National Laboratories

Adaptive Randomized Sketching for Dynamic Nonsmooth Optimization

Robert Baraldi

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Problem Formulation

Goal: Develop efficient algorithms to solve the regularized nonsmooth optimization problem,

$$\min_{\mathbf{z}\in\mathcal{Z}} F(\mathbf{z}) := j(\mathbf{z}) + \phi(\mathbf{z}). \tag{1}$$

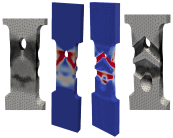
- $\mathcal{Z}$ is a Hilbert space with $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$;
- $\phi : \mathcal{Z} \to [-\infty, \infty]$ is proper, closed, and convex, but may be nonsmooth;
- $j : \mathcal{Z} \to \mathbb{R}$ has Lipschitz continuous gradients on an open set containing $\operatorname{dom}\phi$, but may be inexact;
- $F > -\infty$, bounded below on $\operatorname{dom}\phi$.

Smooth Term

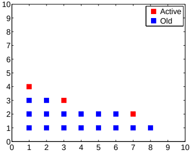
$j(z)$ may be nonconvex and often impossible to compute exactly.

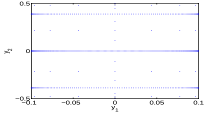
- Stems from discretization, iterative procedures, adaptive model reduction, surrogate models, iterative linear and nonlinear solves, etc.

Adaptive Finite Elements

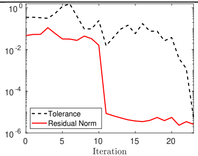


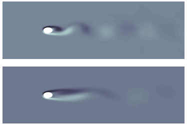
Adaptive Quadrature





Adaptive Compression





Nonsmooth Regularizers

$\phi(\mathbf{z})$ are typically sparsity-inducing and temper model complexity, but lack derivatives.

- **Sparse regularization**: $\mathcal{Z} \hookrightarrow L^1(\Omega)$, $\beta \in L^\infty(\Omega)$, $\beta \geq 0$ a.e., and

$$\phi(\mathbf{z}) = \int_\Omega \beta(\omega)|\mathbf{z}(\omega)|\mathrm{d}\omega.$$

  Applications: Optimal control, data-science, learning, basis-pursuit.

- **Total Variation**: $\mathcal{U} \hookrightarrow BV(\Omega)$, $\beta \in L^\infty(\Omega)$, $\beta \geq 0$ a.e,

$$\phi(\mathbf{u}) = \int_\Omega \beta(\omega)|\mathrm{div}\,\mathbf{u}(\omega)|\mathrm{d}\omega.$$

  Applications: Image processing, digital image correlation, topology optimization.

- **Convex Constraints**: $C \subset \mathcal{Z}$ nonempty, closed and convex,

$$\phi(\mathbf{z}) = \begin{cases} 0, & \text{if } \mathbf{z} \in C \\ +\infty, & \text{otherwise.} \end{cases}$$

  Applications: Optimal control, inverse problems, optimal design.

- **Others**: Matrix completion (rank), phase retrieval.

Motivation: Nonconvexity, Nonsmoothness, Inexactness

- Arise naturally in physical problems, and are useful in enforcing certain solution properties.
  - **Problems:** local min, numerically complex, lacking derivatives, large-scale.
  - **Theory/software** exists for smoothed/convex counterparts (IPOPT, CVX, various Matlab/Julia/Python/... implementations).
  - **Memory** required to store state trajectory (and auxiliary info like Lagrange multipliers) is often prohibitively expensive: $\mathcal{O}(N(M+m))$ for $N \approx 10^5$, $M \approx 10^{10}$.
- **Key Algorithmic Requirements:**
  1. **Nonconvex** functions and **nonsmooth** regularizers.
  2. Handle **large-scale** problems with rapid convergence, mesh independence, and matrix free operations.
  3. **Leverage inexactness** by proving convergence for $j, \nabla j$ computed inexactly via discretization, reduced-order modeling, compression, etc.

Outline

- Dynamical System Reformulation $\rightarrow$ problem assumptions.
- Brief Trust Region Overview:
    - Inexactness Assumptions,
    - Matrix Sketching,
    - Convergence Results.
- Numerical Results:
    - Measure-Valued Parabolic Control.

