SAND2023-06492C

Sandia National Laboratories

Exceptional service in the national interest

The Dirichlet-Neumann Schwarz alternating method for contact problems in elastodynamics

Techniques for reducing artificial oscillations

Daria Koliesnikova[1],
Alejandro Mota[1], Irina Tezaur[2]

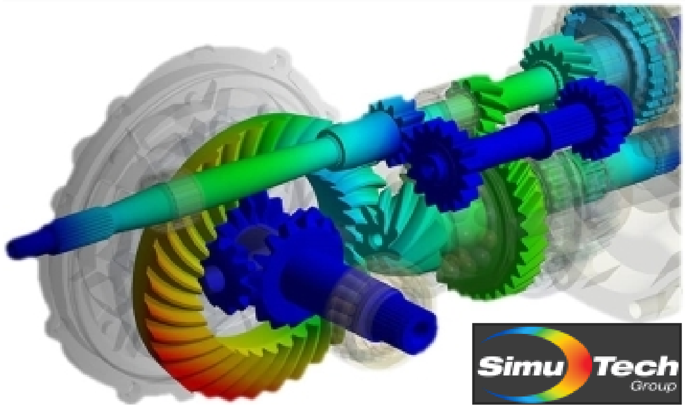
Sandia National Laboratories
[1] Mechanics of materials department
[2] Quantitative modeling and analysis department

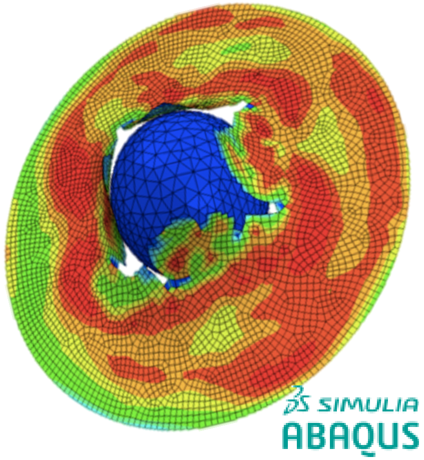
17th U. S. National Congress on Computational Mechanics
Albuquerque, New Mexico, July 23-27, 2023

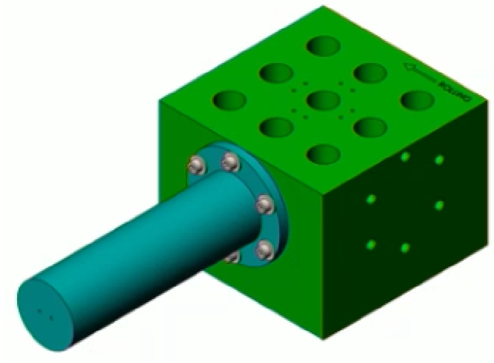
Context and motivations

Numerical simulation of challenging mechanical contact problems

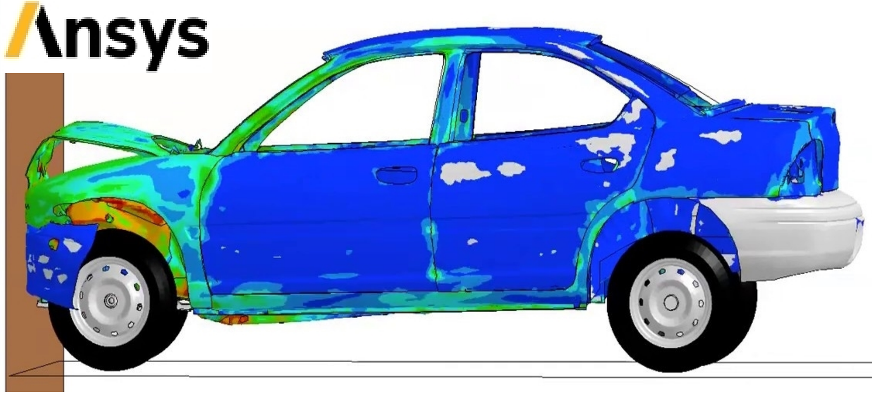
Localized contact, impact, touching surfaces, sliding, bolted/fastener joint, …
→ presence of nonlinearities and lack of smoothness





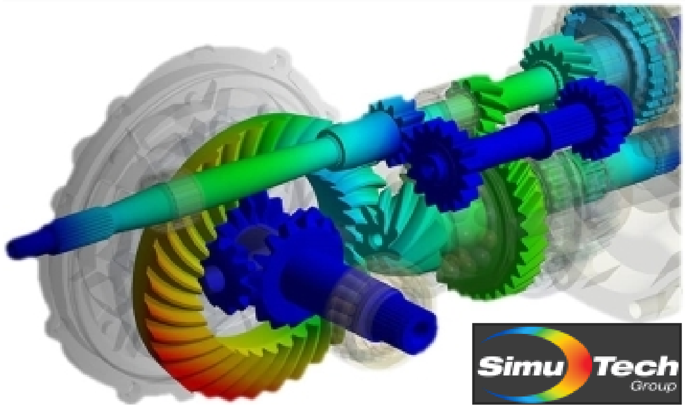


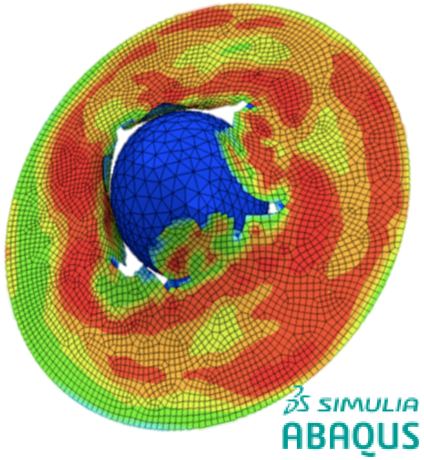
Sandia Mechanics Challenge

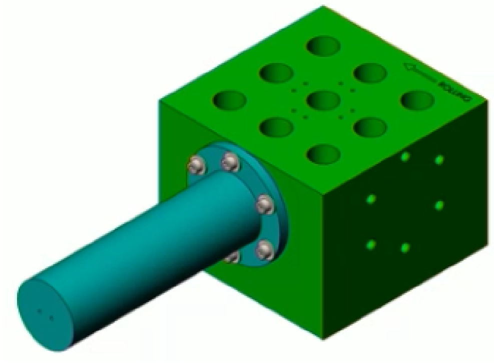


Numerical simulation of challenging mechanical contact problems

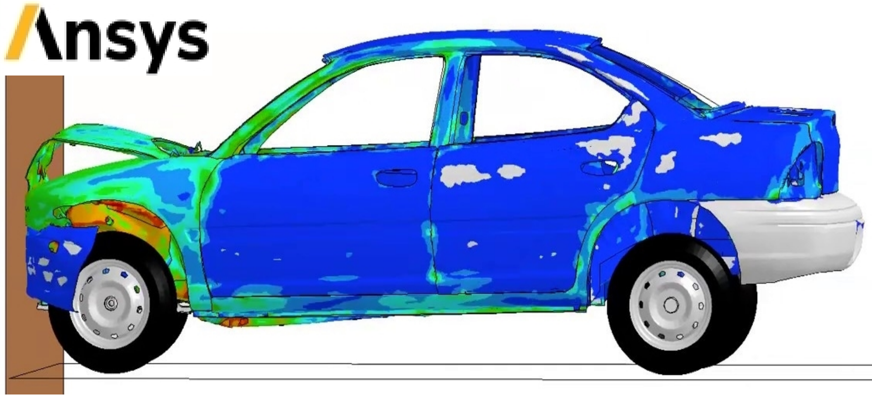
Localized contact, impact, touching surfaces, sliding, bolted/fastener joint, ...
→ presence of nonlinearities and lack of smoothness







Sandia Mechanics Challenge



→ Need a robust, stable and accurate method for mechanical contact

- accurate prediction of quantities of interest (velocities, contact forces, energies, ...)
- multiscale & multiphysics context (different time integrators, solvers, meshes, material models for different bodies)
- numerical efficiency and robustness
- non-intrusive implementation in existent legacy solvers (e.g. Sierra/Solid Mechanics (Sierra/SM), Albany, ...)

Numerical simulation of challenging multiscale contact problems

Conventional contact methods

- introduce **contact constraints** into the variational form
- solve all bodies involved in contact as a **"coupled" system**

Penalty method – amount of interpenetration ~ contact pressure
[G. L. Goudreau et al. (1982) | T. Belytschko et al. (1991)]
Lagrange Multipliers method – additional Lagrange constraints
[T. J. R. Hushes. (1976) | K. J. Bathe et al. (1956)]
Perturbed and **Augmented Lagrangian methods** – constrained optimization theory, hybrid penalty/LM approach
[J. C. Simo et al. (1985, 1992) | R. Glowinski et al. (1989)]
Nitsche method – optimization problem with contact constraints
[F. Chouly et al. (2016) | P. Wriggers et al. (2008)]

+ well-established (proven convergence, ...)
– accuracy and stability often affected by problem-dependent parameters
– all bodies are coupled and solved simultaneously
– intrusive implementation

Numerical simulation of challenging multiscale contact problems

Conventional contact methods

- introduce **contact constraints** into the variational form
- solve all bodies involved in contact as a **"coupled" system**

Penalty method – amount of interpenetration ~ contact pressure
[G. L. Goudreau et al. (1982) | T. Belytschko et al. (1991)]
Lagrange Multipliers method – additional Lagrange constraints
[T. J. R. Hushes. (1976) | K. J. Bathe et al. (1956)]
Perturbed and **Augmented Lagrangian methods** – constrained optimization theory, hybrid penalty/LM approach
[J. C. Simo et al. (1985, 1992) | R. Glowinski et al. (1989)]
Nitsche method – optimization problem with contact constraints
[F. Chouly et al. (2016) | P. Wriggers et al. (2008)]

\+ well-established (proven convergence, ...)
– accuracy and stability often affected by problem-dependent parameters
– all bodies are coupled and solved simultaneously
– intrusive implementation

Schwarz alternating method

- **treats each body separately** (as non-overlapping subdomains)
- prevents interpenetration through an **alternating Dirichlet-Neumann iterative process**

Domain Decomposition context – solving PDEs on irregular domains by splitting them into domains of regular shape, overlapping and non-overlapping approaches *[H. Schwarz (1870) | P. I. Lions (1990)]*
Multiscale coupling – overlapping approach *[A. Mota et al. (2017, 2021)]*
Contact/impact dynamics – non-overlapping approach *[A. Mota et al. (2023)*]*

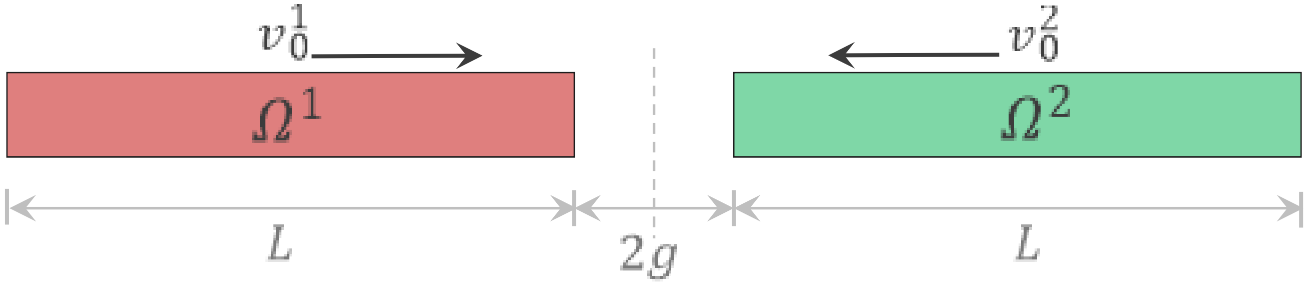
\+ strong theoretical basis (domain decomposition context)
\+ flexible (different time integration schemes, solvers, meshes, material models for different bodies)
\+ non-intrusive implementation
– iterative process and transfer operators

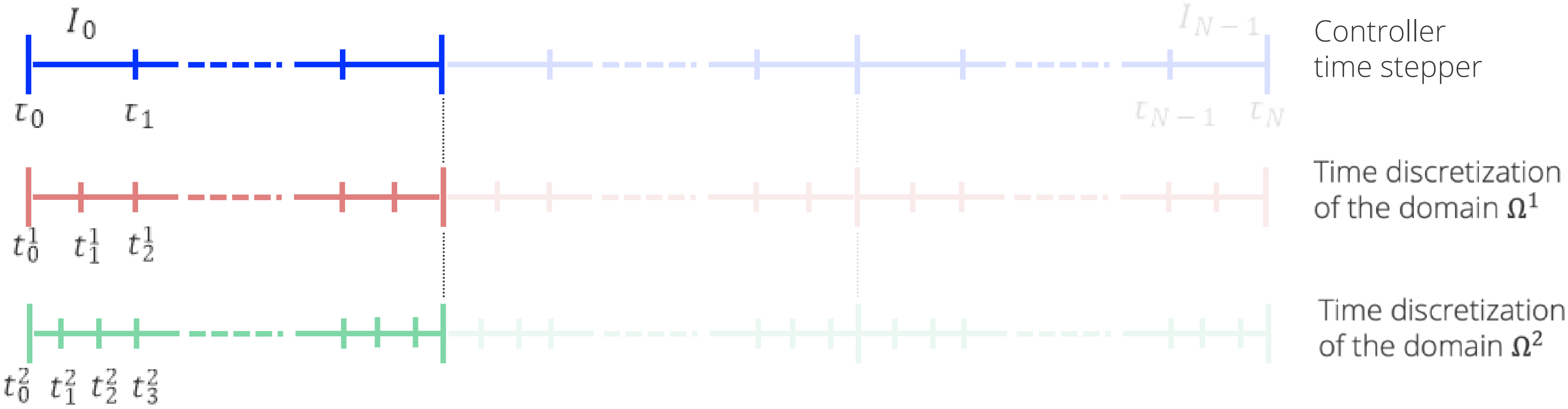
Schwarz alternating method for contact problems

Schwarz alternating method for contact problems

Before the contact phase:

- **solve PDEs in each body separately** (different solvers, discretization techniques, time integrators, ...)
- **check contact criteria** at each controller time step (overlap/distance conditions, compression condition, ...)

