}x$$

$$\text{subject to} \quad a \leq z \leq b \quad \text{a.e.},$$

where $S(z) = u \in H_0^1(\Omega)$ solves

$$\begin{aligned} -\Delta u + u^3 &= z \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega \end{aligned}$$

**Optimal Control**




**Challenges:** Objective function is **nonsmooth**, **nonconvex**, and **expensive**.

# Sparse Control



1. **Subgradient and Bundle Methods:** Iterates $x_{k+1}$ solve the optimization problem

$$\min_{x\in H} \; \frac{t_k}{2}\|x - x_k\|_H^2 + \sup_{j\in I_k}\{f(y_j) + \phi(y_j) + (\nabla f(y_j) + \eta_j, x - y_j)_H\},$$

where $t_k \geq 0$ and $\eta_j \in \partial\phi(y_j)$. Typically, **convergence is slow** (e.g., sublinear).

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2. **Proximal Gradient Methods:** Iterates $x_{k+1}$ solve the optimization problem

$$\min_{x\in H} (\nabla f(x_k), x - x_k)_H + \frac{1}{2\gamma_k}\|x - x_k\|_H^2 + \phi(x) \quad \Longleftrightarrow \quad x_{k+1} = \mathrm{prox}_{\gamma_k\phi}(x_k - \gamma_k\nabla f(x_k)).$$

PG methods are robust, but **slow**. Can use acceleration (Nesterov) or momentum (heavy balls).

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3. **Proximal Newton-Type Methods:** Iterates $x_{k+1}$ solve the optimization problem

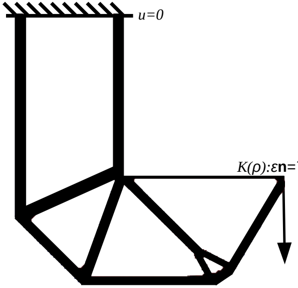
$$\min_{x\in H} (\nabla f(x_k), x-x_k)_H + \frac{1}{2}(B_k(x-x_k), x-x_k)_H + \phi(x),$$

where $B_k \in L(X)$ approximates the Hessian of $f$. PN methods require **positive definite $B_k$** (e.g., convexity) and **nonstandard/nontrivial prox computations**.

# Motivating Application

## Elastic Topology Optimization

**Goal:** Determine a ***binary*** $\rho$ that is maximally stiff and that satisfies the volume constraint.




Given a domain $\Omega \subset \mathbb{R}^d$ and a volume fraction $v \in (0,1)$,

$$\min_{\rho \in L^2(\Omega)} \int_{\Gamma_t} T(x) \cdot [S(\rho)](x)\,\mathrm{d}x$$

$$\text{subject to} \quad \int_\Omega \rho(x)\,\mathrm{d}x \le v|\Omega|, \quad 0 \le \rho \le 1 \ \text{ a.e.},$$

where $S(\rho) = u \in (H^1(\Omega))^d$ solves

$$\begin{aligned} -\nabla \cdot (K(\rho) : \varepsilon) = 0, \quad \varepsilon = \tfrac{1}{2}(\nabla u + \nabla u^\top) \qquad & \text{in } \Omega \\ K(\rho) : \varepsilon \mathbf{n} = T \qquad & \text{on } \Gamma_t \\ u = 0 \qquad & \text{on } \Gamma_d \end{aligned}$$

**Challenges:** Objective function is **expensive** and highly **nonconvex** due to material models like the **S**olid **I**sotropic **M**aterial with **P**enalization (SIMP).

# Elastic Topology Optimization



1. **Optimality Criterion Method:** A **heuristic** fixed-point iteration that is related to a projected gradient method.

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2. **Method of Moving Asymptotes:** A sequential convex optimization approach that uses rational approximations of the objective and constraints. The dual subproblem is commonly solved using nonlinear CG. This method is inherently **finite dimensional**.

