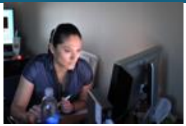
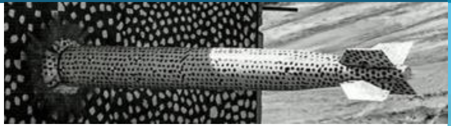
SAND2023-06811C

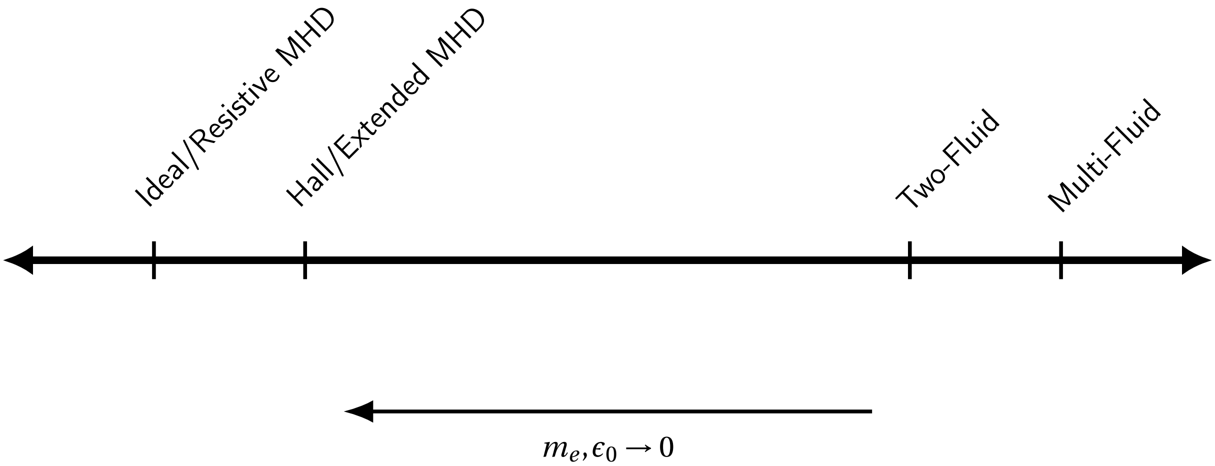
A Non-Neutral Generalized Ohm's Law Model for Magnetohydrodynamics in the Two-Fluid Regime

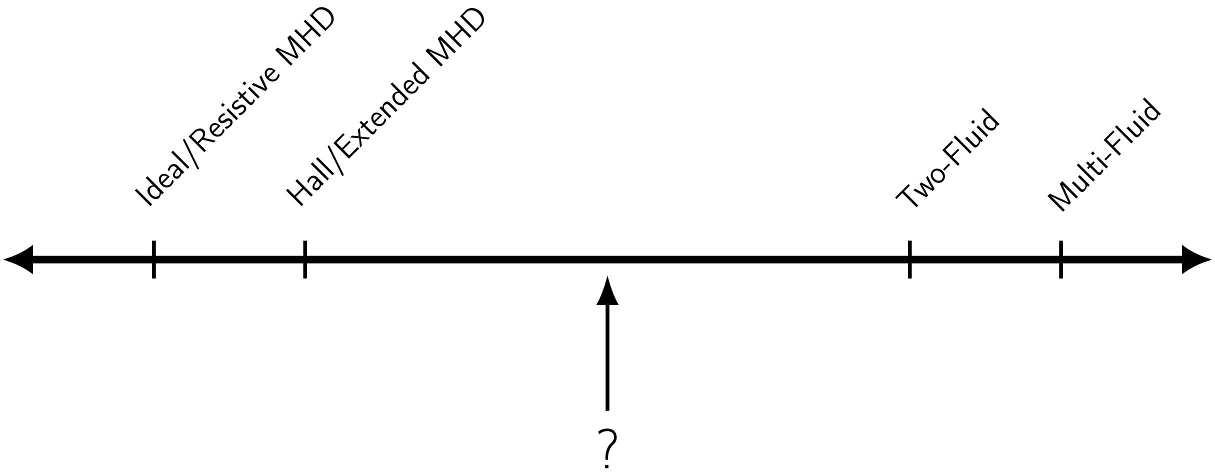
Michael M. Crockatt (SNL) and John N. Shadid (SNL)