Dynamic Optimization Problems I

$$\min_{u_n, z_n} \sum_{n=1}^{N} f_n(u_{n-1}, u_n, z_n) + \phi_n(z_n) \quad \text{s.t.} \quad c_n(u_{n-1}, u_n, z_n) = 0 \quad \forall\, n = 1, \dots, N. \tag{2}$$

- Replace u_n with the unique solution to $c(u_{n-1}, u_n, z_n) = 0$ for fixed u_{n-1}, z_n.
- Stack controls $\mathbf{z} = [z_1^T, \dots, z_N^T]^T \in \mathcal{Z} := \mathbb{R}^{Nm}$ and states $\mathbf{u} = [u_1^T, \dots, u_N^T]^T \in \mathcal{U} := \mathbb{R}^{NM}$

$$c(\mathbf{u}, \mathbf{z}) := \begin{bmatrix} c_1(u_0, u_1, z_1) \\ \vdots \\ c_N(u_{N-1}, u_N, z_N) \end{bmatrix}, \; f(\mathbf{u}, \mathbf{z}) := \sum_{n=1}^{N} f_n(u_n, z_n), \qquad \text{and} \qquad \phi(\mathbf{z}) := \sum_{n=1}^{N} \phi_n(z_n).$$

$$\min_{\mathbf{u} \in \mathcal{U},\, \mathbf{z} \in \mathcal{Z}} f(\mathbf{u}, \mathbf{z}) + \phi(\mathbf{z}) \qquad \text{subject to} \qquad c(\mathbf{u}, \mathbf{z}) = 0. \tag{3}$$

Dynamic Optimization Problems II

Assumption (1 - Function Characteristics)

- *Functions f, c are twice continuously differentiable.*
- *There exists a unique state trajectory $\mathbf{z} \mapsto S(\mathbf{z}) : \mathcal{Z} \to \mathcal{U}$ satisfying $c(S(\mathbf{z}), \mathbf{z}) = 0$ for each $\mathbf{z} \in \mathcal{Z}$.*
- *The state Jacobian of c $\mathrm{d}_{\mathbf{u}} c(\mathbf{u}, \mathbf{z})$, has a bounded inverse for all controls $\mathbf{z} \in \mathcal{Z}$.*

Note: control Jacobian is $\mathrm{d}_{\mathbf{z}} c(\mathbf{u}, \mathbf{z})$ and the partial derivatives are $\mathrm{d}_{\mathbf{u}} f(\mathbf{u}, \mathbf{z})$, $\mathrm{d}_{\mathbf{z}} f(\mathbf{u}, \mathbf{z})$.

Additionally, the Implicit Function Theorem [Hinze et al., 2009, Th. 1.41] ensures S_n and S are continuously differentiable for $S(\mathbf{z})$ stacked (recursively)

$$S(\mathbf{z}) := [S_1(u_0, z_1)^T \dots S_N(S_{N-1}(\dots, z_{N-1}), z_N)^T]^T.$$

Dynamic Optimization Problems III

$$\min_{\mathbf{z}\in\mathcal{Z}} \left\{F(\mathbf{z}) := j(\mathbf{z}) + \phi(\mathbf{z})\right\}, \tag{4}$$

where $j(\mathbf{z}) := f(S(\mathbf{z}), \mathbf{z})$ is the continuously differentiable reduced objective function with gradient

$$\nabla j(\mathbf{z}) = \mathrm{d}_{\mathbf{z}} f(S(\mathbf{z}), \mathbf{z}) + (\mathrm{d}_{\mathbf{z}} c(S(\mathbf{z}), \mathbf{z}))^\top \boldsymbol{\lambda}, \tag{5}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^{MN}$ solves the adjoint equation

$$\mathrm{d}_{\mathbf{u}} c(S(\mathbf{z}), \mathbf{z})^\top \boldsymbol{\lambda} = -\mathrm{d}_{\mathbf{u}} f(S(\mathbf{z}), \mathbf{z}). \tag{6}$$

Note: the adjoint equation (6) is solved backward in time, starting at $n = N$ and requires the entire state trajectory.

General Trust Region Logic & Methodology

Nonsmooth Trust Regions

- **Trust-region methods:** at the k^{th} iteration, use surrogate (quadratic) model of smooth j to make progress:
  - (Approximate) Gradient ∇j, Hessian $B_k = B_k^T$;
  - Valid within a region determined by model performance and accuracy.
- Saves numerical cost for expensive forward solutions.
- **Problem:** nonsmooth trust-region methods are restrictive/impractical and computing gradient curvature information may be prohibitively expensive.
- **Solution:** rework standard trust-region literature for regularized functions with inexact function/gradient calculations (Baraldi and Kouri [2022], Aravkin et al. [2021], Baraldi and Kouri [2023b,a]).

Algorithm 1: Nonsmooth Trust-Region Method (Baraldi and Kouri [2022])

Data: $\mathbf{z}_0 \in \operatorname{dom}\phi$, $\Delta_0 > 0$, $0 < \eta_1 < \eta_2 < 1$, and $0 < \gamma_1 \le \gamma_2 < 1 \le \gamma_3$

for $k = 1, 2, \ldots$ **do**

Model Selection: Choose a subproblem model j_k of j near $\mathbf{z}_k$ (j_k inexact!).