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**It is extremely difficult to handle *inexact* objective values and gradients in these methods!**

# Nonsmooth Trust Regions



**Require:** An initial guess $x_1$, initial trust-region radius $\Delta_1 > 0$, $0 < \eta_1 < \eta_2 < 1$ and $0 < \gamma_1 \le \gamma_2 < 1$

**for** $k = 1, 2, \ldots$ **do**

  **Model Selection:** Choose a subproblem model $f_k$ of $f$ near $x_k$

  **Step Computation:** Compute $x_k^+$ that *approximately* solves

$$\min_{x \in H} \{m_k(x) := f_k(x) + \phi(x)\} \qquad \text{subject to} \qquad \|x - x_k\| \le \Delta_k$$

  **Evaluate Objective:** Compute the actual reduction $\text{ared}_k := F(x_k) - F(x_k^+)$

  **if** $\rho_k := \dfrac{\text{ared}_k}{m_k(x_k) - m_k(x_k^+)} < \eta_1$ **then**

    $x_{k+1} \leftarrow x_k$ and $\Delta_{k+1} \in [\gamma_1 \Delta_k, \gamma_2 \Delta_k]$

  **else**

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    **if** $\rho_k < \eta_2$ **then**

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**Trust-Region Subproblem:** At each iteration, we approximately solve

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where $\Delta_k > 0$ is the radius and $f_k : H \to \mathbb{R}$ is a model of the $f$ near the iterate $x_k$.

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**Recall:** TR methods do **not** solve the subproblem, but rather use a ***Cauchy point*** to ensure **sufficient decrease** of the trial iterate $x_k^+$.

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**Recall:** TR methods do **not** solve the subproblem, but rather use a ***Cauchy point*** to ensure **sufficient decrease** of the trial iterate $x_k^+$.

We generalize the ***Cauchy point*** to nonsmooth problems using the ***proximal gradient path***

$$x_k^{\text{cp}} = p_k(t_k) \qquad \text{with} \qquad p_k(t) := \text{prox}_{t\phi}(x_k - tg_k),$$

where $g_k \approx \nabla f(x_k)$ and the ***proximity operator*** is given by

$$\text{prox}_{t\phi}(x) := \arg\min_{y\in H} \left\{ \tfrac{1}{2t}\|y - x\|^2 + \phi(y) \right\}.$$

# Nonsmooth Trust Regions



We set $x_k^{\mathrm{cp}} = p_k(t_k)$, where the step length $t_k$ satisfies both

1. **Trust-Region Feasibility:**

$$\|x_k^{\mathrm{cp}} - x_k\| \le \nu_1 \Delta_k$$

2. **Sufficient Decrease:**

$$m_k(x_k^{\mathrm{cp}}) - m_k(x_k) \le \mu_1[(g_k, x_k^{\mathrm{cp}} - x_k) + \phi(x_k^{\mathrm{cp}}) - \phi(x_k)]$$

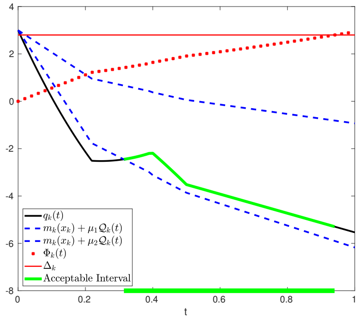
and at least one of the following conditions:

$$t_k \ge \nu_2 t_k' \qquad \text{or} \qquad t_k \ge \nu_3,$$

where $t_k'$ either satisfies

$$m_k(p_k(t_k')) - m_k(x_k) \ge \mu_2[(g_k, p_k(t_k') - x_k) + \phi(p_k(t_k')) - \phi(x_k)]$$

$$\text{or} \qquad \|p_k(t_k') - x_k\| \ge \nu_4 \Delta_k.$$




The interval of **acceptable** $t_k$ is depicted by the **green** line on the horizontal axis.

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- **Consequence of CP:** There exists a trial iterate $x_k^+$ that satisfies

$$\|x_k^+ - x_k\| \le \nu_{\mathrm{rad}}\Delta_k, \quad \nu_{\mathrm{rad}} \ge \nu_1$$
$$m_k(x_k) - m_k(x_k^+) \ge \mu_3[m_k(x_k) - m_k(x_k^{\mathrm{cp}})], \quad 0 < \mu_3 \le 1.$$

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- **Trial Iterate Requirements:** Avoid CP computation by ensuring that $x_k^+$ satisfies

$$\begin{aligned} &\|x_k^+ - x_k\| \le \nu_{\mathrm{rad}}\Delta_k \\ &m_k(x_k) - m_k(x_k^+) \ge \kappa_{\mathrm{fcd}} h_k \min\left\{\frac{h_k}{1+\omega_k}, \Delta_k\right\}, \end{aligned} \tag{FCD}$$

where $h_k := \|p_k(r_0) - x_k\|/r_0$ for fixed $r_0 > 0$ and $\omega_k \ge 0$ measures the curvature of $f_k$.