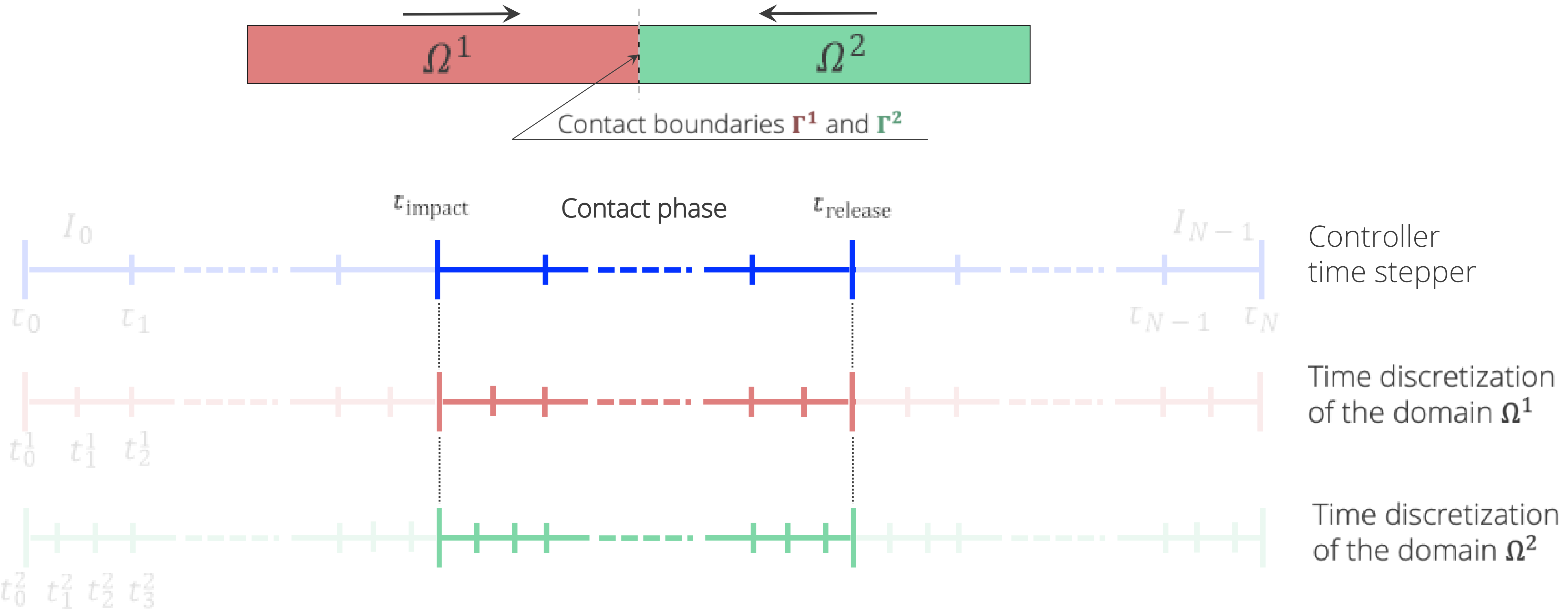




Schwarz alternating method for contact problems

During the contact phase:

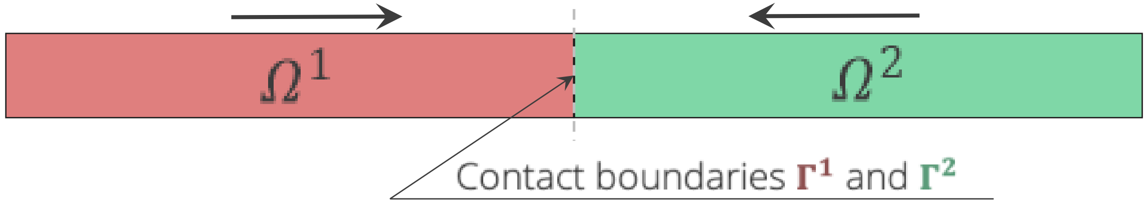
- solve PDEs in each body separately (different solvers, discretization techniques, time integrators, ...)
- prevent interpenetration through an iterative process based on the Dirichlet-Neumann contact boundary conditions
- check contact criteria at each controller time step (overlap/distance conditions, compression condition, ...)

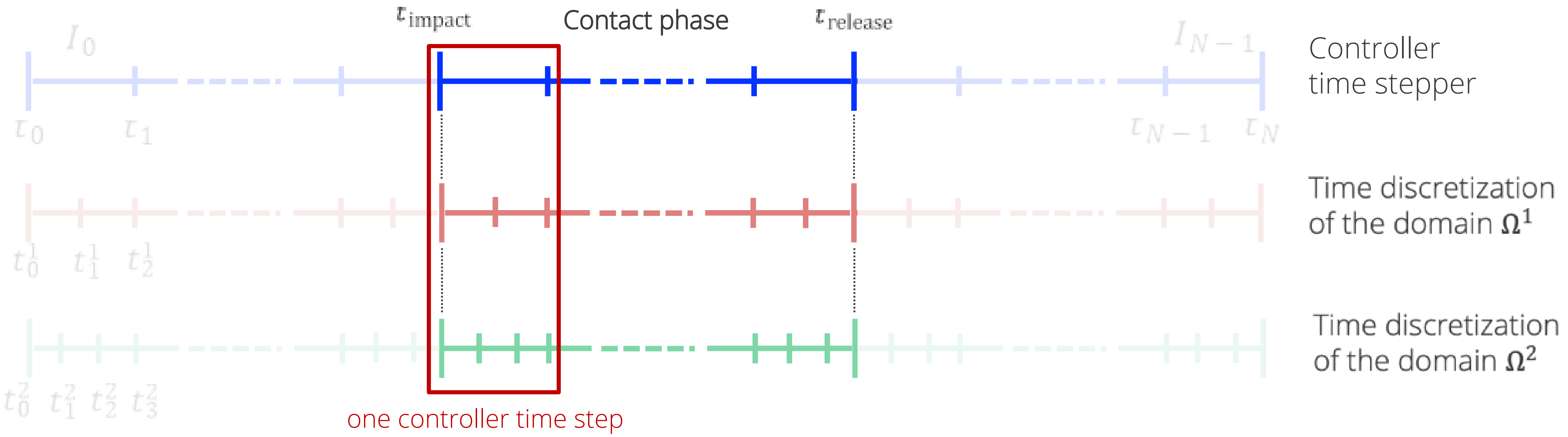


Schwarz alternating method for contact problems

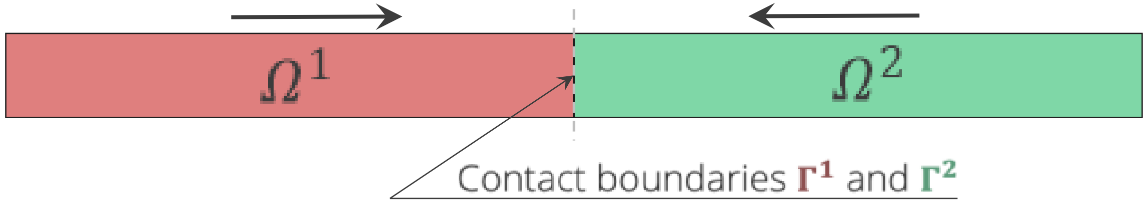
During the contact phase:

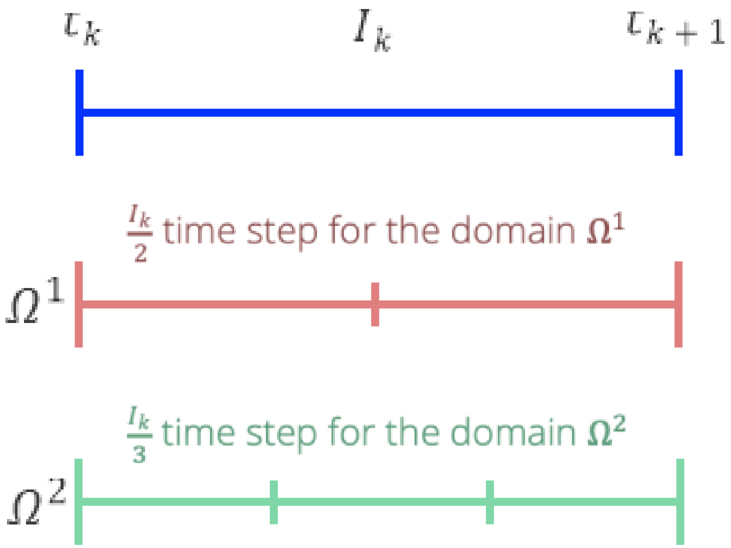
- **solve PDEs in each body separately** (different solvers, discretization techniques, time integrators, ...)
- prevent interpenetration through an iterative process based on the **Dirichlet-Neumann contact boundary conditions**
- **check contact criteria** at each controller time step (overlap/distance conditions, compression condition, ...)





Schwarz alternating method for contact problems



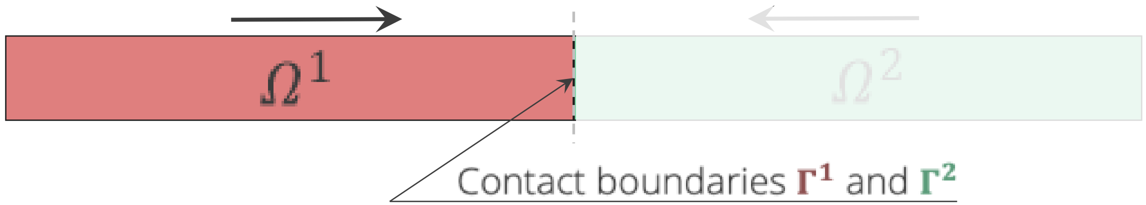


$$\Omega^1 \begin{cases} \operatorname{div} \boldsymbol{P}^n + \rho \boldsymbol{B} = \rho \ddot{\boldsymbol{\varphi}}^n & \text{in } \Omega^1 \times I_k & \text{Dynamic problem} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \boldsymbol{\chi} & \text{on } \partial_{\varphi}\Omega^1 \times I_k & \text{Regular Dirichlet BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \boldsymbol{T} & \text{on } \partial_{T}\Omega^1 \times I_k & \text{Regular Neumann BCs} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \mathcal{P}^{\varphi}_{(\Gamma^2 \to \Gamma^1) \times I_k} \boldsymbol{\varphi}^{n-1}(\Omega^2, t) & \text{on } \Gamma^1 \times I_k & \text{Contact Dirichlet BCs} \end{cases}$$

$$\Omega^2 \begin{cases} \operatorname{div} \boldsymbol{P}^n + \rho \boldsymbol{B} = \rho \ddot{\boldsymbol{\varphi}}^n & \text{in } \Omega^2 \times I_k & \text{Dynamic problem} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \boldsymbol{\chi} & \text{on } \partial_{\varphi}\Omega^2 \times I_k & \text{Regular Dirichlet BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \boldsymbol{T} & \text{on } \partial_{T}\Omega^2 \times I_k & \text{Regular Neumann BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \mathcal{P}^{T}_{(\Gamma^1 \to \Gamma^2) \times I_k} \boldsymbol{T}^n(\Omega^1, t) & \text{on } \Gamma^2 \times I_k & \text{Contact Neumann BCs} \end{cases}$$

Iterate until Schwarz convergence criteria are satisfied

Schwarz alternating method for contact problems

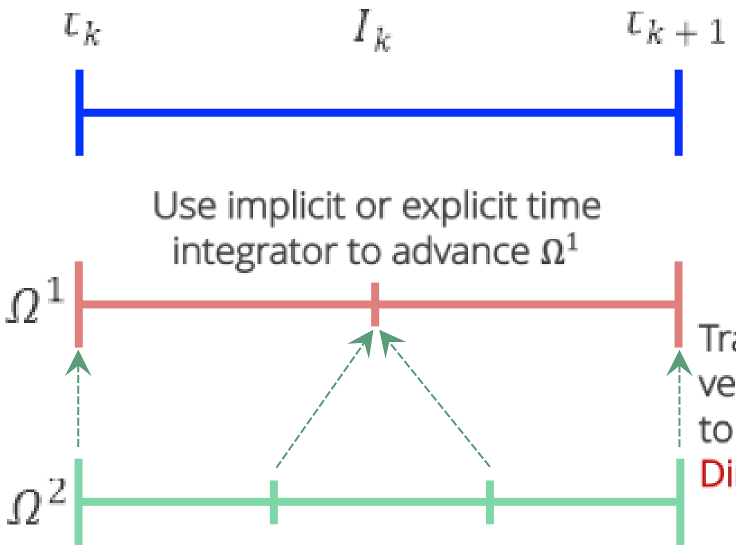


Notations

displacement	φ	mass density	ρ
acceleration	$\ddot{\varphi}$	body force	ρB
Piola-Kirchoff stress	P	traction	T

$$\Omega^1 \begin{cases} \operatorname{div} \boldsymbol{P}^n + \rho \boldsymbol{B} = \rho \ddot{\boldsymbol{\varphi}}^n & \text{in } \Omega^1 \times I_k \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \boldsymbol{\chi} & \text{on } \partial_{\varphi}\Omega^1 \times I_k \\ \boldsymbol{P}^n \boldsymbol{N} = \boldsymbol{T} & \text{on } \partial_{T}\Omega^1 \times I_k \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \mathcal{P}^{\varphi}_{(\Gamma^2 \to \Gamma^1)\times I_k} \boldsymbol{\varphi}^{n-1}(\Omega^2, t) & \text{on } \Gamma^1 \times I_k \end{cases}$$

Dynamic problem
Regular Dirichlet BCs
Regular Neumann BCs
Contact Dirichlet BCs



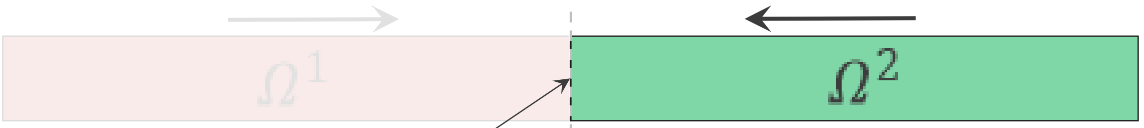
Transfer displacement, velocity and acceleration to impose contact Dirichlet BCs on Γ^1

$$\Omega^2 \begin{cases} \operatorname{div} \boldsymbol{P}^n + \rho \boldsymbol{B} = \rho \ddot{\boldsymbol{\varphi}}^n & \text{in } \Omega^2 \times I_k \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \boldsymbol{\chi} & \text{on } \partial_{\varphi}\Omega^2 \times I_k \\ \boldsymbol{P}^n \boldsymbol{N} = \boldsymbol{T} & \text{on } \partial_{T}\Omega^2 \times I_k \\ \boldsymbol{P}^n \boldsymbol{N} = \mathcal{P}^{T}_{(\Gamma^1 \to \Gamma^2)\times I_k} \boldsymbol{T}^{n}(\Omega^1, t) & \text{on } \Gamma^2 \times I_k \end{cases}$$

Dynamic problem
Regular Dirichlet BCs
Regular Neumann BCs
Contact Neumann BCs

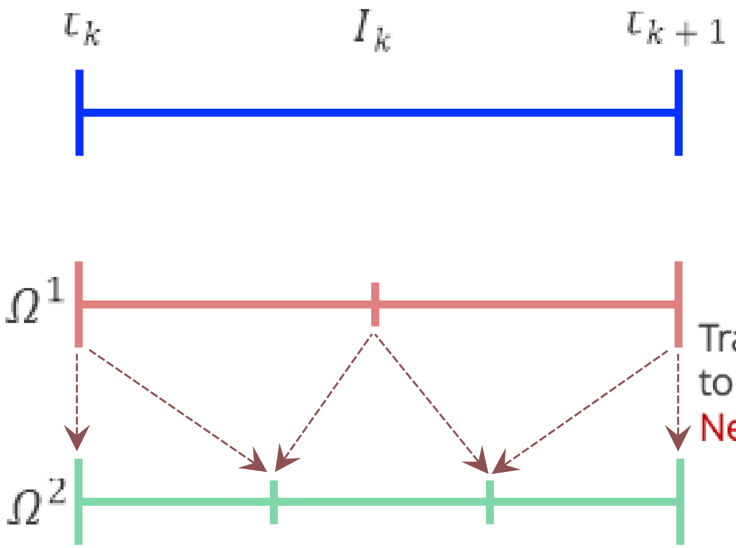
Iterate until Schwarz convergence criteria are satisfied

Schwarz alternating method for contact problems



Notations

displacement	$\boldsymbol{\varphi}$	mass density	ρ
acceleration	$\ddot{\boldsymbol{\varphi}}$	body force	$\rho\boldsymbol{B}$
Piola-Kirchoff stress	$\boldsymbol{P}$	traction	$\boldsymbol{T}$