Step Computation: Compute $\mathbf{z}_{k+1} \in \mathcal{Z}$ that solves

$$\min_{\mathbf{z}\in\mathcal{Z}} m_k(\mathbf{z}) := j_k(\mathbf{z}) + \phi(\mathbf{z}) \quad \text{subject to} \quad \|\mathbf{z} - \mathbf{z}_k\| \le \Delta_k.$$

Computed Reduction: Compute $\mathrm{cred}_k \approx \mathrm{ared}_k$ (inexact!).

Step Acceptance: Compute ratio of computed and predicted reduction

$$\rho_k := \frac{\mathrm{cred}_k}{m_k(\mathbf{z}_k) - m_k(\mathbf{z}_{k+1})} < \eta_1 \quad \Rightarrow \quad \mathbf{z}_{k+1} \leftarrow \mathbf{z}_k.$$

Update Trust-Region Radius:
$$\Delta_{k+1} \in \begin{cases} [\gamma_1\Delta_k, \gamma_2\Delta_k], & \text{if } \rho_k < \eta_1 \\ [\gamma_2\Delta_k, \Delta_k], & \text{if } \rho_k \in [\eta_1, \eta_2) \\ [\Delta_k, \infty), & \text{if } \rho_k > \eta_2. \end{cases}$$

end

Approach & Subproblem

Trust-region subproblem: at each iteration, we solve

$$\min_{\mathbf{z}\in\mathcal{Z}}\{m_k(\mathbf{z}) := j_k(\mathbf{z}) + \phi(\mathbf{z})\} \quad \text{subject to } \|\mathbf{z}-\mathbf{z}_k\|_x \le \Delta_k$$

where $\Delta_k > 0$ is the radius, $j_k : \mathcal{Z} \to \mathbb{R}$ is a model of j near $\mathbf{z}$.

Quadratic model: $j_k(x) = \langle g_k, \mathbf{z}-\mathbf{z}_k\rangle_{\mathcal{Z}} + \frac{1}{2}\langle B_k(\mathbf{z}-\mathbf{z}_k), \mathbf{z}-\mathbf{z}_k\rangle_{\mathcal{Z}}$ where $g_k \approx \nabla j(\mathbf{z}_k)$ and B_k contains some curvature information, $B_k \approx \nabla^2 j(\mathbf{z}_k)$ or some quasi-Newton approximation.

Theoretical Challenges:

1. Ensure convergence with j_k, g_k inexact;
2. Ensure subproblem step yields Fraction of Cauchy Decrease (FCD, shown in Baraldi and Kouri [2022]);
3. Handle nonsmooth subproblems efficiently (Proximal subproblem solvers, show in Baraldi and Kouri [2022]).

Inexactness Conditions

Recall: Infinite-dimensional optimization function and gradient evaluations are often impossible to compute without discretization or even store, leading to inexactness.

When evaluating the reduction of the objective function, we approximate

$$\mathrm{cred}_k \approx \mathrm{ared}_k := (j(\mathbf{z}_k) + \phi(\mathbf{z}_k)) - (j(\mathbf{z}_{k+1}) + \phi(\mathbf{z}_{k+1}))$$

and with $\mathrm{pred}_k := m_k(\mathbf{z}_k) - m_k(\mathbf{z}_{k+1})$ and construct the bound

$$|\mathrm{ared}_k - \mathrm{cred}_k| \leq \kappa_{\mathrm{obj}} \left[\eta \min\{\mathrm{pred}_k, \theta_k\}\right]^{\zeta}, \tag{7}$$

$\eta < \min\{\eta_1, (1-\eta_2)\}$, $\{\theta_k\}_{k=1}^{+\infty} \subset [0, +\infty)$, $\lim_{k\to+\infty} \theta_k = 0$.
Similarly for the gradient with $\kappa_{\mathrm{grad}} > 0$ and $h_k := t_0^{-1} \|\mathrm{Prox}_{t_0\phi}(\mathbf{z}_k - t_0 g_k) - \mathbf{z}_k\|_{\mathcal{Z}}$:

$$\|g_k - \nabla j(\mathbf{z}_k)\| \leq \kappa_{\mathrm{grad}} \min\{\Delta_k, h_k\} \quad \forall\, k. \tag{8}$$