**Model:** For the SPG subproblem solver, we employ the models

$$f_k(x) = \tfrac{1}{2}(B_k(x - x_k), x - x_k) + (g_k, x - x_k) \quad \text{and} \quad \phi_k(x) = \begin{cases} \phi(x) & \text{if } \|x - x_k\| \le \Delta_k \\ +\infty & \text{otherwise} \end{cases}$$

**Algorithm:** Set $x_{k,0} = x_k^{\mathrm{cp}}$, $s_{k,0} = x_{k,0} - x_k$ to ensure (FCD) is satisfied and compute

$$x_{k,\ell+1} = x_{k,\ell} + \alpha_{k,\ell} s_{k,\ell} \quad \text{with} \quad s_{k,\ell} = \mathrm{prox}_{\lambda_{k,\ell}\phi_k}(x_{k,\ell} - \lambda_{k,\ell}\nabla f_k(x_{k,\ell})) - x_{k,\ell},$$

where $\alpha_{k,\ell} \in [0,1]$ minimizes the quadratic upper bound

$$\begin{aligned} q_{k,\ell}(t) &:= f_k(x_{k,\ell} + t s_{k,\ell}) + t[\phi_k(x_{k,\ell} + s_{k,\ell}) - \phi_k(x_{k,\ell})] + \phi_k(x_{k,\ell}) \\ &\ge f_k(x_{k,\ell} + t s_{k,\ell}) + \phi_k(x_{k,\ell} + t s_{k,\ell}) = m_k(x_{k,\ell} + t s_{k,\ell}) \end{aligned}$$

and $\lambda_{k,\ell}$ is the the safeguarded spectral step length given by

$$\lambda_{k,\ell} = \max\left\{\lambda_{\min}, \min\left\{\lambda_{\max}, \frac{(s_{k,\ell-1}, s_{k,\ell-1})}{(B_k s_{k,\ell-1}, s_{k,\ell-1})}\right\}\right\}.$$

Birgin, et al., Nonmonotone spectral projected gradient methods on convex sets, SIOPT, 2000.
Kouri, A matrix-free trust-region Newton algorithm for convex-constrained optimization, Opt. Letters, 2022.
Baraldi & Kouri, A proximal trust-region method for nonsmooth optimization with inexact function and gradient evaluations, Math. Prog., 2022.

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  **Model Selection:** Choose a subproblem model $f_k$ of $f$ near $x_k$ — **Inexact!**

  **Step Computation:** Compute a trial step $x_k^+$ that satisfies (FCD)

  **Evaluate Objective:** Evaluate the computed reduction $\text{cred}_k \approx \text{ared}_k$ — **Inexact!**

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When evaluating the of reduction of the objective function, we approximate

$$\mathrm{cred}_k \approx \mathrm{ared}_k := (f(x_k) + \phi(x_k)) - (f(x_{k+1}) - \phi(x_{k+1})),$$

where $\mathrm{cred}_k$ satsifies:

$$\exists\, \kappa_{\mathrm{obj}} > 0, \quad \zeta > 1, \quad \eta < \min\{\eta_1, 1-\eta_2\}, \quad \text{and} \quad \theta_k \searrow 0 \quad \text{such that}$$
$$|\mathrm{ared}_k - \mathrm{cred}_k| \le \kappa_{\mathrm{obj}}[\eta \min\{m_k(x_k) - m_k(x_{k+1}), \theta_k\}]^{\zeta} \quad \forall\, k.$$

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We also require that the model gradient $g_k$ must satisfy:

$$\exists\, \kappa_{\mathrm{grad}} > 0 \qquad \text{such that} \qquad \|\nabla f(x_k) - g_k\| \le \kappa_{\mathrm{grad}} \min\{h_k, \Delta_k\} \quad \forall\, k.$$

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$$\mathrm{cred}_k \approx \mathrm{ared}_k := (f(x_k) + \phi(x_k)) - (f(x_{k+1}) - \phi(x_{k+1})),$$

where $\mathrm{cred}_k$ satsifies:

$$\exists\, \kappa_{\mathrm{obj}} > 0, \quad \zeta > 1, \quad \eta < \min\{\eta_1, 1 - \eta_2\}, \quad \text{and} \quad \theta_k \searrow 0 \quad \text{such that}$$
$$|\mathrm{ared}_k - \mathrm{cred}_k| \le \kappa_{\mathrm{obj}}[\eta \min\{m_k(x_k) - m_k(x_{k+1}), \theta_k\}]^{\zeta} \quad \forall\, k.$$