Transfer traction to impose contact Neumann BCs on Γ^2

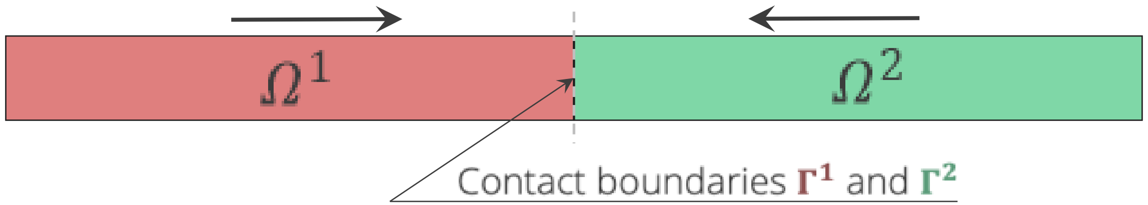
Use implicit or explicit time integrator to advance Ω^2

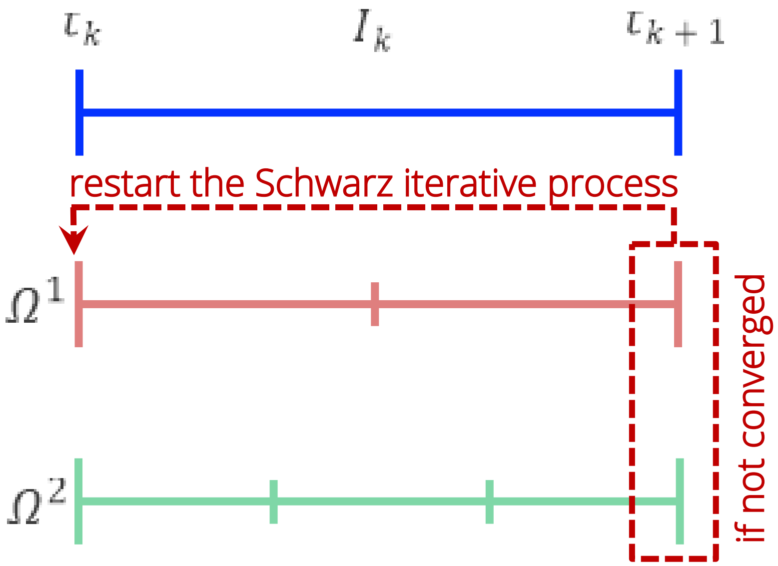
$$\Omega^1 \begin{cases} \operatorname{div} \boldsymbol{P}^n + \rho\boldsymbol{B} = \rho\ddot{\boldsymbol{\varphi}}^n & \text{in } \Omega^1 \times I_k & \text{Dynamic problem} \\ \boldsymbol{\varphi}^n(\boldsymbol{x},t) = \boldsymbol{\chi} & \text{on } \partial_{\varphi}\Omega^1 \times I_k & \text{Regular Dirichlet BCs} \\ \boldsymbol{P}^n\boldsymbol{N} = \boldsymbol{T} & \text{on } \partial_T\Omega^1 \times I_k & \text{Regular Neumann BCs} \\ \boldsymbol{\varphi}^n(\boldsymbol{x},t) = \mathcal{P}^{\varphi}_{(\Gamma^2\to\Gamma^1)\times I_k}\boldsymbol{\varphi}^{n-1}(\Omega^2,t) & \text{on } \Gamma^1 \times I_k & \text{Contact Dirichlet BCs} \end{cases}$$

$$\Omega^2 \begin{cases} \operatorname{div} \boldsymbol{P}^n + \rho\boldsymbol{B} = \rho\ddot{\boldsymbol{\varphi}}^n & \text{in } \Omega^2 \times I_k & \text{Dynamic problem} \\ \boldsymbol{\varphi}^n(\boldsymbol{x},t) = \boldsymbol{\chi} & \text{on } \partial_{\varphi}\Omega^2 \times I_k & \text{Regular Dirichlet BCs} \\ \boldsymbol{P}^n\boldsymbol{N} = \boldsymbol{T} & \text{on } \partial_T\Omega^2 \times I_k & \text{Regular Neumann BCs} \\ \boldsymbol{P}^n\boldsymbol{N} = \mathcal{P}^{\boldsymbol{T}}_{(\Gamma^1\to\Gamma^2)\times I_k}\boldsymbol{T}^n(\Omega^1,t) & \text{on } \Gamma^2 \times I_k & \text{Contact Neumann BCs} \end{cases}$$

Iterate until Schwarz convergence criteria are satisfied

Schwarz alternating method for contact problems



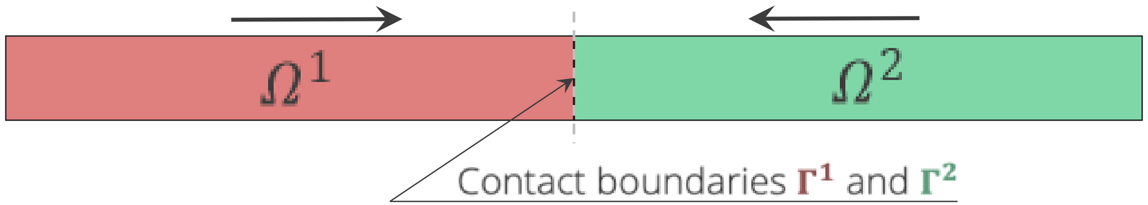


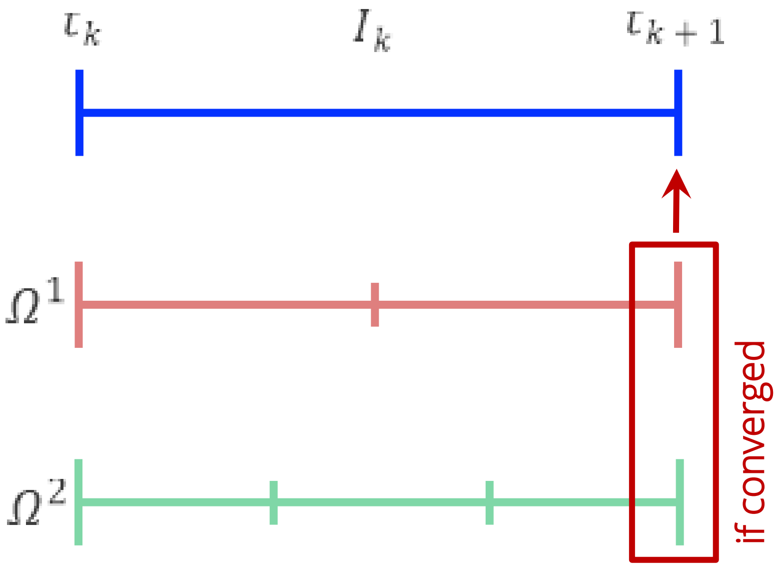
$$
\Omega^1 \begin{cases} \operatorname{div} \boldsymbol{P}^n + \rho \boldsymbol{B} = \rho \ddot{\boldsymbol{\varphi}}^n & \text{in } \Omega^1 \times I_k & \text{Dynamic problem} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \chi & \text{on } \partial_{\boldsymbol{\varphi}} \Omega^1 \times I_k & \text{Regular Dirichlet BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \boldsymbol{T} & \text{on } \partial_{\boldsymbol{T}} \Omega^1 \times I_k & \text{Regular Neumann BCs} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \mathcal{P}^{\boldsymbol{\varphi}}_{(\Gamma^2 \to \Gamma^1) \times I_k} \boldsymbol{\varphi}^{n-1}(\Omega^2, t) & \text{on } \Gamma^1 \times I_k & \text{Contact Dirichlet BCs} \end{cases}
$$

$$
\Omega^2 \begin{cases} \operatorname{div} \boldsymbol{P}^n + \rho \boldsymbol{B} = \rho \ddot{\boldsymbol{\varphi}}^n & \text{in } \Omega^2 \times I_k & \text{Dynamic problem} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \chi & \text{on } \partial_{\boldsymbol{\varphi}} \Omega^2 \times I_k & \text{Regular Dirichlet BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \boldsymbol{T} & \text{on } \partial_{\boldsymbol{T}} \Omega^2 \times I_k & \text{Regular Neumann BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \mathcal{P}^{\boldsymbol{T}}_{(\Gamma^1 \to \Gamma^2) \times I_k} \boldsymbol{T}^n(\Omega^1, t) & \text{on } \Gamma^2 \times I_k & \text{Contact Neumann BCs} \end{cases}
$$

Iterate until Schwarz convergence criteria are satisfied

Schwarz alternating method for contact problems



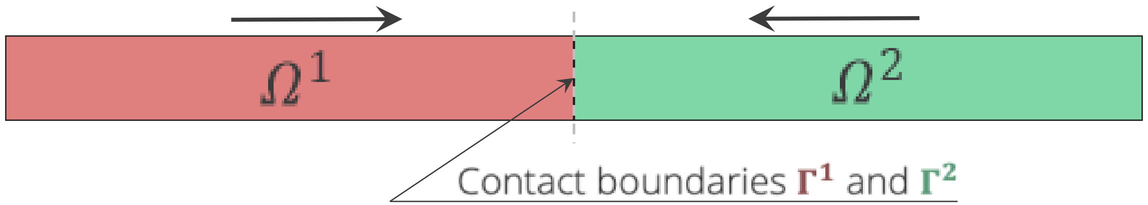


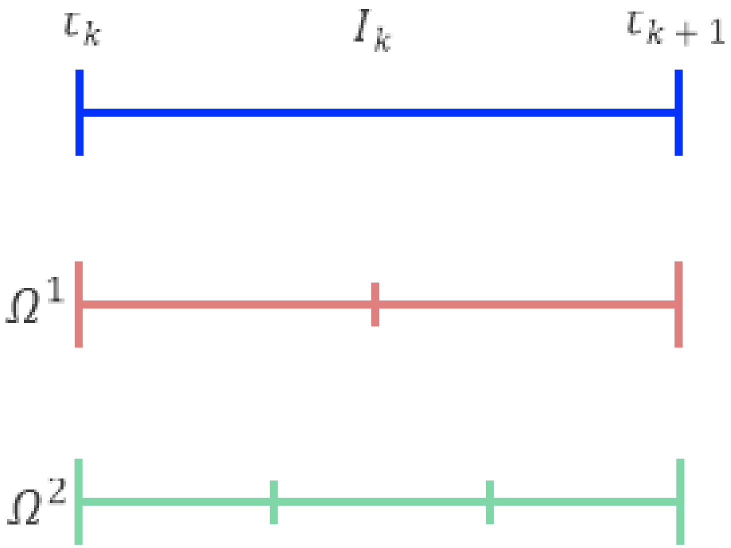
$$\Omega^1 \begin{cases} \operatorname{div} \boldsymbol{P}^n + \rho \boldsymbol{B} = \rho \ddot{\boldsymbol{\varphi}}^n & \text{in } \Omega^1 \times I_k & \text{Dynamic problem} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \chi & \text{on } \partial_{\boldsymbol{\varphi}} \Omega^1 \times I_k & \text{Regular Dirichlet BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \boldsymbol{T} & \text{on } \partial_{\boldsymbol{T}} \Omega^1 \times I_k & \text{Regular Neumann BCs} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \mathcal{P}^{\boldsymbol{\varphi}}_{(\Gamma^2 \to \Gamma^1) \times I_k} \boldsymbol{\varphi}^{n-1}(\Omega^2, t) & \text{on } \Gamma^1 \times I_k & \text{Contact Dirichlet BCs} \end{cases}$$

$$\Omega^2 \begin{cases} \operatorname{div} \boldsymbol{P}^n + \rho \boldsymbol{B} = \rho \ddot{\boldsymbol{\varphi}}^n & \text{in } \Omega^2 \times I_k & \text{Dynamic problem} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \chi & \text{on } \partial_{\boldsymbol{\varphi}} \Omega^2 \times I_k & \text{Regular Dirichlet BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \boldsymbol{T} & \text{on } \partial_{\boldsymbol{T}} \Omega^2 \times I_k & \text{Regular Neumann BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \mathcal{P}^{\boldsymbol{T}}_{(\Gamma^1 \to \Gamma^2) \times I_k} \boldsymbol{T}^n(\Omega^1, t) & \text{on } \Gamma^2 \times I_k & \text{Contact Neumann BCs} \end{cases}$$

Iterate until Schwarz convergence criteria are satisfied

Schwarz alternating method for contact problems



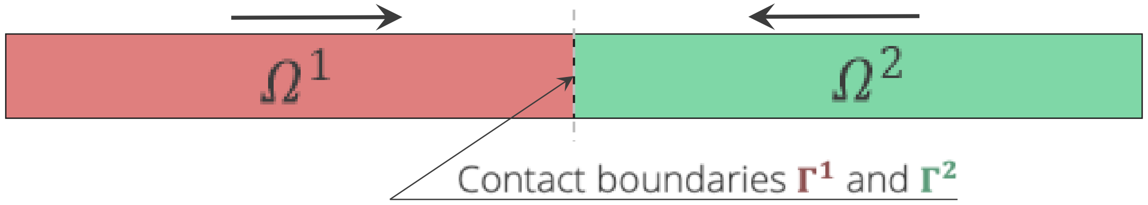


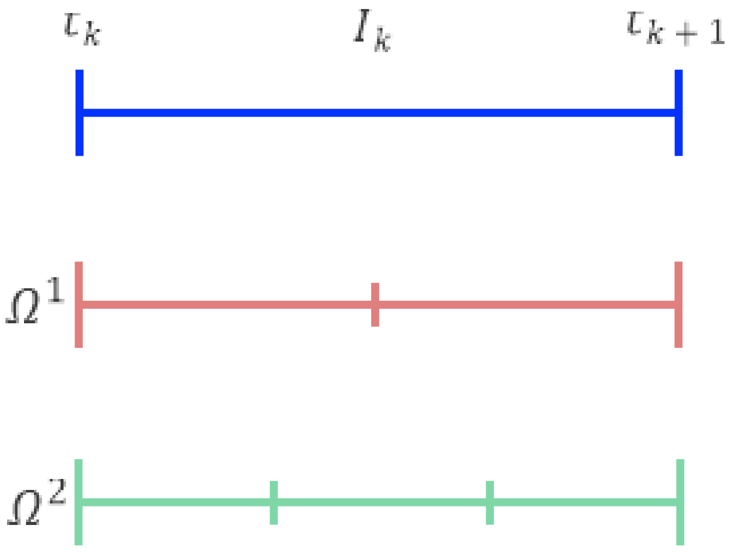
$$\Omega^1 \begin{cases} \operatorname{div} \boldsymbol{P}^n + \rho \boldsymbol{B} = \rho \ddot{\boldsymbol{\varphi}}^n & \text{in } \Omega^1 \times I_k & \text{Dynamic problem} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \chi & \text{on } \partial_{\boldsymbol{\varphi}} \Omega^1 \times I_k & \text{Regular Dirichlet BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \boldsymbol{T} & \text{on } \partial_{\boldsymbol{T}} \Omega^1 \times I_k & \text{Regular Neumann BCs} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \mathcal{P}^{\boldsymbol{\varphi}}_{(\Gamma^2 \to \Gamma^1) \times I_k} \boldsymbol{\varphi}^{n-1}(\Omega^2, t) & \text{on } \Gamma^1 \times I_k & \text{Contact Dirichlet BCs} \end{cases}$$

$$\Omega^2 \begin{cases} \operatorname{div} \boldsymbol{P}^n + \rho \boldsymbol{B} = \rho \ddot{\boldsymbol{\varphi}}^n & \text{in } \Omega^2 \times I_k & \text{Dynamic problem} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \chi & \text{on } \partial_{\boldsymbol{\varphi}} \Omega^2 \times I_k & \text{Regular Dirichlet BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \boldsymbol{T} & \text{on } \partial_{\boldsymbol{T}} \Omega^2 \times I_k & \text{Regular Neumann BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \mathcal{P}^{\boldsymbol{T}}_{(\Gamma^1 \to \Gamma^2) \times I_k} \boldsymbol{T}^n(\Omega^1, t) & \text{on } \Gamma^2 \times I_k & \text{Contact Neumann BCs} \end{cases}$$