Randomized Sketching

Idea: use randomized sketching to compress $M \times N$ state trajectory matrix

$$U = (u_1 | \dots | u_N) \in \mathbb{R}^{M \times N}$$

and use the sketched state U^r for gradient evaluations and Hessian applications.
Goal: Generate an accurate rank-r approximation of U using $\mathcal{O}(r(M+N))$ storage from [Tropp et al., 2019, Muthukumar et al., 2021].
Matrix Sketching: Given the four random linear dimension reduction maps and $r \geq k \geq s$, $k = 2r+1$, and $s = 2k+1$,

$$\Upsilon \in \mathbb{R}^{k \times M}, \quad \Omega \in \mathbb{R}^{k \times N}, \quad \Phi \in \mathbb{R}^{s \times M}, \quad \text{and} \quad \Psi \in \mathbb{R}^{s \times N}$$

$$\begin{aligned} X &:= \Upsilon U \in \mathbb{R}^{k \times N}, && \text{co-range sketch (row space)},\\ Y &:= U \Omega^* \in \mathbb{R}^{M \times k}, && \text{range sketch (column space)},\\ Z &:= \Phi U \Psi^* \in \mathbb{R}^{s \times s}, && \text{core sketch (singular values)}. \end{aligned}$$

Randomized Sketching II

Algorithm 2: Online State Sketching

Require: $X = 0, Y = 0$

1: **for** $n = 1, \ldots, N$ **do**
2: Given u_{n-1} and z_n, solve $c_n(u_{n-1}, u_n, z_n) = 0$ for u_n
3: $X \leftarrow X + \Upsilon u_n e_n^T$
4: $Y \leftarrow Y + u_n(\Omega e_n)$
5: $Z \leftarrow Z + (\Phi u_n)(\Psi e_n)^T$
6: **end for**

where e_i is the i^{th} unit vector.

Storage Requirement: Sketching requires $k(M+N)+s^2$ FLOPs.

Randomized Sketching III

Algorithm 3: Recovery

1: $(Q, R_2) \leftarrow \mathrm{qr}(Y, 0)$, $\quad Q \in \mathbb{R}^{M\times k}, R_2 \in \mathbb{R}^{k\times k}$.
2: $(P, R_1) \leftarrow \mathrm{qr}(X^T, 0)$, $\quad P \in \mathbb{R}^{N\times k}, R_1 \in \mathbb{R}^{k\times k}$.
3: $C \leftarrow (\Phi Q)^\dagger Z((\Psi P)^\dagger)^T$, $\quad C \in \mathbb{R}^{k\times k}$.
4: $W \leftarrow CP^T W \in \mathbb{R}^{k\times N}$.
5: $u_n \approx QWe_n$.

- Approximately recover U via QR factorizations of X^T and Y and solve 2 small LS problems (Muthukumar et al. [2021]).
- Storage Requirement: Recovery requires $k(M+N)+k^2$ FLOPs.
- Further reductions can be gained by overwriting X and Y with Q and W: j^{th} column then becomes $U^r[:,j] = (QW)[:,j]$. This reduces total complexity to $\mathcal{O}(k(M+N))$.

State Compression Error

Lemma (Sketching Error - Muthukumar et al. [2021])

Let U^r denote sketch of U associated with rank parameter r. Then

$$\mathbb{E}_{\Upsilon,\Omega,\Phi,\Psi}\|U - U^r\|_F \leq \sqrt{6}\left(\sum_{i\geq r+1}\sigma_i^2(U)\right)^{\frac{1}{2}}.$$

The error is *expected* to be slightly larger than the best rank r approximation!

Similar results exist for the probability of large deviation in Tropp et al. [2019].

Adaptive Compression: Increase sketch rank r until dynamical system residual, $\|c(U^r, z)\|$ satisfies required tolerance.

Algorithm 4: Inexact Gradient Computation with Adaptive Rank

Require: $\mathbf{z}_k \in \mathbb{R}^{mN}$, initial r, sketch for state $\mathbf{u}_k^r := \mathrm{vec}(U^r)$, $\Delta_k > 0$, $\kappa_{\mathrm{scale}} > 0$, and tolerance $\mu_{\mathrm{grad}} > 1$.

Set $\tau_k^- \leftarrow \kappa_{\mathrm{scale}} \Delta_k$
Compute $g_k \leftarrow g(\Lambda(\mathbf{u}_k^r, \mathbf{z}_k), \mathbf{u}_k^r, \mathbf{z}_k)$, $h_k \leftarrow t^{-1} \|\mathrm{Prox}_{r\phi}(\mathbf{z}_k - t g_k) - \mathbf{z}_k\|$, and $\tau_k^+ \leftarrow \kappa_{\mathrm{scale}} \min\{h_k, \Delta_k\}$
while $\tau_k^- > \mu_{\mathrm{grad}} \tau_k^+$ **do**
  while $r < \min\{M, N\}$ **do**
    Compute norm of the constraint residual `rnorm` $\leftarrow \|c(\mathbf{u}_k^r, \mathbf{z}_k)\|$
    if `rnorm` $< \tau_k^+$ **then**
      Compute gradient $g_k^r \leftarrow g(\Lambda(\mathbf{u}_k^r, \mathbf{z}_k), \mathbf{u}_k^r, \mathbf{z}_k)$, **break**
    end if
    $r \leftarrow 2r$ and solve state equation at $\mathbf{z}_k$ and resketch $\mathbf{u}_k^r$
  end while
  Set $g_k \leftarrow g_k^r$ and compute $h_k \leftarrow t^{-1} \|\mathrm{Prox}_{r\phi}(\mathbf{z}_k - t g_k) - \mathbf{z}_k\|$
  Set $\tau_k^- \leftarrow \tau_k^+$ and $\tau_k^+ \leftarrow \kappa_{\mathrm{scale}} \min\{h_k, \Delta_k\}$
end while
return Approximate gradient $g_k \approx \nabla j(\mathbf{z}_k)$ using $\mathcal{O}(r(M+N) + mN)$ storage for $r \leq \min\{M, N\}$. Is this guaranteed to converge in finite time?