We also require that the model gradient $g_k$ must satisfy:

$$\exists\, \kappa_{\mathrm{grad}} > 0 \qquad \text{such that} \qquad \|\nabla f(x_k) - g_k\| \le \kappa_{\mathrm{grad}} \min\{h_k, \Delta_k\} \quad \forall\, k.$$

**Can compute model gradient $g_k$ in *finitely* many iterations!**

# Convergence Theory

Under the stated assumptions, the iterates produced by the TR algorithm satisfy

$$\liminf_{k\to\infty} h_k = 0 \qquad \Longrightarrow \qquad \liminf_{k\to\infty} h(x_k, t) = 0 \quad \forall\, t > 0,$$

where $\quad h_k := \frac{1}{r_0}\|\mathrm{prox}_{r_0\phi}(x_k - r_0 g_k) - x_k\| \quad$ and $\quad h(x,t) := \frac{1}{t}\|\mathrm{prox}_{t\phi}(x - t\nabla f(x)) - x\|.$

**Finite Termination:** $\forall\,\tau > 0 \quad \exists\, K_\tau \in \mathbb{N}$ such that $h_{K_\tau} \le \tau h_1.$

**Tikhonov Regularization:** If $f(x) = f_0(x) + \frac{\alpha}{2}\|x - x_0\|^2$, where $\alpha > 0$, $x_0 \in H$, $\nabla f_0$ is **completely continuous** and $r_0 \ge \alpha^{-1}$, then any **weak accumulation point** of $\{x_k\}$ is a **critical point** of $f + \phi$. See, e.g., **sparse control**.


Baraldi & Kouri, A proximal trust-region method for nonsmooth optimization with inexact function and gradient evaluations, Math. Prog., 2022.
Kouri, A matrix-free trust-region Newton algorithm for convex-constrained optimization, Opt. Letters, 2022.

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**Strong Local Convergence:** Suppose $f$ is **strongly convex** on a convex set $U \subseteq H$ with $U \cap \mathrm{dom}\phi \neq \emptyset$ and $\exists\, K_0 \in \mathbb{N}$ such that $x_k \in U$ for $k \ge K_0$. If $\exists\, \bar{x} \in U$ satisfying $h(\bar{x}, t) = 0\ \forall\, t > 0$, then $x_k \to \bar{x}$. That is, $\{x_k\}$ **converges strongly to a critical point**.

Baraldi & Kouri, A proximal trust-region method for nonsmooth optimization with inexact function and gradient evaluations, Math. Prog., 2022.
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**Finite Termination:** $\forall\, \tau > 0 \quad \exists\, K_\tau \in \mathbb{N}$ such that $h_{K_\tau} \le \tau h_1$.

**Convergence Rates:** Further, suppose $f_k$ is the quadratic model and $\nabla^2 f$ is Lipschitz.*

1. If $\frac{1}{r_0}\|\mathrm{prox}_{r_0\phi}(x_k^+ - r_0\nabla f_k(x_k^+)) - x_k^+\| \le \tau_k h_k$ and $\tau_k \to 0$, then $x_k$ converges ***superlinearly***.
2. If $\tau_k \le \tau h_k^{1+\alpha}$ for $\tau > 0$ and $\alpha \ge 0$, then $x_k$ converges ***quadratically***.

*Also requires modest assumptions on the iterates generated by the subproblem solver.

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# Numerical Results



**Goals:** **1. Comparison of TR method with modern nonsmooth methods.**
**2. Demonstration of mesh independence for TR method.**

Let $\Omega = (0,1)^2$, $w \equiv -1$, $a \equiv -25$, $b \equiv 25$, $\alpha = 10^{-4}$ and $\beta = 10^{-2}$, and consider

$$\min_{z \in L^2(\Omega)} \int_\Omega |S(z) - w|^2(x)\,\mathrm{d}x + \frac{\alpha}{2}\int_\Omega |z|^2(x)\,\mathrm{d}x + \beta \int_\Omega |z|(x)\,\mathrm{d}x$$
$$\text{subject to} \qquad a \le z \le b \quad \text{a.e.,}$$

where $S(z) = u \in H_0^1(\Omega)$ solves

$$\begin{aligned} -\Delta u + u^3 &= z \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega \end{aligned}$$