- Dirichlet-Neumann BCs can be swapped
- Can be easily coupled with different time integrators – implicit, explicit Newmark-β, ... schemes
- Challenge: appropriate transfer operators for displacement and traction

Schwarz alternating method for contact problems



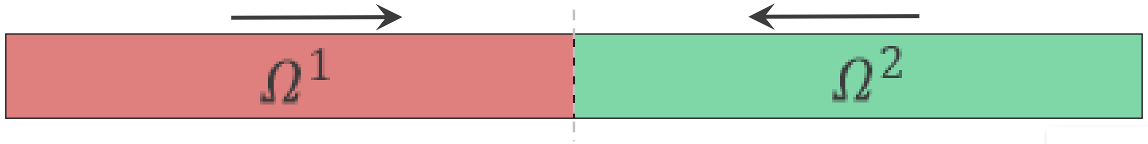


$$\begin{cases} \operatorname{div} \boldsymbol{P}^n + \rho \boldsymbol{B} = \rho \ddot{\boldsymbol{\varphi}}^n & \text{in } \Omega^1 \times I_k & \text{Dynamic problem} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \boldsymbol{\chi} & \text{on } \partial_{\boldsymbol{\varphi}} \Omega^1 \times I_k & \text{Regular Dirichlet BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \boldsymbol{T} & \text{on } \partial_{\boldsymbol{T}} \Omega^1 \times I_k & \text{Regular Neumann BCs} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \boxed{\mathcal{P}^{\boldsymbol{\varphi}}_{(\Gamma^2 \to \Gamma^1) \times I_k}} \boldsymbol{\varphi}^{n-1}(\Omega^2, t) & \text{on } \Gamma^1 \times I_k & \text{Contact Dirichlet BCs} \end{cases}$$

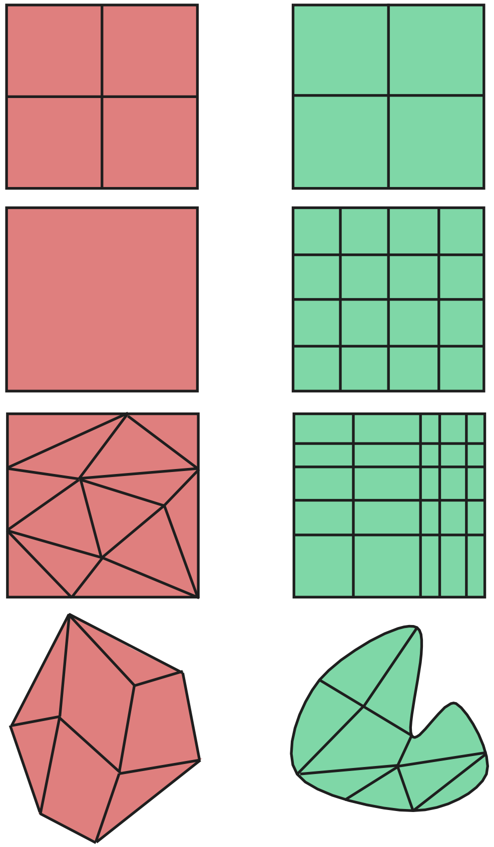
$$\Omega^2 \begin{cases} \operatorname{div} \boldsymbol{P}^n + \rho \boldsymbol{B} = \rho \ddot{\boldsymbol{\varphi}}^n & \text{in } \Omega^2 \times I_k & \text{Dynamic problem} \\ \boldsymbol{\varphi}^n(\boldsymbol{x}, t) = \boldsymbol{\chi} & \text{on } \partial_{\boldsymbol{\varphi}} \Omega^2 \times I_k & \text{Regular Dirichlet BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \boldsymbol{T} & \text{on } \partial_{\boldsymbol{T}} \Omega^2 \times I_k & \text{Regular Neumann BCs} \\ \boldsymbol{P}^n \boldsymbol{N} = \boxed{\mathcal{P}^{\boldsymbol{T}}_{(\Gamma^1 \to \Gamma^2) \times I_k}} \boldsymbol{T}^n(\Omega^1, t) & \text{on } \Gamma^2 \times I_k & \text{Contact Neumann BCs} \end{cases}$$

- Dirichlet-Neumann BCs can be swapped
- Can be easily coupled with different time integrators – implicit, explicit Newmark-β, ... schemes
- Challenge: appropriate transfer operators for displacement and traction

Schwarz alternating method for contact problems



Contact boundaries Γ^{src} and Γ^{dst}



- Displacement transfer operator from Γ^{src} to Γ^{dst}

$$\mathcal{P}^{\varphi} = \left[\int_{\Gamma^{\mathrm{dst}}} \mathcal{N}^{\mathrm{dst}} \left(\mathcal{N}^{\mathrm{dst}}\right)^{\mathrm{T}} \mathrm{d}S\right]^{-1} \left[\int_{\Gamma^{\mathrm{dst}}} \mathcal{N}^{\mathrm{dst}} \left(\mathcal{N}^{\mathrm{src}}\right)^{\mathrm{T}} \mathrm{d}S\right]$$

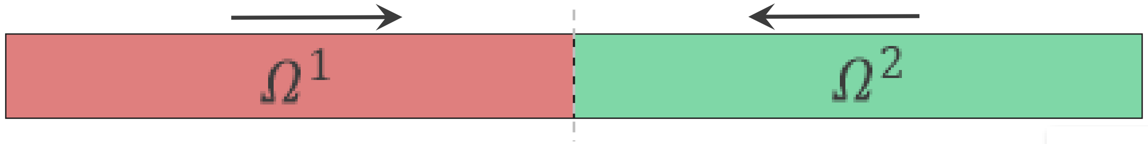
- Traction transfer operator from Γ^{src} to Γ^{dst}

$$\mathcal{P}^{T} = \left[\int_{\Gamma^{\mathrm{dst}}} \mathcal{N}^{\mathrm{dst}} \left(\mathcal{N}^{\mathrm{src}}\right)^{\mathrm{T}} \mathrm{d}S\right] \left[\int_{\Gamma^{\mathrm{src}}} \mathcal{N}^{\mathrm{src}} \left(\mathcal{N}^{\mathrm{src}}\right)^{\mathrm{T}} \mathrm{d}S\right]^{-1}$$

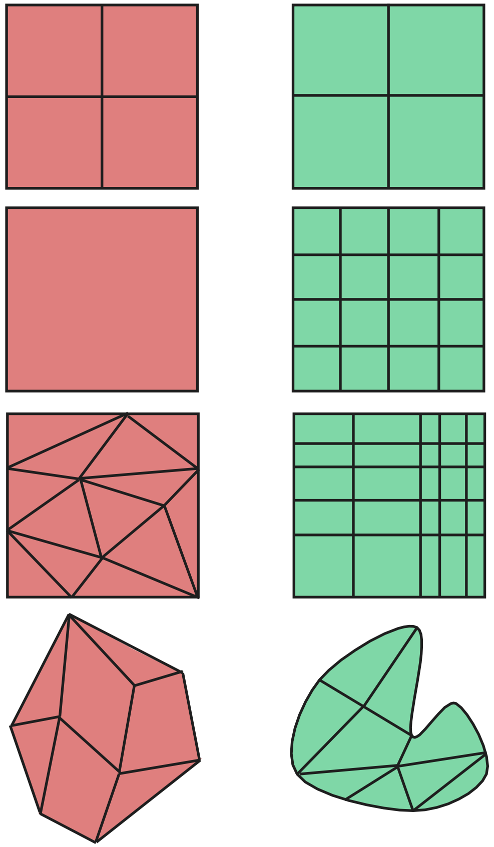
$\mathcal{N}^{\mathrm{src}}$ and $\mathcal{N}^{\mathrm{dst}}$ - finite elements interpolation functions defined on Γ^{src} and Γ^{dst}

- Generic (appropriate for different type of geometries, mesh types and sizes, …)
- Same operator used for all Schwarz iterations within one controller time step

Schwarz alternating method for contact problems



Contact boundaries $\boldsymbol{\Gamma}^{\mathrm{src}}$ and $\boldsymbol{\Gamma}^{\mathrm{dst}}$



- Displacement transfer operator from $\boldsymbol{\Gamma}^{\mathrm{src}}$ to $\boldsymbol{\Gamma}^{\mathrm{dst}}$

$$\mathcal{P}^{\boldsymbol{\varphi}} = \left[\int_{\Gamma^{\mathrm{dst}}} \mathcal{N}^{\mathrm{dst}} \left(\mathcal{N}^{\mathrm{dst}}\right)^{\mathrm{T}} \mathrm{d}S\right]^{-1} \left[\int_{\Gamma^{\mathrm{dst}}} \mathcal{N}^{\mathrm{dst}} \left(\mathcal{N}^{\mathrm{src}}\right)^{\mathrm{T}} \mathrm{d}S\right]$$

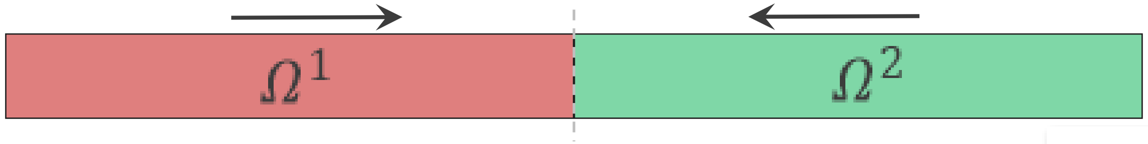
- Traction transfer operator from $\boldsymbol{\Gamma}^{\mathrm{src}}$ to $\boldsymbol{\Gamma}^{\mathrm{dst}}$

$$\mathcal{P}^{\boldsymbol{T}} = \left[\int_{\Gamma^{\mathrm{dst}}} \mathcal{N}^{\mathrm{dst}} \left(\mathcal{N}^{\mathrm{src}}\right)^{\mathrm{T}} \mathrm{d}S\right] \left[\int_{\Gamma^{\mathrm{src}}} \mathcal{N}^{\mathrm{src}} \left(\mathcal{N}^{\mathrm{src}}\right)^{\mathrm{T}} \mathrm{d}S\right]^{-1}$$

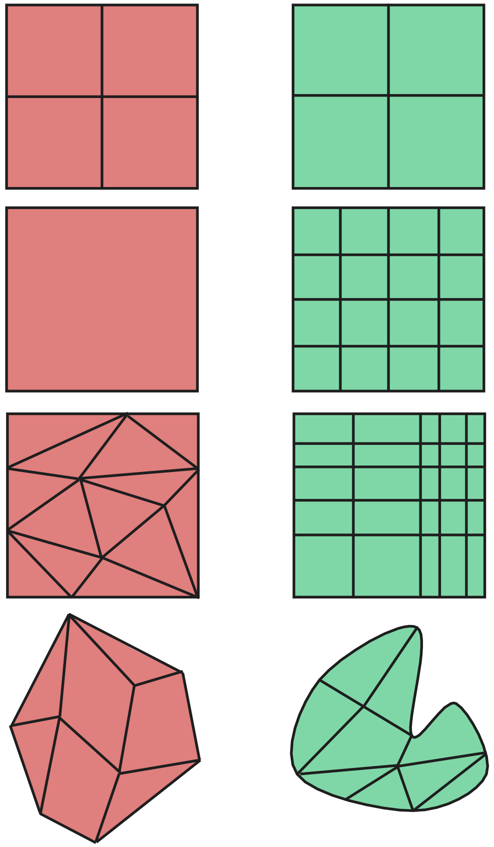
$\mathcal{N}^{\mathrm{src}}$ and $\mathcal{N}^{\mathrm{dst}}$ - finite elements interpolation functions defined on $\boldsymbol{\Gamma}^{\mathrm{src}}$ and $\boldsymbol{\Gamma}^{\mathrm{dst}}$

- Generic (appropriate for different type of geometries, mesh types and sizes, …)
- Same operator used for all Schwarz iterations within one controller time step

Schwarz alternating method for contact problems



Contact boundaries $\boldsymbol{\Gamma}^{\mathbf{src}}$ and $\boldsymbol{\Gamma}^{\mathbf{dst}}$



- Displacement transfer operator from $\boldsymbol{\Gamma}^{\mathbf{src}}$ to $\boldsymbol{\Gamma}^{\mathbf{dst}}$

$$\mathcal{P}^{\boldsymbol{\varphi}} = \left[\int_{\Gamma^{\mathrm{dst}}} \mathcal{N}^{\mathrm{dst}} \left(\mathcal{N}^{\mathrm{dst}}\right)^{\mathrm{T}} \, \mathrm{d}\mathcal{S} \right]^{-1} \left[\int_{\Gamma^{\mathrm{dst}}} \mathcal{N}^{\mathrm{dst}} \left(\mathcal{N}^{\mathrm{src}}\right)^{\mathrm{T}} \, \mathrm{d}\mathcal{S} \right]$$

- Traction transfer operator from $\boldsymbol{\Gamma}^{\mathbf{src}}$ to $\boldsymbol{\Gamma}^{\mathbf{dst}}$

$$\mathcal{P}^{\boldsymbol{T}} = \left[\int_{\Gamma^{\mathrm{dst}}} \mathcal{N}^{\mathrm{dst}} \left(\mathcal{N}^{\mathrm{src}}\right)^{\mathrm{T}} \, \mathrm{d}\mathcal{S} \right] \left[\int_{\Gamma^{\mathrm{src}}} \mathcal{N}^{\mathrm{src}} \left(\mathcal{N}^{\mathrm{src}}\right)^{\mathrm{T}} \, \mathrm{d}\mathcal{S} \right]^{-1}$$

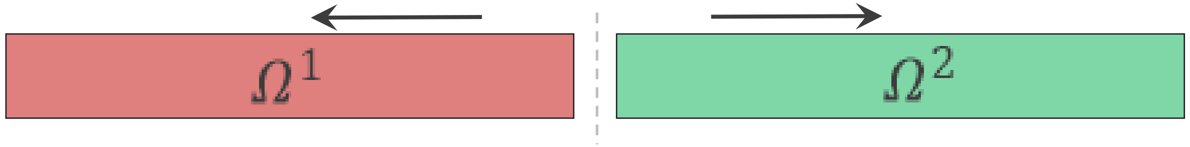
$\mathcal{N}^{\mathrm{src}}$ and $\mathcal{N}^{\mathrm{dst}}$ - finite elements interpolation functions defined on $\boldsymbol{\Gamma}^{\mathbf{src}}$ and $\boldsymbol{\Gamma}^{\mathbf{dst}}$