Based on: M. M. Crockatt and J. N. Shadid, *A non-neutral generalized Ohm's law model for magnetohydrodynamics in the two-fluid regime*, Physics of

Wide range of fluid models available



Wide range of fluid models available



Ideal/Resistive MHD

$$
\begin{aligned}
\partial_t \rho + \nabla \cdot \left(\rho \mathbf{u}\right) &= 0 \\
\partial_t \left(\rho \mathbf{u}\right) + \nabla \cdot \left[\rho \mathbf{u} \otimes \mathbf{u} + p \underline{\mathbf{I}}\right] &= \mathbf{J} \times \mathbf{B} \\
\partial_t \mathcal{E} + \nabla \cdot \left[\mathbf{u}\left(\mathcal{E} + p\right)\right] &= \mathbf{J} \cdot \mathbf{E} \\
q &= 0 \\
0 &= \mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{\sigma}\mathbf{J} \\
\partial_t \mathbf{B} + \nabla \times \mathbf{E} &= 0 \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J}
\end{aligned}
$$

$$
\mathcal{E} = \varepsilon + \frac{1}{2}\rho \mathbf{u}^2
$$

- Magnetosonic wave speed unbounded as $\rho \to 0$ $\left[v_A^2 = \mathbf{B}^2/(\mu_0 \rho)\right]$.
- No Hall term (asymmetry, fast magnetic reconnection).

Ref: N. Shen, Y. Li, D. I. Pullin, R. Samtaney, and V. Wheatley, *On the magnetohydrodynamic limits of the ideal two-fluid plasma equations*, Physics of Plasmas, 25 (2018), p. 122113, doi:10.1063/1.5067387.

Hall/Extended MHD

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$
$$\partial_t (\rho \mathbf{u}) + \nabla \cdot [\rho \mathbf{u} \otimes \mathbf{u} + p \underline{\mathbf{I}}] = \mathbf{J} \times \mathbf{B}$$
$$\partial_t \mathcal{E} + \nabla \cdot [\mathbf{u}(\mathcal{E} + p)] = \mathbf{J} \cdot \mathbf{E}$$
$$q = 0$$
$$0 = \mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{\sigma}\mathbf{J} + \frac{1}{e n_e}\mathbf{J} \times \mathbf{B}$$
$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\mathcal{E} = \varepsilon + \frac{1}{2}\rho \mathbf{u}^2$$

- Magnetosonic wave speed unbounded as $\rho \to 0$ $[v_A^2 = \mathbf{B}^2/(\mu_0 \rho)]$.
- Hall term introduces Whistler wave (quadratic dispersion, unbounded).

Ref: N. Shen, Y. Li, D. I. Pullin, R. Samtaney, and V. Wheatley, *On the magnetohydrodynamic limits of the ideal two-fluid plasma equations*, Physics of Plasmas, 25 (2018), p. 122113, doi:10.1063/1.5067387.

PERSEUS quasi-neutral GOL model

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\partial_t (\rho \mathbf{u}) + \nabla \cdot [\rho \mathbf{u} \otimes \mathbf{u} + p \underline{\mathbf{I}}] = \mathbf{J} \times \mathbf{B}$$

$$\partial_t \mathcal{E} + \nabla \cdot [\mathbf{u} (\mathcal{E} + p)] = \mathbf{J} \cdot \mathbf{E}$$

$$q = 0$$

$$\partial_t \mathbf{J} + \nabla \cdot \left[\mathbf{u} \otimes \mathbf{J} + \mathbf{J} \otimes \mathbf{u} - \frac{\mathbf{J} \otimes \mathbf{J}}{\mathrm{e} n_e} - \frac{\mathrm{e}}{m_e} p_e \underline{\mathbf{I}} \right] = \frac{\mathrm{e}^2 n_e}{m_e} \left[\mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{\sigma} \mathbf{J} + \frac{1}{\mathrm{e} n_e} \mathbf{J} \times \mathbf{B} \right]$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$$

$$\frac{1}{c^2} \partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J}$$

$$\mathcal{E} = \varepsilon + \frac{1}{2} \rho \mathbf{u}^2$$

- Magnetosonic/Whistler modes bounded by light wave.
- Whistler mode bounded by electron cyclotron resonance.
- Quasi-neutrality requires $\nabla \cdot \mathbf{J} = 0$ (not guaranteed).