**Discretization:** P1 FEM for state variables and piecewise constant for controls.

**Problem Size:** 131,072 control degrees of freedom.

# Numerical Results

Sparse Control

| method | iter | fval | grad | hess | phi | prox | time (s) | TR speedup* |
|---|---|---|---|---|---|---|---|---|
| TR | 4 | 5 | 5 | 39 | 57 | 142 | 22.88 | 1.0000 |
| PG | 59 | 149 | 60 | 0 | 149 | 209 | 498.56 | 21.79 |
| SPG | 30 | 46 | 31 | 0 | 46 | 62 | 168.26 | 7.35 |
| R2 | 106 | 107 | 46 | 0 | 107 | 153 | 368.27 | 16.10 |
| nmAPG | 93 | 194 | 186 | 0 | 194 | 196 | 1018.66 | 44.52 |
| iPiano | 103 | 240 | 104 | 0 | 104 | 344 | 816.96 | 35.71 |
| FISTA | 141 | 430 | 283 | 0 | 430 | 290 | 1532.58 | 66.98 |
| PANOC | 83 | 285 | 108 | 0 | 272 | 287 | 948.04 | 41.44 |
| ZeroFPR | 21 | 70 | 43 | 0 | 45 | 93 | 247.39 | 10.81 |

**Proximal Gradient Methods** **Accelerated Methods** **Proximal Quasi-Newton Methods**

*TR speedup is the ratio of the wallclock time for TR divided by the times for the other methods.

# Numerical Results

Sparse Control

| $\tau_{\text{op}}$ | 1e-4 | | | | 1e-6 | | | | 1e-8 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| mesh | iter | npde | lpde | prox | iter | npde | lpde | prox | iter | npde | lpde | prox |
| 64x64 | 3 | 4 | 56 | 80 | 5 | 6 | 108 | 129 | 7 | 8 | 186 | 181 |
| 128x128 | 3 | 4 | 54 | 79 | 4 | 5 | 79 | 102 | 6 | 7 | 129 | 151 |
| 256x256 | 3 | 4 | 56 | 80 | 5 | 6 | 108 | 129 | 6 | 7 | 133 | 153 |
| 512x512 | 3 | 4 | 54 | 78 | 5 | 6 | 102 | 123 | 6 | 7 | 127 | 147 |

Trust-region algorithm demonstrates **mesh independence** with respect to the number of iterations and the number of PDE solves!

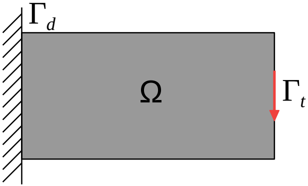
**Requires only modest additional computational work to achieve tight tolerances!**

# Numerical Results



**Goals: 1. Comparison of TR method with modern projected and AL methods.**
**2. Demonstration of TR inexactness control for 3D problems.**

Let $\Omega = (0,2)\times(0,1)^d$, $d = 1,2$, and $v = 0.4$, and consider




$$\min_{\rho\in L^2(\Omega)} \int_{\Gamma_t} T(x)\cdot[S(\rho)](x)\,\mathrm{d}x$$

$$\text{subject to} \quad \int_\Omega \rho(x)\,\mathrm{d}x = v|\Omega|, \quad 0\le\rho\le 1 \;\text{ a.e.},$$

where $S(\rho) = u \in (H^1(\Omega))^{d+1}$ solves

$$\begin{aligned} -\nabla\cdot(K(\rho):\varepsilon) &= 0 && \text{in } \Omega \\ \varepsilon &= \frac{1}{2}(\nabla u + \nabla u^\top) && \text{in } \Omega \\ K(\rho):\varepsilon\mathbf{n} &= T && \text{on } \Gamma_t \\ u &= 0 && \text{on } \Gamma_d \end{aligned}$$

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# Numerical Results



**Formulation:** SIMP power $p = 3$ with Helmholtz filtering (radius$= 0.1$).

**Discretization:** Q1 FEM for displacement variables and piecewise constant for density.

**Problem Size:** 26,880 density degrees of freedom.

| method | iter | fval | grad | hess | proj | time(s) | TR speedup* |
|---|---|---|---|---|---|---|---|
| TR | 9 | 10 | 10 | 236 | 1200 | 16.49 | 1.0000 |
| LMTR | 33 | 34 | 31 | 418 | 391 | 32.42 | 1.9660 |
| PQN | 126 | 235 | 127 | 0 | 4972 | 164.49 | 9.9751 |
| SPG | 84 | 90 | 85 | 0 | 170 | 52.36 | 3.1753 |
| AL-TR | 9 | 52 | 51 | 1153 | 0 | 61.98 | 3.7586 |
| AL-LMTR | 11 | 276 | 263 | 4368 | 0 | 280.77 | 17.0267 |

**Projected Newton-Type Methods** **Spectral Projected Gradient** **AL Methods**

*TR speedup is the ratio of the wallclock time for TR divided by the times for the other methods.