Convergence for Algorithm 4

Assumption (2 - Regularity Properties for (1))

The following conditions hold for the data in (1):

1. *There exists* $\mathcal{U}_0 \subset \mathcal{U}$ *open and bounded such that* $\{\mathbf{u} \in \mathcal{U} | \exists \mathbf{z} \in \mathcal{Z}_0, c(\mathbf{u}, \mathbf{z}) = 0\} \subseteq \mathcal{U}_0$.
2. *There exists singular value thresholds* $0 < \sigma_0 \le \sigma_1 < +\infty$ *such that for any* $\mathbf{u} \in \mathcal{U}_0$ *and* $\mathbf{z} \in \mathcal{Z}_0$, $\sigma_0 \le \sigma_{\min}(\mathrm{d}_{\mathbf{u}} c(\mathbf{u}, \mathbf{z})) \le \sigma_{\max}(\mathrm{d}_{\mathbf{u}} c(\mathbf{u}, \mathbf{z})) \le \sigma_1$.
3. *The following functions are Lipschitz continuous on* $\mathcal{U}_0 \times \mathcal{Z}_0$ *with respect to their first arguments, and their respective Lipschitz moduli are independent of* $\mathbf{z} \in \mathcal{Z}_0$:
   - 3.1 *the state Jacobian of the constraint* $\mathrm{d}_{\mathbf{u}} c(\mathbf{u}, \mathbf{z})$;
   - 3.2 *the control Jacobian of the constraint* $\mathrm{d}_{\mathbf{u}} c(\mathbf{u}, \mathbf{z})$;
   - 3.3 *the state gradient of the smooth objective term* $\mathrm{d}_{\mathbf{u}} f(\mathbf{u}, \mathbf{z})$;
   - 3.4 *the control gradient of the smooth objective term* $\mathrm{d}_{\mathbf{z}} f(\mathbf{u}, \mathbf{z})$.

Convergence for Algorithm 4 II

Using Assumption 2, we can bound the state, adjoint and gradient errors as in [Muthukumar et al., 2021, Prop. 4.1].
We use this to show our gradient approximation algorithm satisfies (8).

Lemma (Adaptive Rank Gradient Approximation)

If Assumption 2 holds, then Algorithm 4 produces a gradient approximation $g_k = g(\Lambda(\mathbf{u}_k^r, \mathbf{z}_k), \mathbf{u}_k^r, \mathbf{z}_k)$*, in finitely many iterations, that satisfies the gradient error bound* (8) *with* $\kappa_{\mathrm{grad}} = \kappa_4 \kappa_{\mathrm{scale}} \mu_{\mathrm{grad}}$.

Algorithm 5: Final Sketched Nonsmooth Trust-Region Algorithm

Data: $\mathbf{z}_0 \in \operatorname{dom} \phi$, $\Delta_0 > 0$, $0 < \eta_1 < \eta_2 < 1$, and $0 < \gamma_1 \le \gamma_2 < 1 \le \gamma_3$

for $k = 1, 2, \ldots$ **do**

Model Selection: Use Algorithm 4 with rank r_k to compute g_k and choose B_k

Step Computation: Compute $\mathbf{z}_{k+1} \in \mathcal{Z}$ (efficient) that solves

$$\min_{\mathbf{z} \in \mathcal{Z}} m_k(\mathbf{z}) := j_k(\mathbf{z}) + \phi(\mathbf{z}) \quad \text{subject to} \quad \|\mathbf{z} - \mathbf{z}_k\| \le \Delta_k.$$

Computed Reduction: Compute $\mathrm{cred}_k \approx \mathrm{ared}_k$ (inexact!).