Ref 1: C. E. Seyler and M. R. Martin, *Relaxation model for extended magnetohydrodynamics: Comparison to magnetohydrodynamics for dense Z-pinches*, Physics of Plasmas, 18 (2011), p. 012703, doi:10.1063/1.3543799.

Ref 2: M. Martin, *Generalized Ohm's Law at the Plasma-Vacuum Interface*, PhD, Cornell University, 2010.

PERSEUS non-neutral GOL model

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0$$

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot\left[\rho\mathbf{u}\otimes\mathbf{u} + p\underline{\mathbf{I}} + \frac{m_e}{\mathrm{e}^2 n_e}(q\mathbf{u}-\mathbf{J})\otimes(q\mathbf{u}-\mathbf{J})\right] = q\mathbf{E} + \mathbf{J}\times\mathbf{B}$$

$$\partial_t \mathcal{E} + \nabla\cdot\left[\mathbf{u}(\mathcal{E}+p) + \frac{1}{\mathrm{e}n_e}(q\mathbf{u}-\mathbf{J})(\mathcal{E}_e+p_e)\right] = \mathbf{J}\cdot\mathbf{E}$$

$$\partial_t q + \nabla\cdot\mathbf{J} = 0$$

$$\partial_t \mathbf{J} + \nabla\cdot\left[\xi(\mathbf{u}\otimes\mathbf{J} + \mathbf{J}\otimes\mathbf{u} - q\mathbf{u}\otimes\mathbf{u}) - \frac{\mathbf{J}\otimes\mathbf{J}}{\mathrm{e}n_e} - \frac{\mathrm{e}}{m_e}p_e\underline{\mathbf{I}}\right] = \frac{\mathrm{e}^2 n_e}{m_e}\left[\mathbf{E} + \xi\mathbf{u}\times\mathbf{B} + \frac{1}{\sigma}(q\mathbf{u}-\mathbf{J}) + \frac{1}{\mathrm{e}n_e}\mathbf{J}\times\mathbf{B}\right]$$

$$\partial_t \mathcal{E}_e + \nabla\cdot\left[\left(\xi\mathbf{u} - \frac{\mathbf{J}}{\mathrm{e}n_e}\right)(\mathcal{E}_e+p_e)\right] = \left[\mathbf{J} - (\mathrm{e}n_e+q)\mathbf{u}\right]\cdot\mathbf{E} + \mathcal{C}_e$$

$$\partial_t \mathbf{B} + \nabla\times\mathbf{E} = 0$$

$$\xi = 1 + \frac{q}{\mathrm{e}n_e} \qquad \frac{1}{c^2}\partial_t\mathbf{E} - \nabla\times\mathbf{B} = -\mu_0\mathbf{J}$$

$$\mathcal{E} = \varepsilon + \frac{1}{2}\rho\mathbf{u}^2 + \frac{1}{2}\frac{m_e}{\mathrm{e}^2 n_e}(q\mathbf{u}-\mathbf{J})^2$$

$$\mathcal{E}_e = \varepsilon_e + \frac{1}{2}m_e n_e\left(\xi\mathbf{u} - \frac{\mathbf{J}}{\mathrm{e}n_e}\right)^2$$

- Derived but not originally implemented.
- Unstable for significant charge separation.

Ref: M. Martin, *Generalized Ohm's Law at the Plasma-Vacuum Interface*, PhD, Cornell University, 2010.

Derive an improved non-neutral model

- Only assumption is that $m_e << m_\alpha$. Model is simplified using the approximation:

$$1 + \frac{\overline{Z}_\alpha m_e}{m_\alpha} \approx 1.$$

- Previous derivations assumed that

$$\frac{\rho_\alpha}{\rho} \approx 1, \qquad \frac{\rho_e}{\rho} = O\left(\frac{m_e}{m_\alpha}\right) \to 0.$$

  This approximation is unsuitable for significant charge separation (limit where $\rho_\alpha \to 0$).