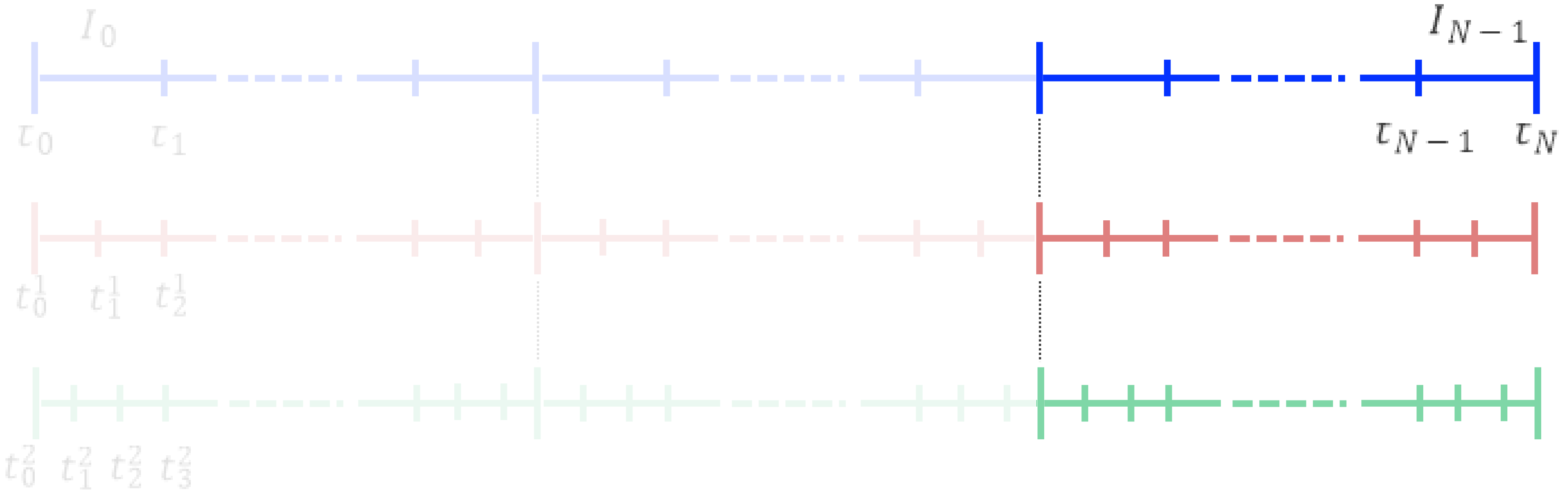
- Generic (appropriate for different type of geometries, mesh types and sizes, ...)
- Same operator used for all Schwarz iterations within one controller time step

Schwarz alternating method for contact problems

After the contact phase:

- **solve PDEs in each body separately** (different solvers, discretization techniques, time integrators, ...)
- prevent interpenetration through an iterative process based on the **Dirichlet-Neumann contact boundary conditions**
- **check contact criteria** at each controller time step (overlap/distance conditions, compression condition, ...)

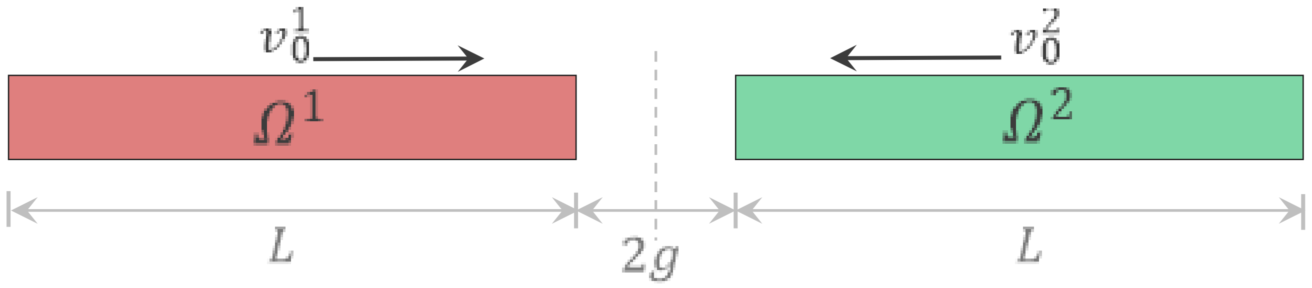




Academic benchmark

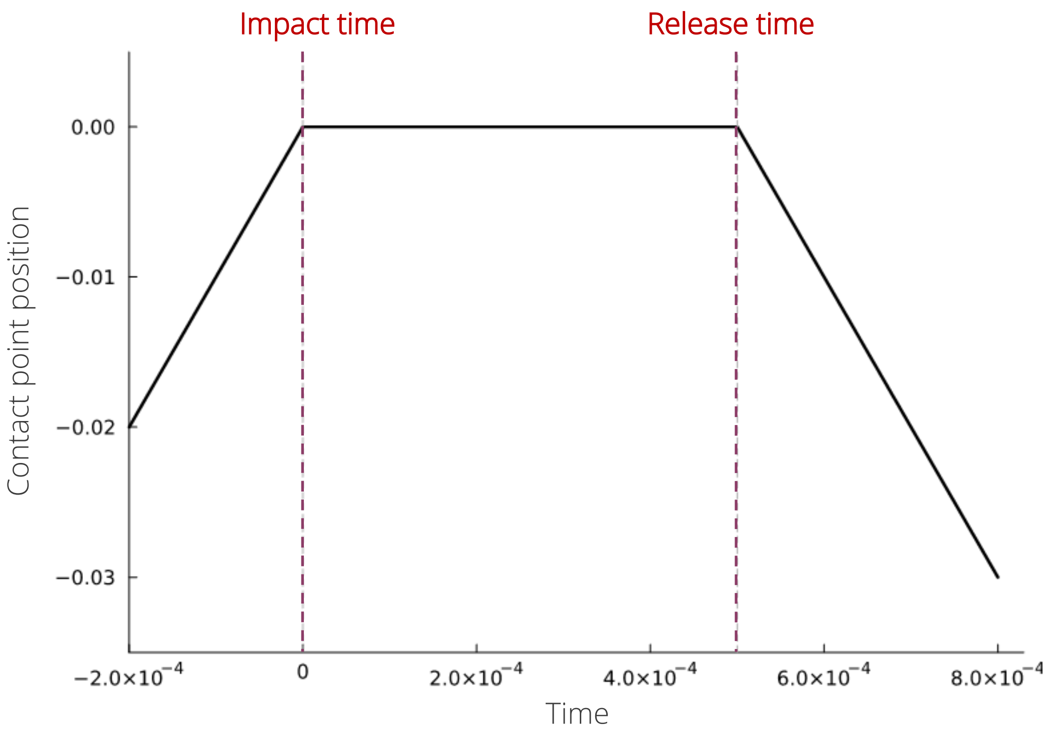
Schwarz contact method vs conventional methods

Impact of two 1D linear elastic prismatic rods



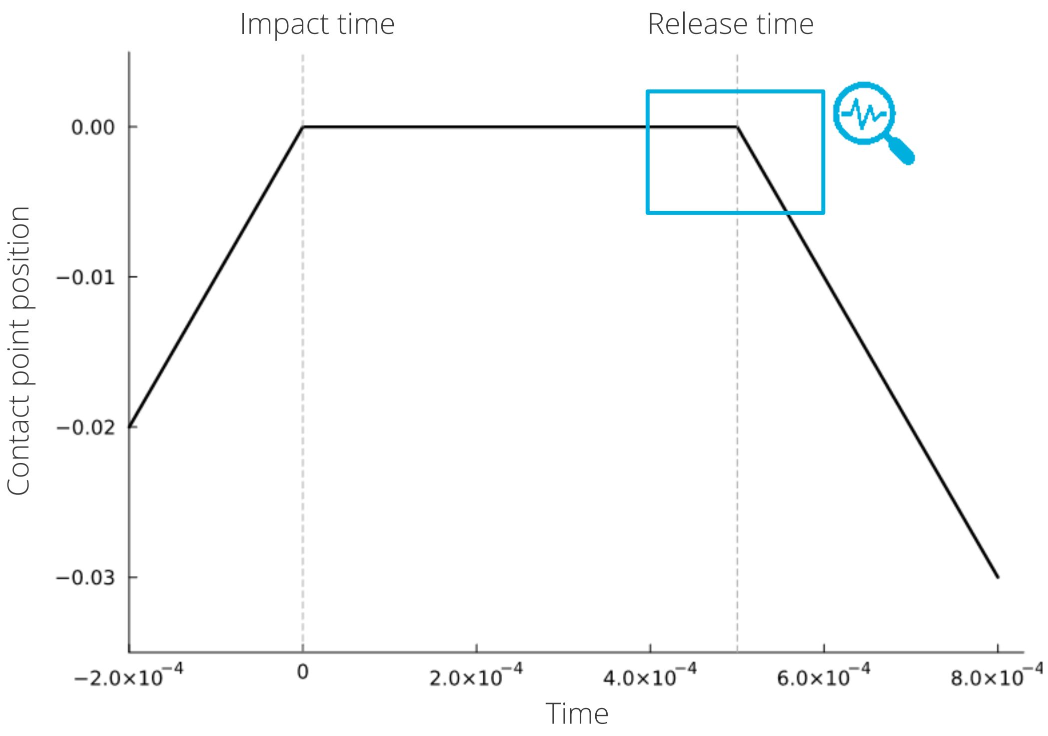
- **Analytical solution** available in *[Carpenter et al., 1991]*
- **Newmark-β** time integrator
- **Numerical comparison**: Schwarz method vs conventional contact methods
  - implicit and explicit Lagrange multiplier methods
  - implicit and explicit penalty methods
  - explicit-explicit, implicit-implicit, implicit-explicit Schwarz methods