Step Acceptance: Compute ratio of computed and predicted reduction

$$\rho_k := \frac{\mathrm{cred}_k}{m_k(\mathbf{z}_k) - m_k(\mathbf{z}_{k+1})} < \eta_1 \quad \Rightarrow \quad \mathbf{z}_{k+1} \leftarrow \mathbf{z}_k.$$

Update Trust-Region Radius:
$$\Delta_{k+1} \in \begin{cases} [\gamma_1 \Delta_k, \gamma_2 \Delta_k], & \text{if } \rho_k < \eta_1 \\ [\gamma_2 \Delta_k, \Delta_k], & \text{if } \rho_k \in [\eta_1, \eta_2) \\ [\Delta_k, \infty), & \text{if } \rho_k > \eta_2. \end{cases}$$

end

Convergence of Algorithm 1

Recall: $h_k := r_0^{-1}\,\|\mathrm{Prox}_\phi(x_k - r_0 g_k) - x_k\|_X$

Theorem (Algorithm Convergence)

Let $\{x_k\}$ be the sequence of iterates generated

$$\liminf_{k\to\infty} h_k = 0 \quad \Rightarrow \quad \liminf_{k\to\infty} r_0^{-1}\,\|\mathrm{Prox}_\phi(x_k - r_0\nabla f(x_k)) - x_k\|_X = 0. \tag{9}$$

Note: Permits unbounded model curvature.

Given $\varepsilon > 0$ and bounded model curvature, then Trust-Region Algorithm satisfies $h_k \le \min\{\varepsilon, 1\}$ in at most $\mathcal{O}(\varepsilon^{-2})$ iterations.

Note: This is a worst-case bound; we find much better performance in practice.

Goals: **Demonstrate Algorithm 1 inexactness control for adaptive compression**.
Let $\Omega = (0,1)^2$, $T = 2$ and $\alpha = 0.1$, and solve

$$\min_{z \in M(\Omega)} \frac{1}{2} \int_0^T \|S(z) - w\|^2_{L^2(\Omega)} \, \mathrm{d}t \quad \text{subject to} \quad \|z\|_{M(\Omega)} \leq \alpha, \quad z \succeq 0,$$

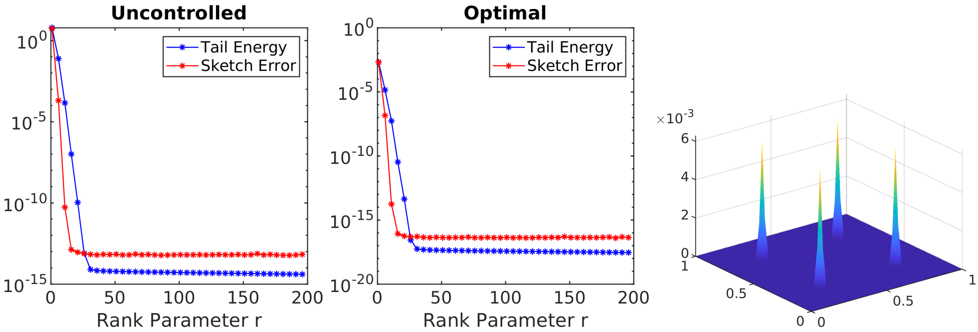
for $w = |(\sin(2\pi x_1)\sin(2\pi x_2)|^{10}$ and $S(z) = u$ solves

$$\begin{aligned} \partial u_t - \Delta u &= 0 \quad \text{in } \Omega \times (0, T) \\ \nabla u \cdot n &= 0 \quad \text{on } \partial\Omega \times (0, T) \\ u(0) &= z \quad \text{in } \Omega. \end{aligned}$$

Discretization: P1 FEM + implicit Euler for states and variational discretization for controls.
Problem size: 4225 control degress of freedom.

State Tail Energy Decay

Figure 1 depicts the tail energy and sketching error averaged over 20 realizations for uncontrolled (random initial point) and optimal **z**.



The sketching error averaged over 20 realizations and the tail energy for the uncontrolled state (left) and the optimal state(right). recall that the rank of the sketch is $k = 2r + 1$.

Measure-Valued Parabolic Control

Termination Criteria: $h_k \leq 10^{-4} h_1$ or $\left\| z_k^+ - z_k \right\| \leq 10^{-6} h_1$ or $k = 40$.
Rank Increase Function: $r \leftarrow 2r$.

rank	objective	niter	nobjs	ngrad	nhess	nobjn	nprox	ζ
*1	2.680962e-02	16	17	9	521	1036	1726	148.78
2	2.680946e-02	37	38	38	1948	4597	3873	89.12
3	2.680946e-02	31	32	32	1635	3759	3248	63.55
4	2.680946e-02	22	23	23	1163	2654	2308	49.35
5	2.680946e-02	22	23	23	1160	2640	2305	40.31
Adaptive	2.680946e-02	22	23	25	1162	2530	2309	25.95
Full	2.680946e-02	23	24	24	1212	2793	2409	---

$*$ rank 1 experiment failed to converge due to step-size tolerance.