- Instead, introduce parameter ψ such that

$$\frac{\rho_\alpha}{\rho} \approx \psi, \qquad \frac{\rho_e}{\rho} \approx 1 - \psi.$$

Two-temperature non-neutral GOL MHD model

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\partial_t (\rho \mathbf{u}) + \nabla \cdot \left[\rho \mathbf{u} \otimes \mathbf{u} + p \underline{\mathbf{I}} + \frac{m_e}{\mathrm{e}^2 n_e} \psi^{-1} (q\mathbf{u} - \mathbf{J}) \otimes (q\mathbf{u} - \mathbf{J}) \right] = q\mathbf{E} + \mathbf{J} \times \mathbf{B}$$

$$\partial_t \mathcal{E} + \nabla \cdot \left[\mathbf{u} (\mathcal{E} + p) + \frac{\psi^{-1}}{\mathrm{e} n_e} (q\mathbf{u} - \mathbf{J}) \Big((\psi - 1)(\mathcal{E} + p) + \mathcal{E}_e + p_e \Big) \right] = \mathbf{J} \cdot \mathbf{E}$$

$$\partial_t q + \nabla \cdot \mathbf{J} = 0$$

$$\partial_t \mathbf{J} + \nabla \cdot \left[\xi (\mathbf{u} \otimes \mathbf{J} + \mathbf{J} \otimes \mathbf{u} - q \mathbf{u} \otimes \mathbf{u}) - \frac{\mathbf{J} \otimes \mathbf{J}}{\mathrm{e} n_e} - \frac{\mathrm{e}}{m_e} p_e \underline{\mathbf{I}} + \frac{\xi}{\mathrm{e} n_e} \frac{1 - \psi}{\psi^2} (q\mathbf{u} - \mathbf{J}) \otimes (q\mathbf{u} - \mathbf{J}) \right] = \frac{\mathrm{e}^2 n_e}{m_e} \left[\mathbf{E} + \xi \mathbf{u} \times \mathbf{B} + \frac{1}{\sigma} (q\mathbf{u} - \mathbf{J}) + \frac{1}{\mathrm{e} n_e} \mathbf{J} \times \mathbf{B} \right]$$

$$\partial_t \mathcal{E}_e + \nabla \cdot \left[\left(\xi \mathbf{u} - \frac{\mathbf{J}}{\mathrm{e} n_e} \right) (\mathcal{E}_e + p_e) \right] = [\mathbf{J} - (\mathrm{e} n_e + q) \mathbf{u}] \cdot \mathbf{E}$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$$

$$\frac{1}{c^2} \partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J}$$

$$\xi = 1 + \frac{q}{\mathrm{e} n_e}$$

$$\psi = 1 + \frac{m_e q}{\mathrm{e} \rho}$$

$$\mathcal{E} = \varepsilon + \frac{1}{2} \rho \mathbf{u}^2 + \frac{1}{2} \frac{m_e}{\mathrm{e}^2 n_e} \psi^{-1} (q\mathbf{u} - \mathbf{J})^2$$

$$\mathcal{E}_e = \varepsilon_e + \frac{1}{2} m_e n_e \left(\xi \mathbf{u} - \frac{\mathbf{J}}{\mathrm{e} n_e} \right)^2$$

Ref: M. M. Crockatt and J. N. Shadid, *A non-neutral generalized Ohm's law model for magnetohydrodynamics in the two-fluid regime*, Physics of Plasmas, 30 (2023), p. 053902, doi:10.1063/5.0138673.