Contact point position



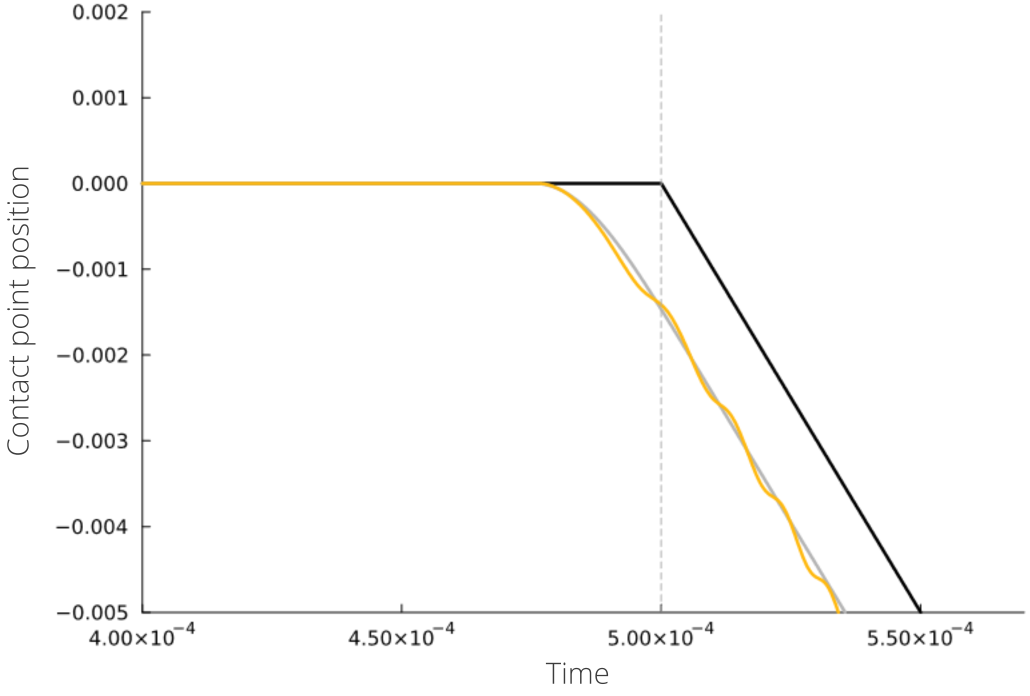
Analytical solution

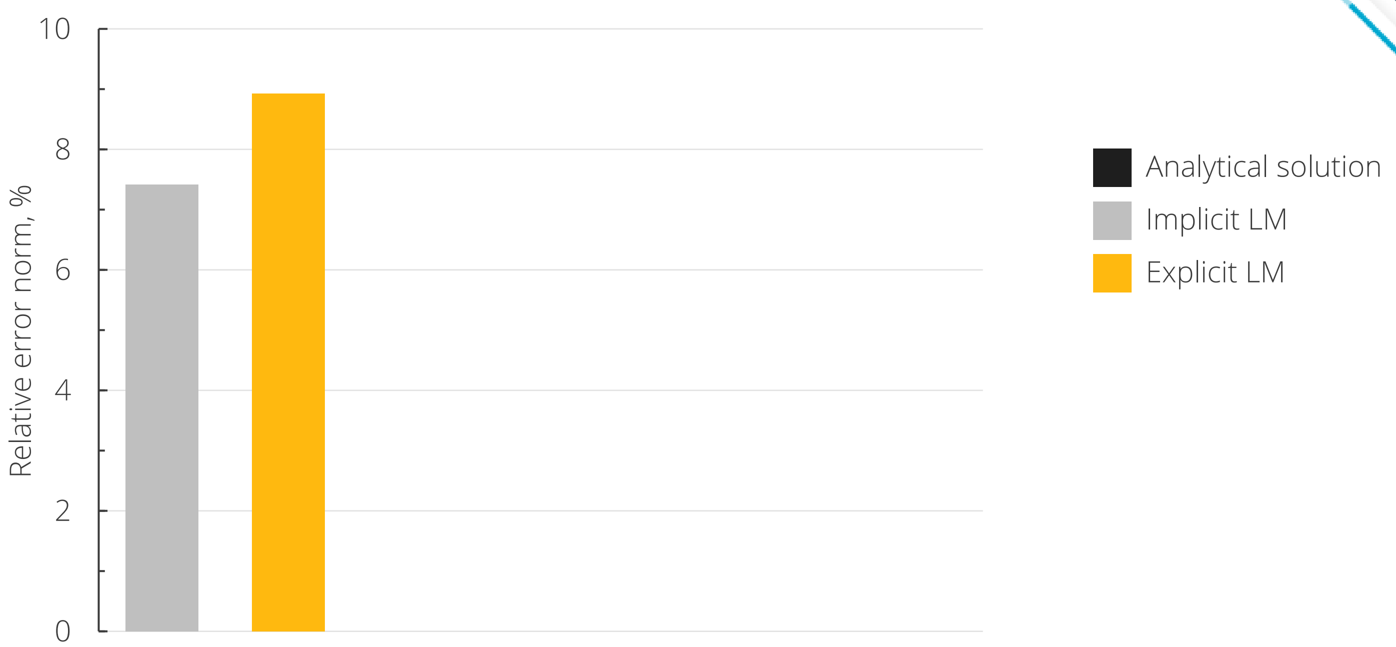
Contact point position



Analytical solution

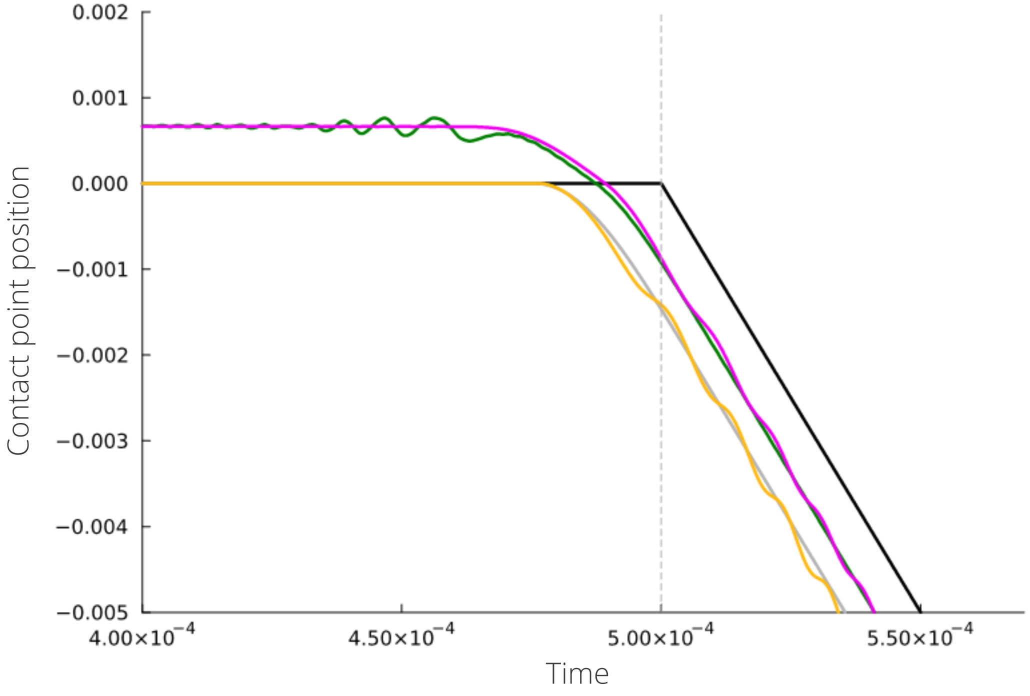
Contact point position

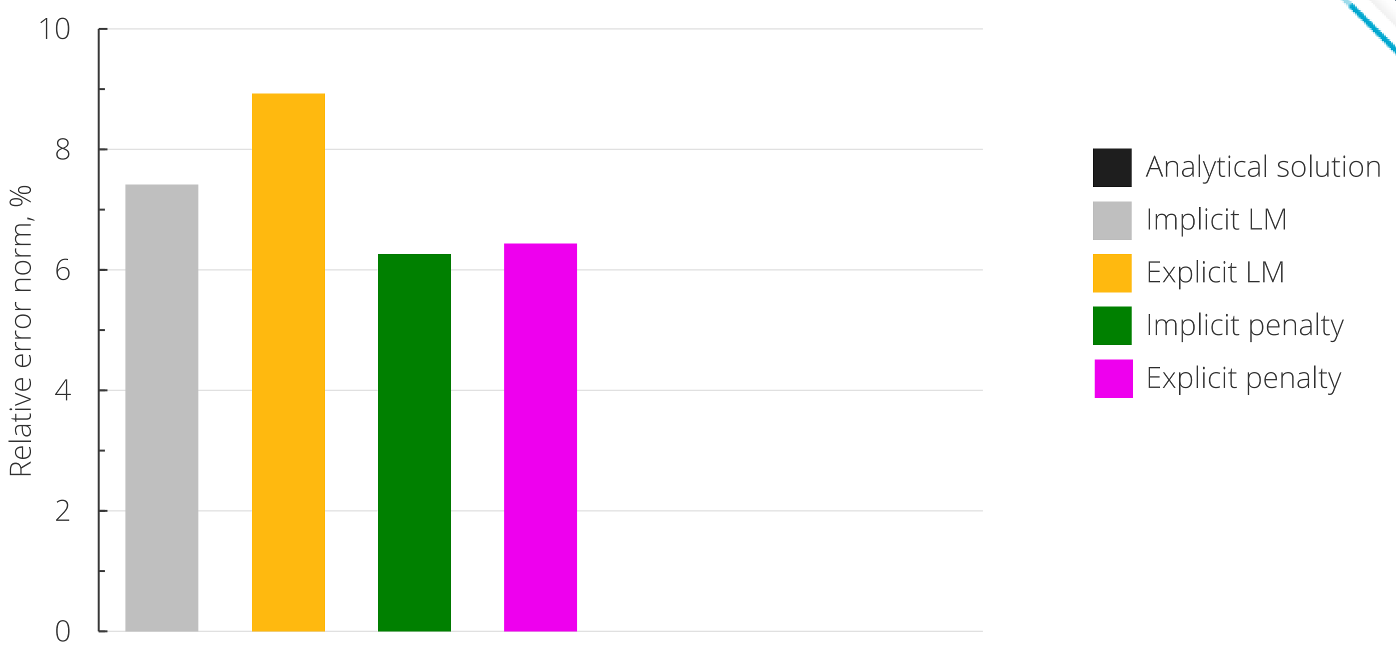




- **Lagrange Multiplier methods:** under-predict the release time

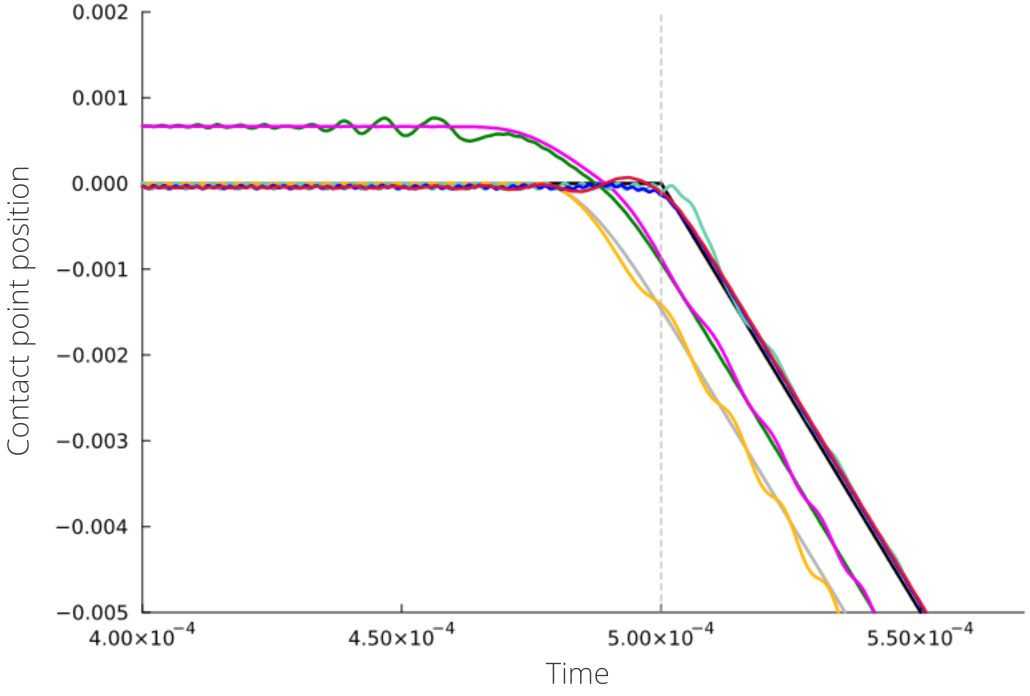
Contact point position

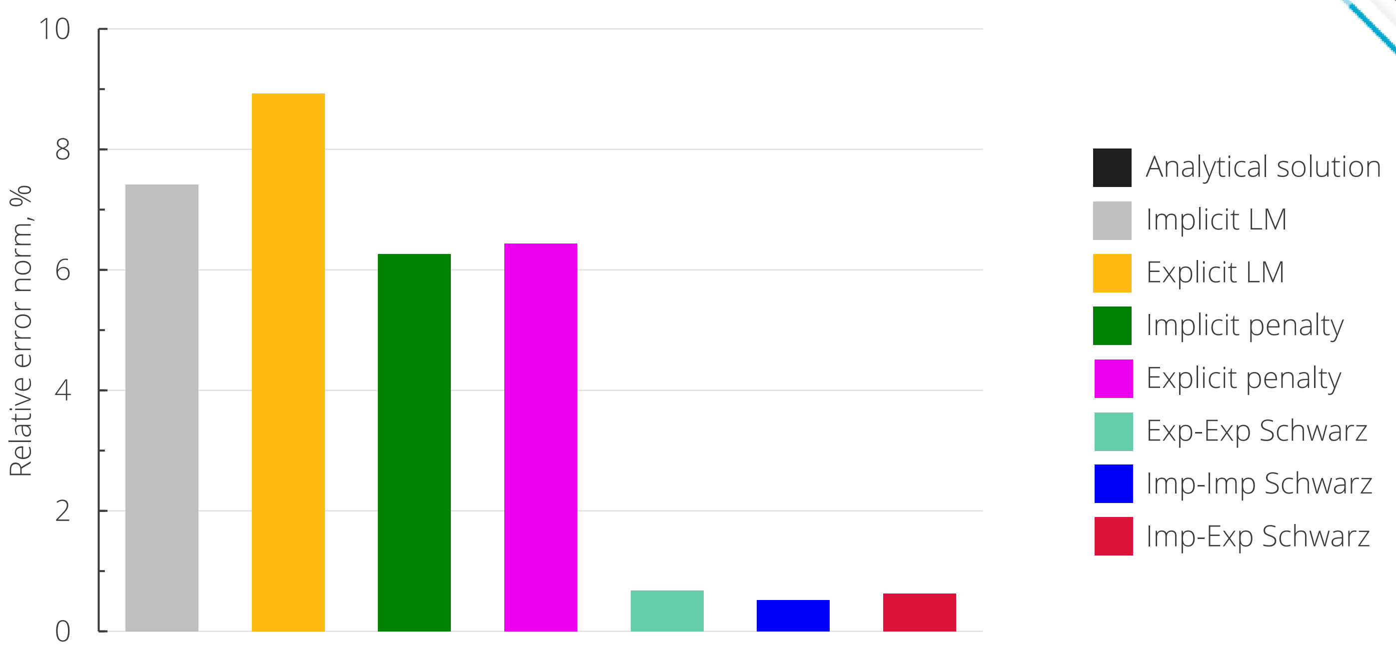




- **Lagrange Multiplier methods:** under-predict the release time
- **Penalty methods**: over-predict the contact point position and under-predict the release time

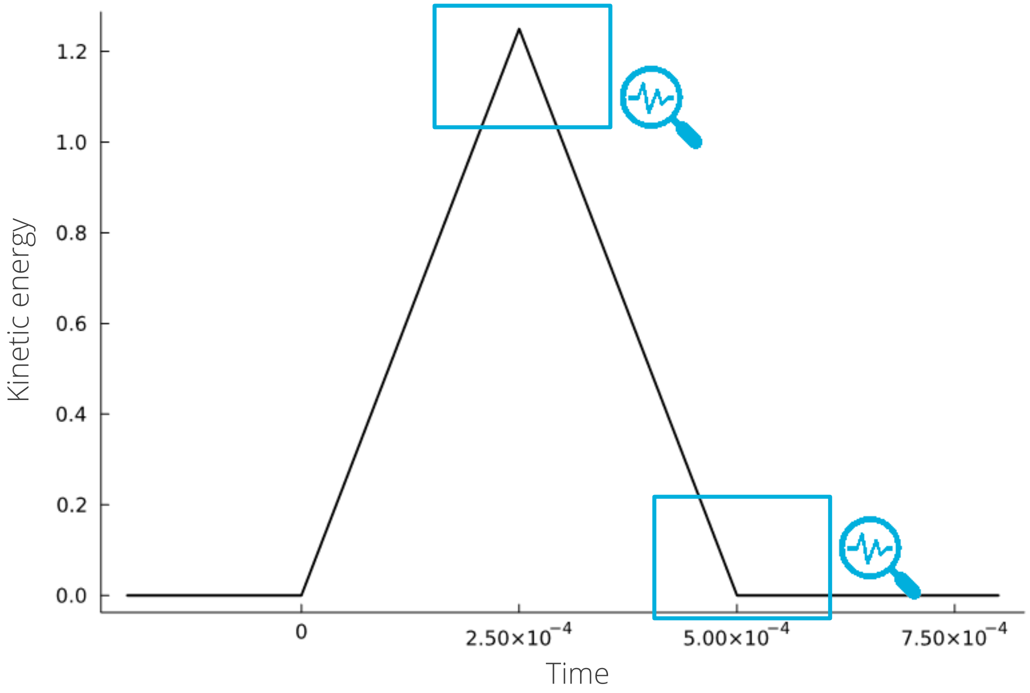
Contact point position





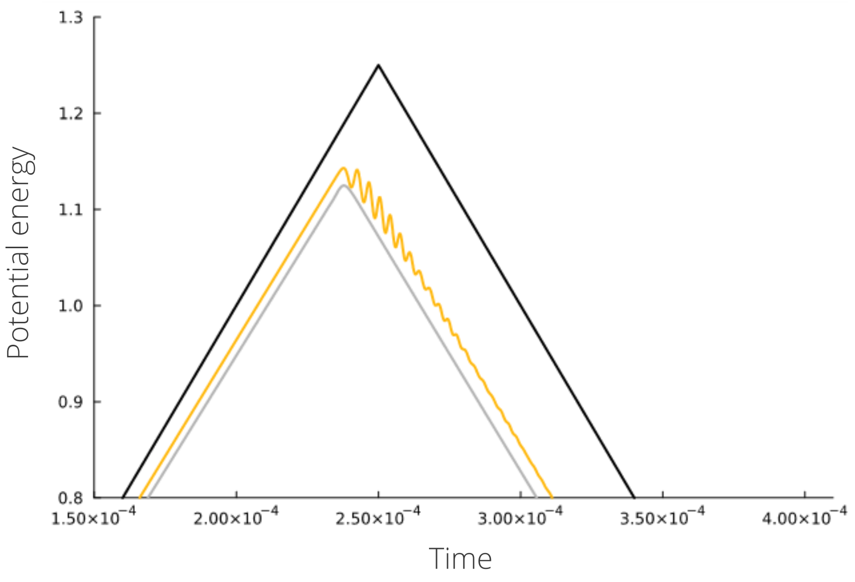
- Lagrange Multiplier methods: under-predict the release time
- Penalty methods: over-predict the contact point position and under-predict the release time
- Schwarz methods: accurately predict the contact point position and release time