The final adapted rank was $r = 8$ leading to 26x compression.

Conclusions

- **Numerical solutions** of infinite-dimensional problems requires expensive approximations.
  - Objectives and gradients can only be compute inexactly.
- Nonsmooth trust-region is provably convergent even with inexact computation via matrix sketching/compression.
  - Next: incorporate other compression techniques, harder examples (fluid flow around a cylinder).

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Questions?
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Proximal Operator

Definition (Moreau-Yosida Envelope)

For a proper, lower semicontinuous function $\phi : \mathcal{Z} \to \overline{\mathbb{R}}$ and a parameter $t > 0$, the *Moreau envelope* $e_{t\phi} : \mathcal{Z} \to \mathbb{R}$ and the *proximal operator* $\mathrm{Prox}_{t\phi} : \mathcal{Z} \to \mathcal{Z}$ are defined by

$$e_{t\phi}(x) := \inf_{z \in \mathcal{Z}} \frac{1}{2t} \|z - x\|^2 + \phi(z), \tag{10a}$$

$$\mathrm{Prox}_{t\phi}(x) := \mathrm{argmin}_{z \in \mathcal{Z}} \frac{1}{2t} \|z - x\|^2 + \phi(z). \tag{10b}$$

- **Interpretation**: extension of cost function to minimizing ϕ and near x.
- **Utility**: many proximal operators have analytic solutions;
  - **L^1-Norm**: $\mathcal{Z} = L^2(\Omega)$, $\phi(\mathbf{z}) = \beta \|\mathbf{z}\|_{L^1(\Omega)} \Rightarrow \mathrm{Prox}_{t\phi} = \mathrm{sign}(\mathbf{z}) \odot (|x| - t\beta)_+$.
  - **ReLU**: $\mathcal{Z} = \mathbb{R}$, $\phi(\mathbf{z}) = \max\{0, \mathbf{z}\} \Rightarrow \mathrm{Prox}_{t\phi} = \max\{\mathbf{z} - t, \min\{0, \mathbf{z}\}\}$.

First-Order Necessary Optimality Conditions

Nonsmooth Stationarity

Definition (Local Minimizer)

$\bar{\mathbf{z}} \in \mathcal{Z}$ is a **local minimizer** of $(j+\phi)$ if there exists a neighborhood U of $\bar{\mathbf{z}}$ on which $(j+\phi)(\bar{\mathbf{z}}) \le (j+\phi)(\mathbf{z}) \quad \forall \mathbf{z} \in U$. Additionally, $\bar{\mathbf{z}}$ satisfies

$$-\nabla j(\bar{\mathbf{z}}) \in \partial\phi(\bar{\mathbf{z}}) \quad \Leftrightarrow \quad \bar{\mathbf{z}} = \mathrm{Prox}_{t\phi}(\bar{\mathbf{z}} - t\nabla j(\bar{\mathbf{z}}))\, \forall t > 0. \tag{11}$$

Lemma (Generalized Gradient & Stationary Point)

If $\mathbf{z} \in \mathcal{Z}$ is a stationary point of $(j+\phi)$, then $h(\bar{\mathbf{z}}) = 0$ where

$$h(\mathbf{z}) := t^{-1} \|\mathrm{Prox}_{t\phi}(\mathbf{z} - t\nabla j(\mathbf{z})) - \mathbf{z}\|$$

for arbitrary fixed $t > 0$.

Inexact Function Algorithm

Assume $\tau_k^{\mathrm{obj}} \le \mu_{\mathrm{obj}} \tau_{k+1}^{\mathrm{obj}}$, then $\kappa_{\mathrm{obj}} = (1+\mu_{\mathrm{obj}})\bar{\kappa}_{\mathrm{obj}} C_{\mathrm{obj}}$ since

$$|\mathrm{ared}_k - \mathrm{cred}_k| \le (1+\mu_{\mathrm{obj}})\bar{\kappa}_{\mathrm{obj}} C_{\mathrm{obj}} [\eta \min\{\mathrm{pred}_k, \theta_k\}]^{\zeta}.$$

Algorithm 6: Inexact Objective Function Computation

Require: Constants $\bar{\kappa}_{\mathrm{obj}} > 0$ and $\mu_{\mathrm{obj}} \ge 1$, the current objective function tolerance τ_k, the current iterate $x_k \in X$ and approximate objective value $v_k = \bar{f}(x_k, \tau_k)$, the new iterate $\mathbf{z}_{k+1} \in X$, and the predicted reduction pred_k.