Numerical tests

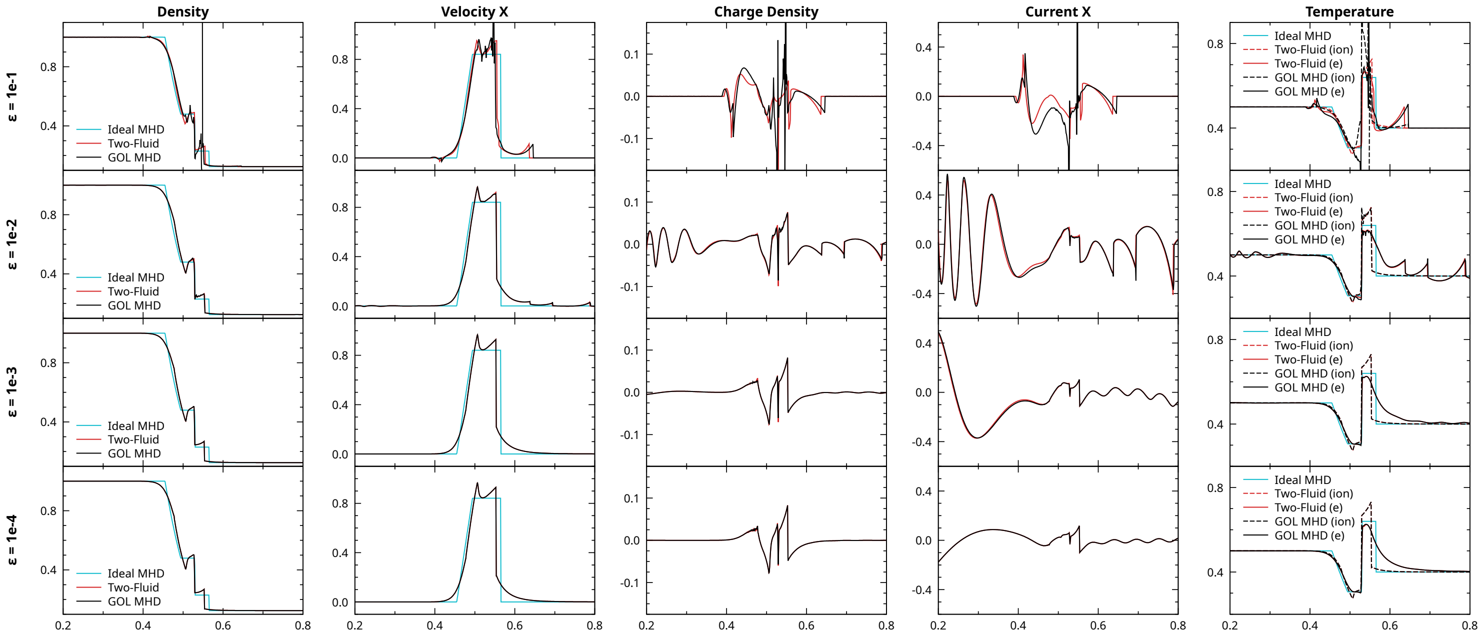
- Problems non-dimensionalized such that

$$1 = m_\alpha = m_e + m_i = \frac{\varepsilon}{1+\varepsilon} + \frac{1}{1+\varepsilon}.$$

- Simplified model valid in the limit where $\varepsilon \to 0$.
- Problems parameterized by plasma frequency, Debye length, and cyclotron frequency.
- Compare to existing two-fluid implementation:

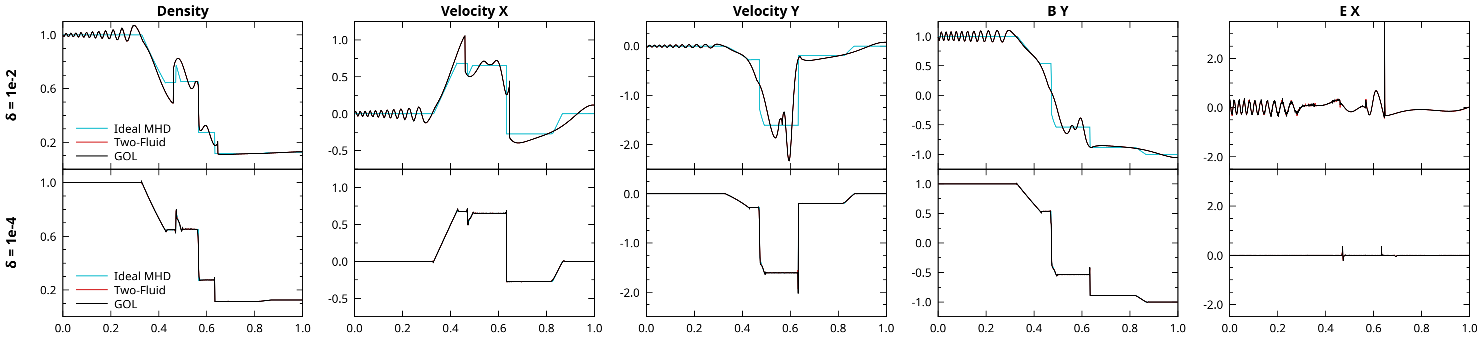
M. M. Crockatt, S. Mabuza, J. N. Shadid, S. Conde, T. M. Smith, and R. P. Pawlowski, *An implicit monolithic AFC stabilization method for the CG finite element discretization of the fully-ionized ideal multifluid electromagnetic plasma system*, Journal of Computational Physics, 464 (2022), p. 111228, doi:10.1016/j.jcp.2022.111228.