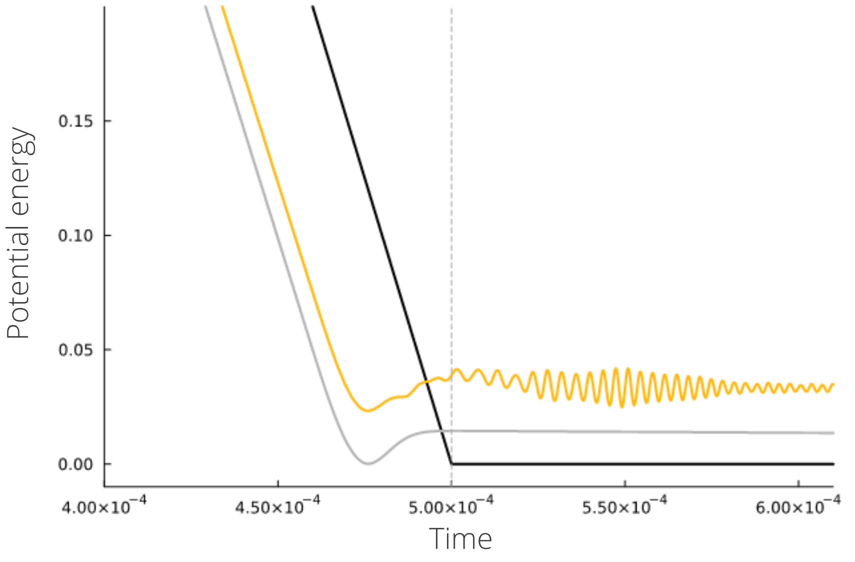
Potential energy



Analytical solution

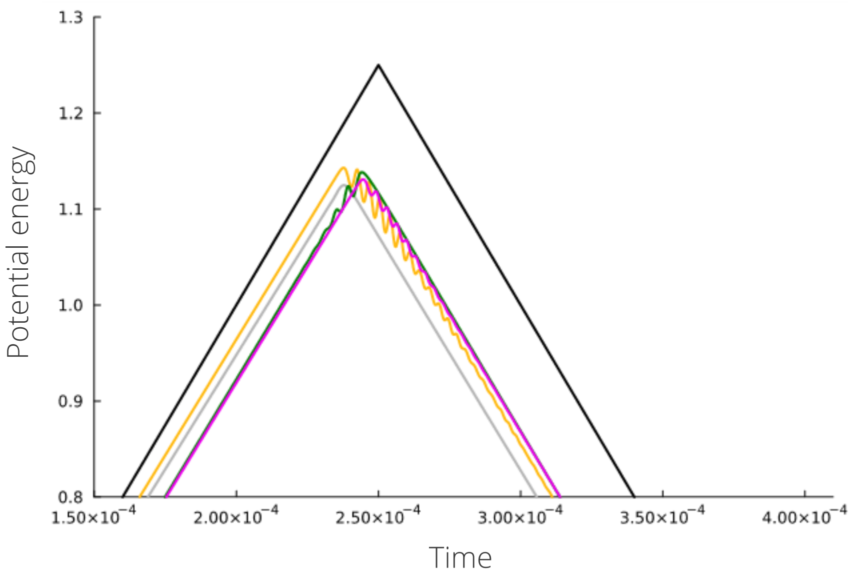
Potential energy

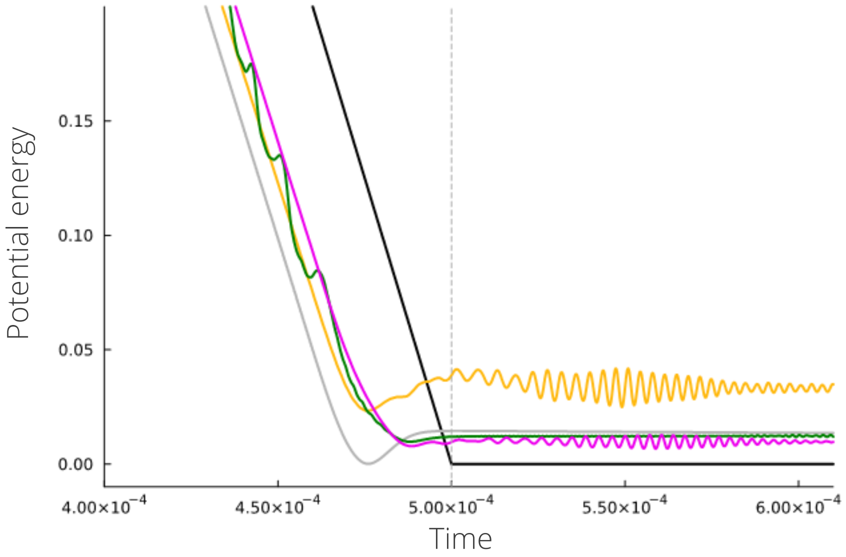




- **Lagrange Multiplier methods:** under-predict the maximum potential energy peak and over-predicts the energy after release

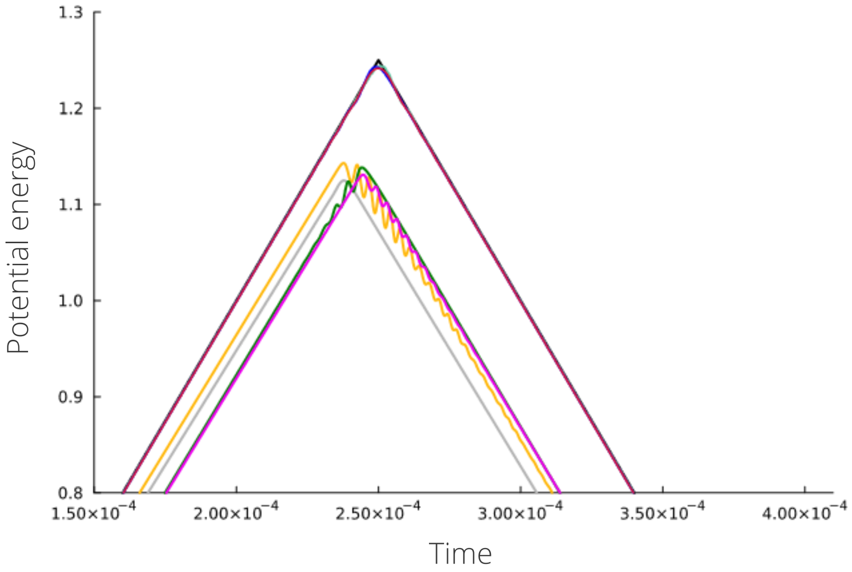
Potential energy

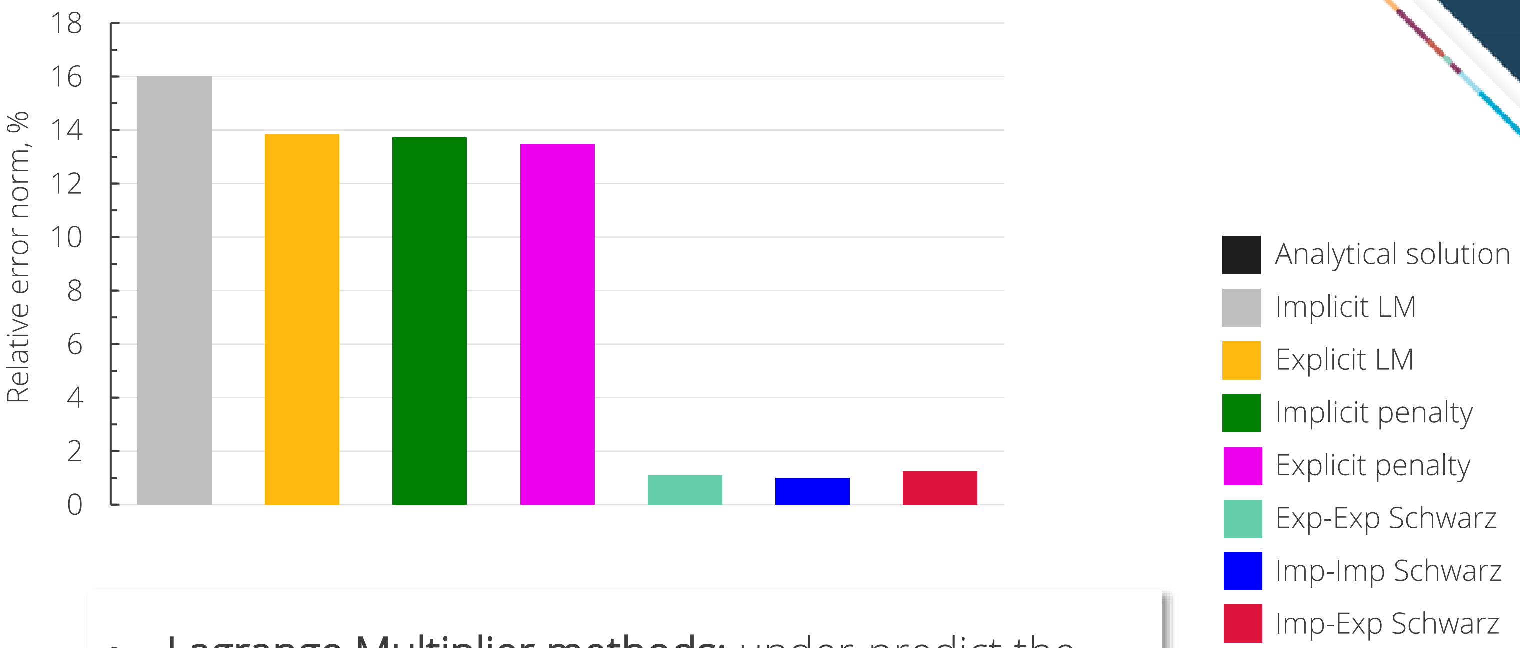


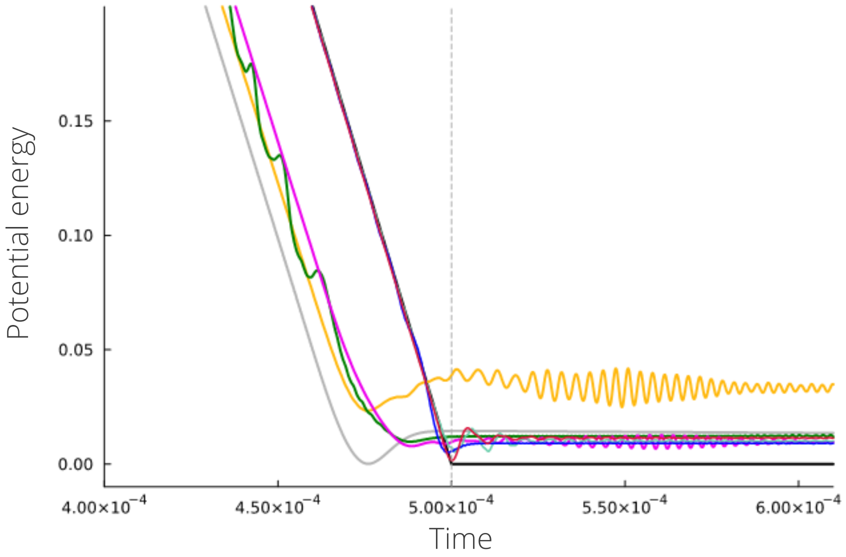


- **Lagrange Multiplier methods:** under-predict the maximum potential energy peak and over-predicts the energy after release
- **Penalty methods**: similar behavior as LM methods

Potential energy

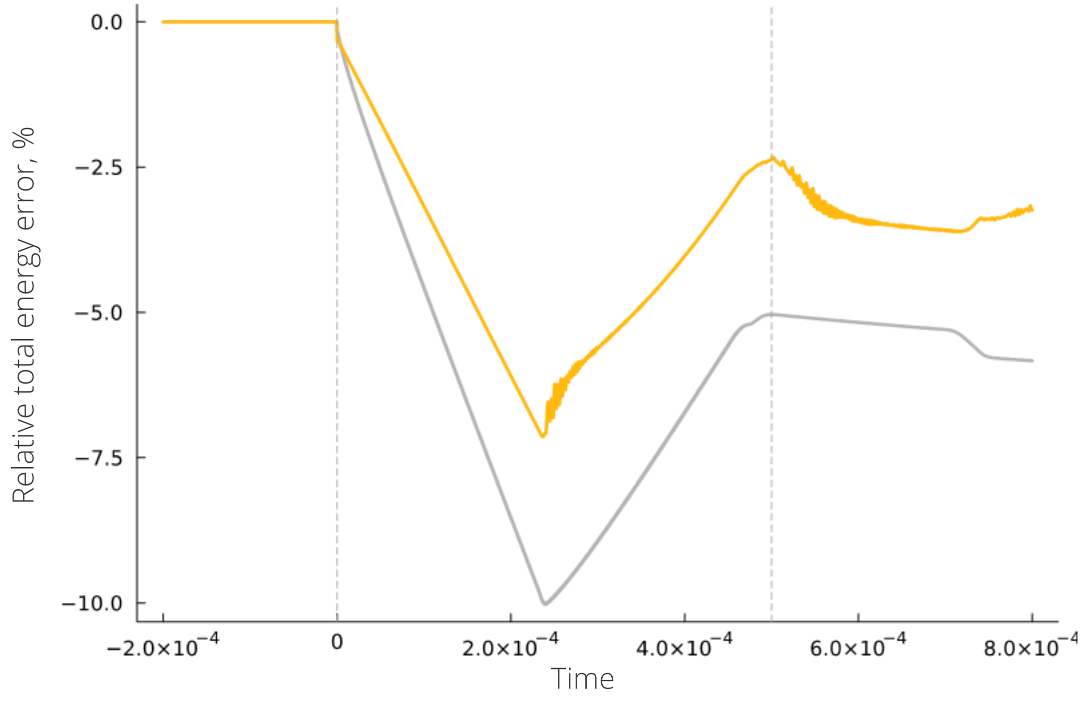






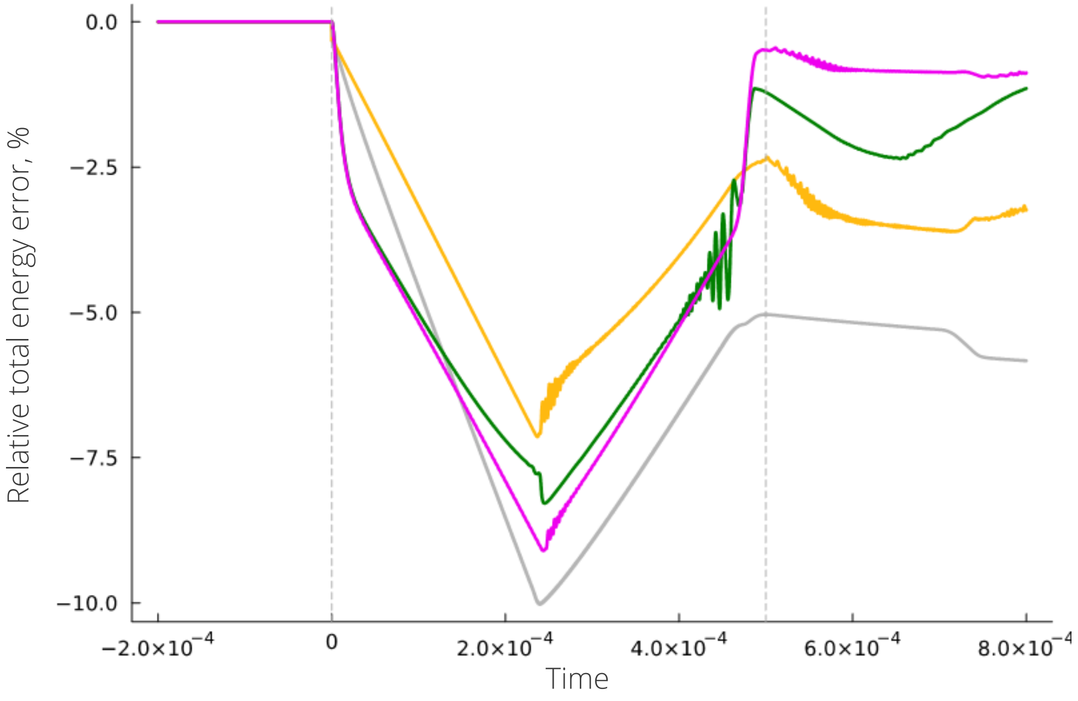
- **Lagrange Multiplier methods:** under-predict the maximum potential energy peak and over-predicts the energy after release
- **Penalty methods**: similar behavior as LM methods
- **Schwarz methods**: accurately predicts the maximum energy peak and better capture the energy after release