Set $\tau_{k+1} \leftarrow \bar{\kappa}_{\mathrm{obj}} [\eta \min\{\mathrm{pred}_k, \theta_k\}]^{\zeta}$

if $\mu_{\mathrm{obj}} \tau_{k+1} < \tau_k$ **then**

  Compute $v_k \leftarrow \bar{f}(x_k, \tau_{k+1})$

end if

Compute $v_{k+1} \leftarrow \bar{f}(\mathbf{z}_{k+1}, \tau_{k+1})$ and set $\mathrm{cred}_k = (v_k + \phi(x_k)) - (v_{k+1} + \phi(\mathbf{z}_{k+1}))$

Inexact Gradient Algorithm

Assume that we have an approximation $\bar{g} : X \times [0, +\infty) \to X$ that satisfies $C_{\mathrm{grad}} \geq 0$ such that

$$\bar{g}(x, 0) = \nabla f(x) \quad \text{and} \quad \|\nabla f(x) - \bar{g}(x, \tau)\| \leq C_{\mathrm{grad}} \tau \quad \forall \tau \in \mathbb{R}_+. \tag{12}$$

Algorithm 7: Inexact Gradient Computation

Require: Constant $\bar{\kappa}_{\mathrm{grad}} > 0$, a tolerance $\mu_{\mathrm{grad}} > 1$, current iterate $x_k \in X$, and Δ_k.

Set $\tau_k^- \leftarrow \bar{\kappa}_{\mathrm{grad}} \Delta_k$

Compute $g_k \leftarrow \bar{g}(x_k, \tau_k^-)$ and $h_k \leftarrow r_0^{-1} \|\mathrm{Prox}_{r_0 \phi}(x_k - r_0 g_k) - x_k\|$

Set $\tau_k^+ \leftarrow \bar{\kappa}_{\mathrm{grad}} \min\{h_k, \Delta_k\}$

while $\mu_{\mathrm{grad}} \tau_k^+ < \tau_k^-$ **do**

  Compute $g_k \leftarrow \bar{g}(x_k, \tau_k^+)$ and $h_k \leftarrow r_0^{-1} \|\mathrm{Prox}_{r_0 \phi}(x_k - r_0 g_k) - x_k\|$

  Set $\tau_k^- \leftarrow \tau_k^+$ and $\tau_k^+ \leftarrow \bar{\kappa}_{\mathrm{grad}} \min\{h_k, \Delta_k\}$

end while

If $h(x_k) > 0$, Algorithm 7 terminates finitely and the inexact gradient condition holds with $\kappa_{\mathrm{grad}} = \mu_{\mathrm{grad}} \bar{\kappa}_{\mathrm{grad}} C_{\mathrm{grad}}$.

Inexact Gradient Computation

Define the adjoint equation residual $G : \mathcal{U} \times \mathcal{U} \times \mathcal{Z} \to \mathcal{U}$ by

$$G(\boldsymbol{\lambda}, \mathbf{u}, \mathbf{z}) := \mathrm{d}_{\mathbf{u}} f(\mathbf{u}, \mathbf{z}) + (\mathrm{d}_{\mathbf{u}} c(\mathbf{u}, \mathbf{z}))^* \boldsymbol{\lambda}$$

with $\Lambda(\mathbf{u}, \mathbf{z}) \in \mathcal{U}$ the solution of $G(\Lambda(\mathbf{u}, \mathbf{z}), \mathbf{u}, \mathbf{z}) = 0$ for the fixed $\mathbf{u}, \mathbf{z}$. Next define the equation

$$g(\boldsymbol{\lambda}, \mathbf{u}, \mathbf{z}) := \mathrm{d}_{\mathbf{z}} f(\mathbf{u}, \mathbf{z}) + (\mathrm{d}_{\mathbf{z}} c(\mathbf{u}, \mathbf{z}))^* \boldsymbol{\lambda}.$$

When evaluated at $\mathbf{u} = S(\mathbf{z})$ and $\boldsymbol{\lambda} = \Lambda(\mathbf{u}, \mathbf{z})$, $g(\boldsymbol{\lambda}, \mathbf{u}, \mathbf{z})$ is the gradient of the reduced objective function j as in (5).

Evaluating $g(\boldsymbol{\lambda}, \mathbf{u}, \mathbf{z})$ at the sketched state $\mathbf{u}^r = \mathrm{vec}(U^r)$ instead of the full state trajectory $\mathbf{u} = S(\mathbf{z})$ reduces the memory burden‘ for gradient computation but $g^r(\mathbf{z}) = g(\Lambda(\mathbf{u}^r, \mathbf{z}), \mathbf{u}^r, \mathbf{z})$ an approximation of true gradient $g(\Lambda(S(\mathbf{z}), \mathbf{z}), S(\mathbf{z}), \mathbf{z})$.