Sod electrostatic shock

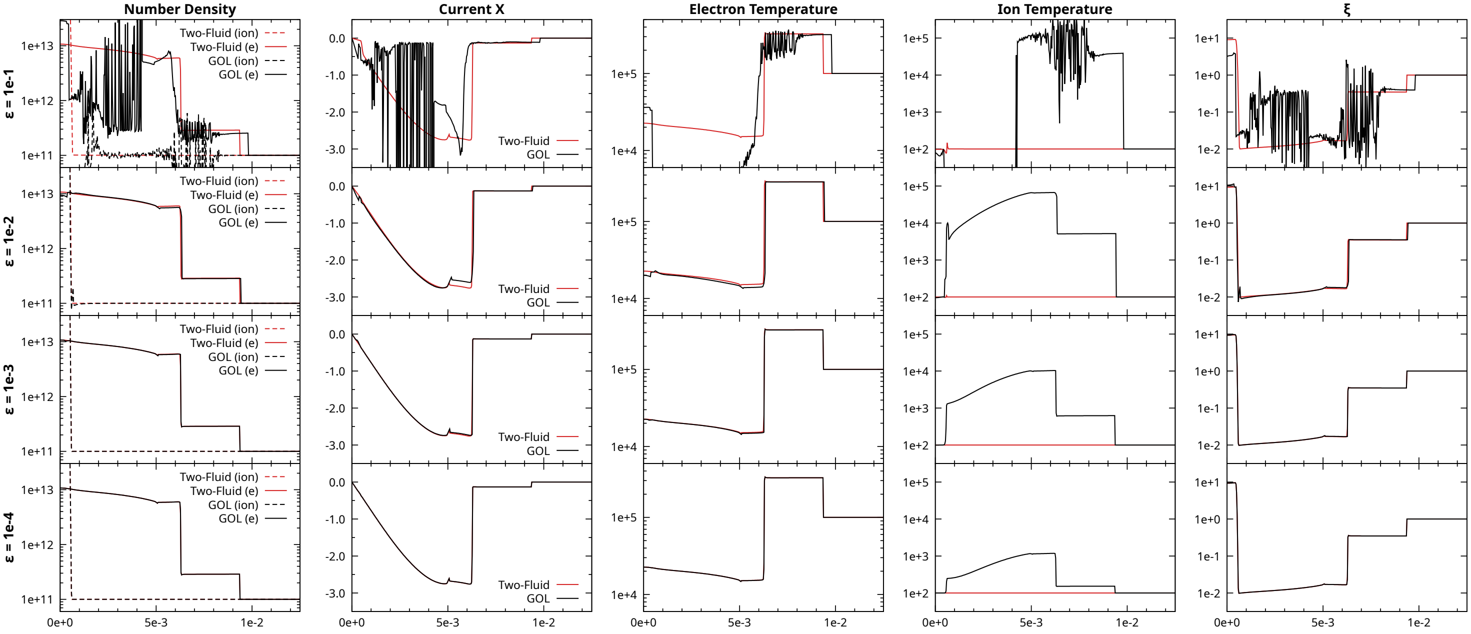


Brio-Wu electromagnetic shock

- $\varepsilon = 10^{-4}$
- δ scales cyclotron radii/frequency.



Expanding slab



Thank you