# Numerical Results

## 3D Elastic Topology Optimization

**Formulation:** SIMP power $p = 3$ with Helmholtz filtering (radius$= 0.1$).

**Discretization:** Q1 FEM for displacement variables and piecewise constant for density.

**Problem Size:** 221,184 density degrees of freedom.

**Inexact Solves:** Solve using CG with AMG preconditioning.

- **Helmholtz Filter:** Requires ∼8 iterations to achieve the relative error of $\sim 10^{-12}$

  $\Longrightarrow$ Considered to be **exact**.

- **Elasticity Equations:** Trust-region algorithm controls accuracy of linear solver.

| $k$ | $F(x_k)$ | $h_k$ | $\|x_k - x_{k-1}\|$ | $\Delta_k$ | fval | grad | hess | proj | obj tol | grad tol |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1.0000 | 4.017e-2 | --- | 1e1 | 1 | 1 | 0 | 3 | 1.000e-2 | 1.000e-2 |
| 1 | 0.8157 | 1.927e-2 | 1.000e1 | 1e2 | 2 | 2 | 12 | 44 | 1.000e-2 | 1.000e-2 |
| 2 | 0.4716 | 1.279e-2 | 5.420e1 | 1e3 | 3 | 3 | 25 | 75 | 1.000e-2 | 1.000e-2 |
| 3 | 0.4144 | 6.280e-3 | 1.260e1 | 1e4 | 4 | 4 | 39 | 103 | 4.632e-3 | 1.000e-2 |
| 4 | 0.1600 | 3.101e-3 | 1.990e2 | 1e4 | 5 | 5 | 52 | 132 | 1.000e-2 | 1.000e-2 |
| 5 | 0.1300 | 1.226e-3 | 1.085e2 | 1e5 | 6 | 6 | 65 | 161 | 2.970e-3 | 1.000e-2 |
| 6 | 0.1262 | 1.242e-5 | 6.044e1 | 1e6 | 7 | 7 | 78 | 190 | 3.539e-4 | 1.000e-2 |
| 7 | 0.1254 | 6.590e-6 | 5.821e1 | 1e7 | 8 | 8 | 91 | 220 | 6.971e-5 | 6.590e-3 |
| 8 | 0.1251 | 3.221e-6 | 3.599e1 | 1e8 | 9 | 9 | 104 | 249 | 1.942e-5 | 3.221e-3 |

## Conclusions:

- **Numerical solution** of infinite-dimensional problems requires **expensive approximations**
- Often, the objective function and its gradient can only be computed **inexactly**
- Nonsmooth trust region is **provably convergent** even with **inexact computations**
- **We can efficiently compute a trial step using the spectral proximal gradient method**
- SPG trust-region subproblem solver is **matrix free**, but may **require** many prox computations
  **Future:** Can we incorporate inexact prox computations? Can we handle nonconvex $\phi$?
- Nonsmooth trust-region method **outperforms** existing nonsmooth methods!

## References:


- R. J. Baraldi & D. P. Kouri, Local Convergence Analysis of an Inexact Trust-Region Method for Nonsmooth Optimization, Submitted, 2023.
- R. J. Baraldi & D. P. Kouri, A proximal trust-region method for nonsmooth optimization with inexact function and gradient evaluations, Math Programming, 2022.
- D. P. Kouri, A matrix-free trust-region Newton algorithm for convex-constrained optimization, Optimization Letters, 2022.
- D. P. Kouri & D. Ridzal, Inexact trust-region methods for PDE-constrained optimization, Frontiers in PDE-Constrained Optimization, 2018.

# Rapid Optimization Library

rol.sandia.gov

Trilinos package for **large-scale optimization**. Uses: optimal design, optimal control and inverse problems in engineering applications; mesh optimization; image processing.

RAPID OPTIMIZATION LIBRARY

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