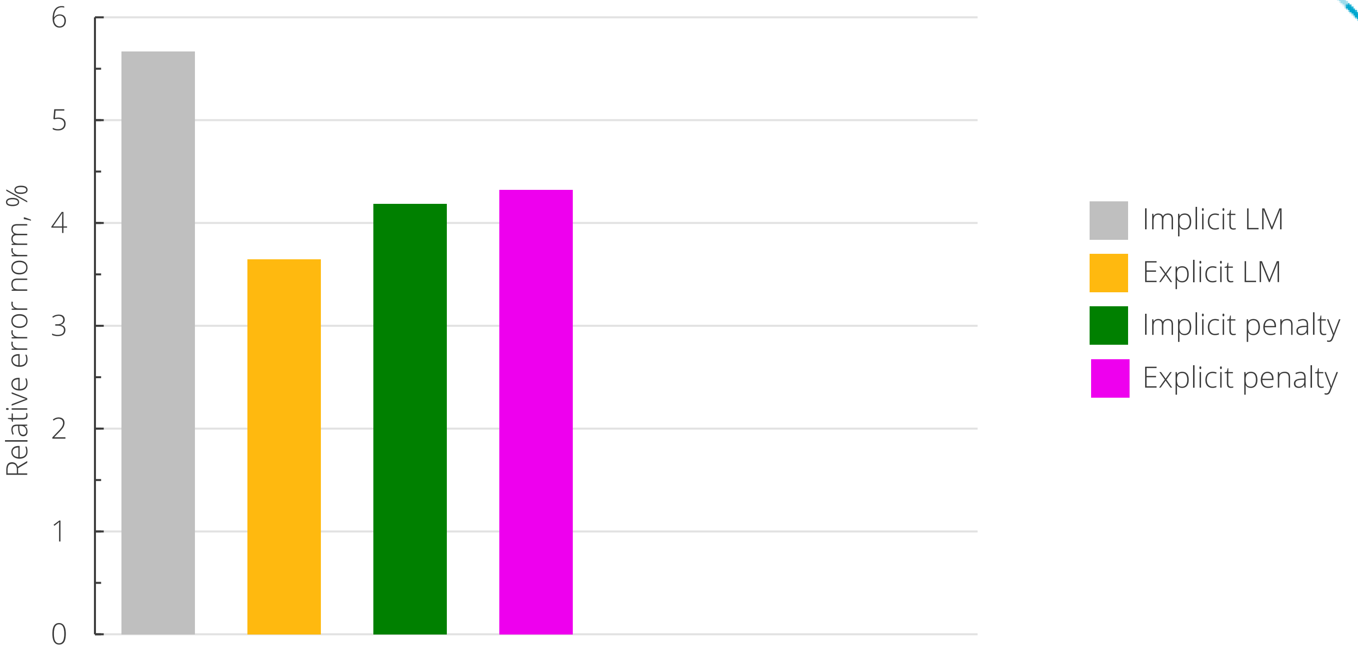
Total energy conservation



- **Lagrange Multiplier methods:** important energy loss up to 7% (explicit) and 10% (implicit)

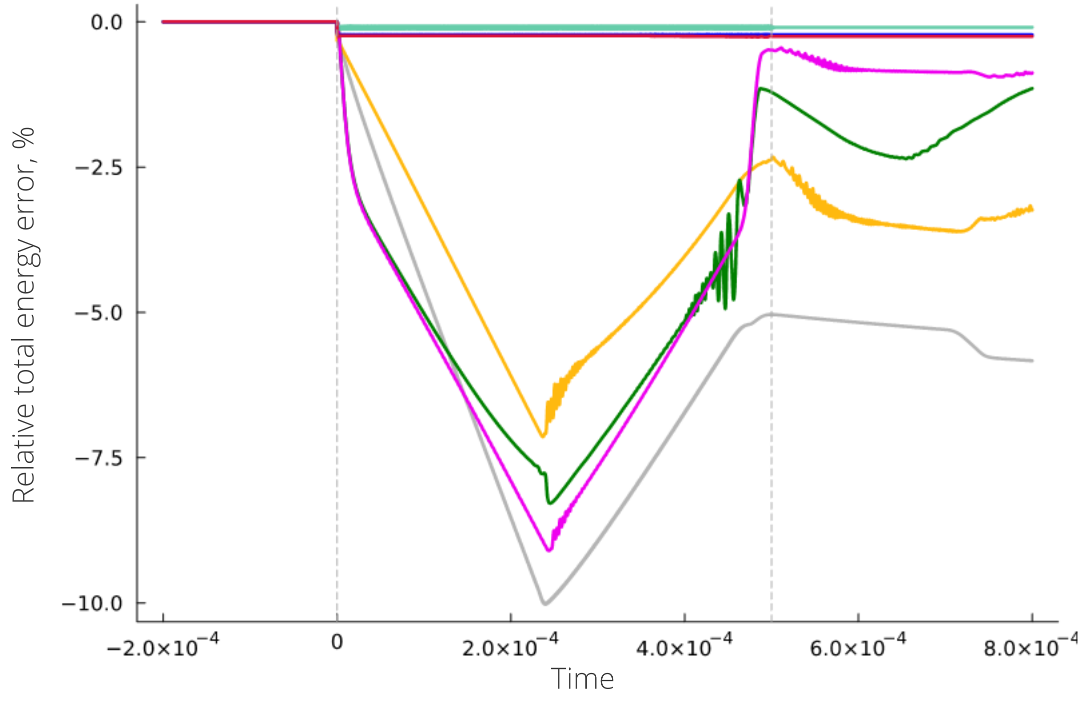
Total energy conservation

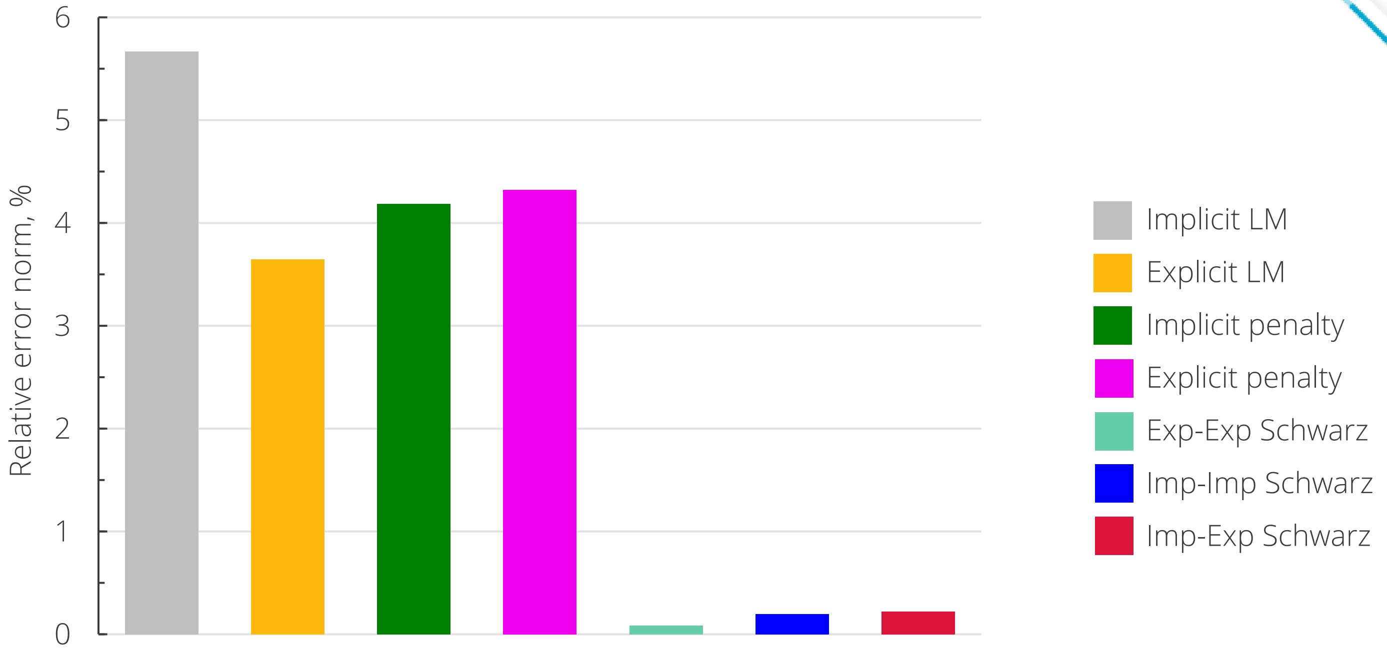




- **Lagrange Multiplier methods:** important energy loss up to 7% (explicit) and 10% (implicit)
- **Penalty methods**: important energy loss up to 8% (implicit) and 9% (implicit)

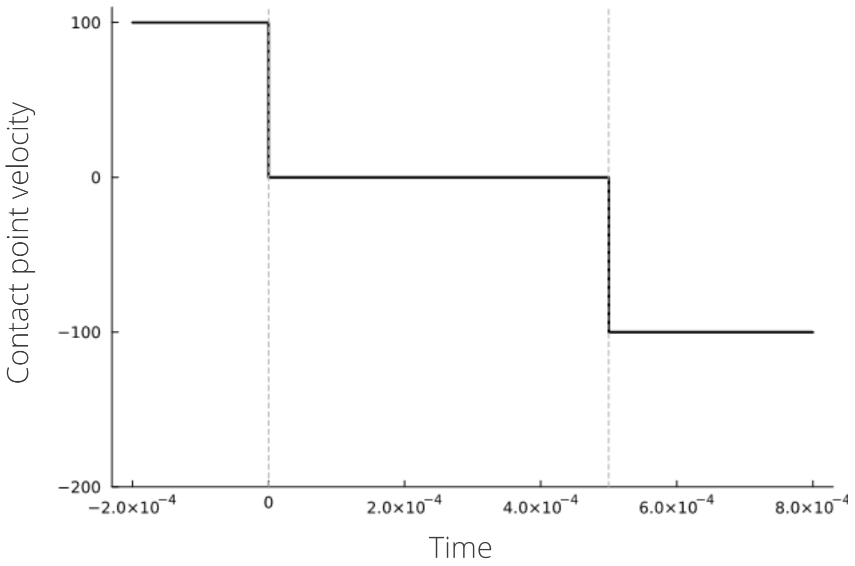
Total energy conservation





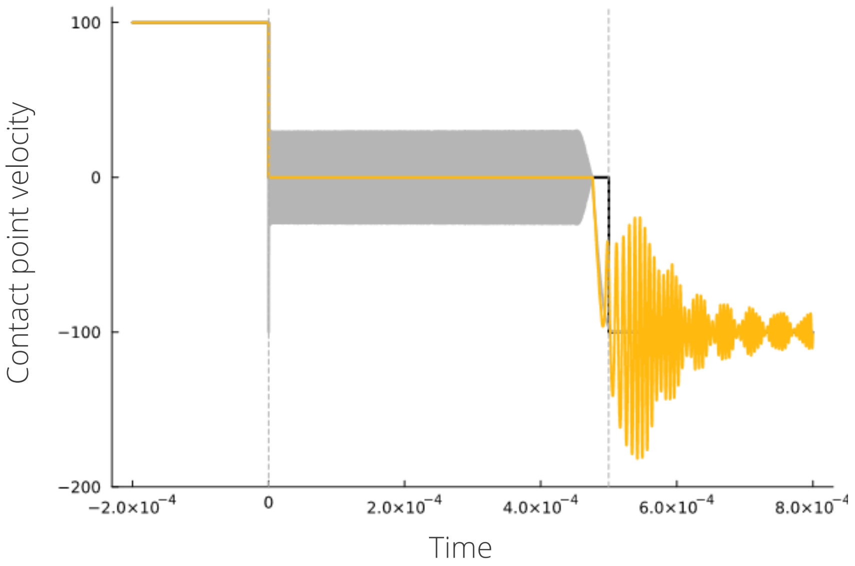
- Lagrange Multiplier methods: important energy loss up to 7% (explicit) and 10% (implicit)
- Penalty methods: important energy loss up to 8% (implicit) and 9% (implicit)
- Schwarz methods: remarkable energy conservation properties – energy loss less than 0.3%

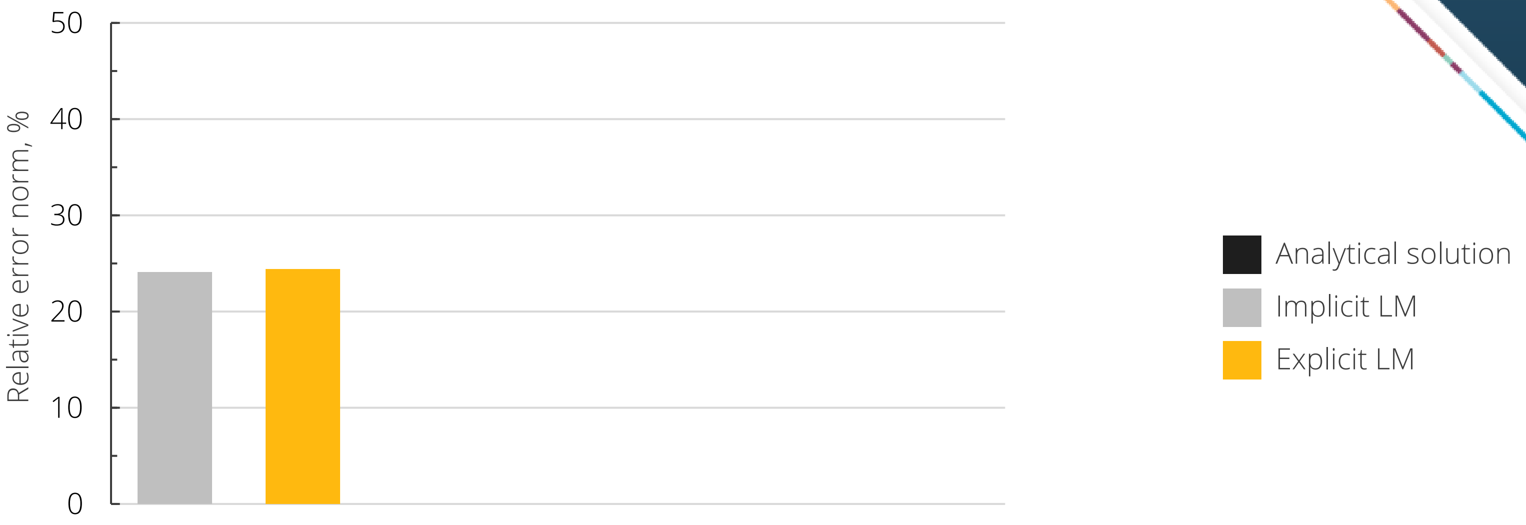
Contact point velocity



Analytical solution

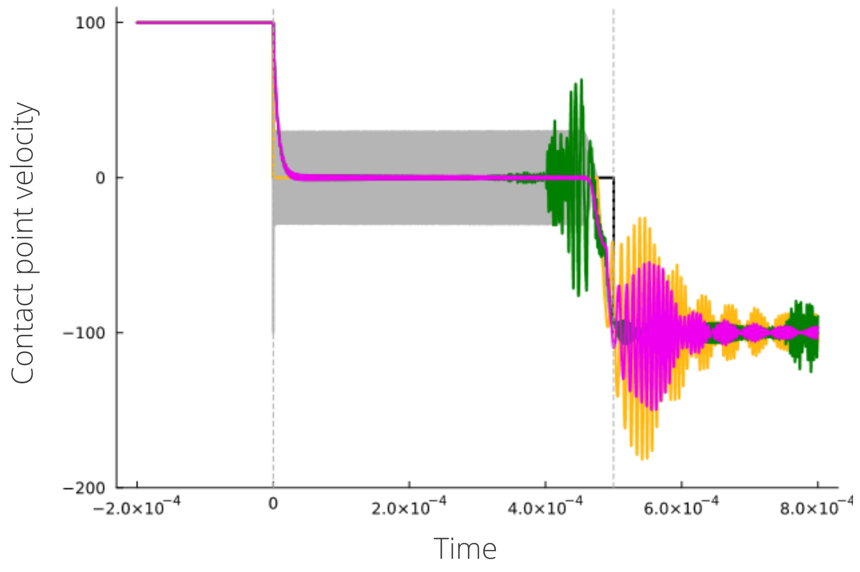
Contact point velocity

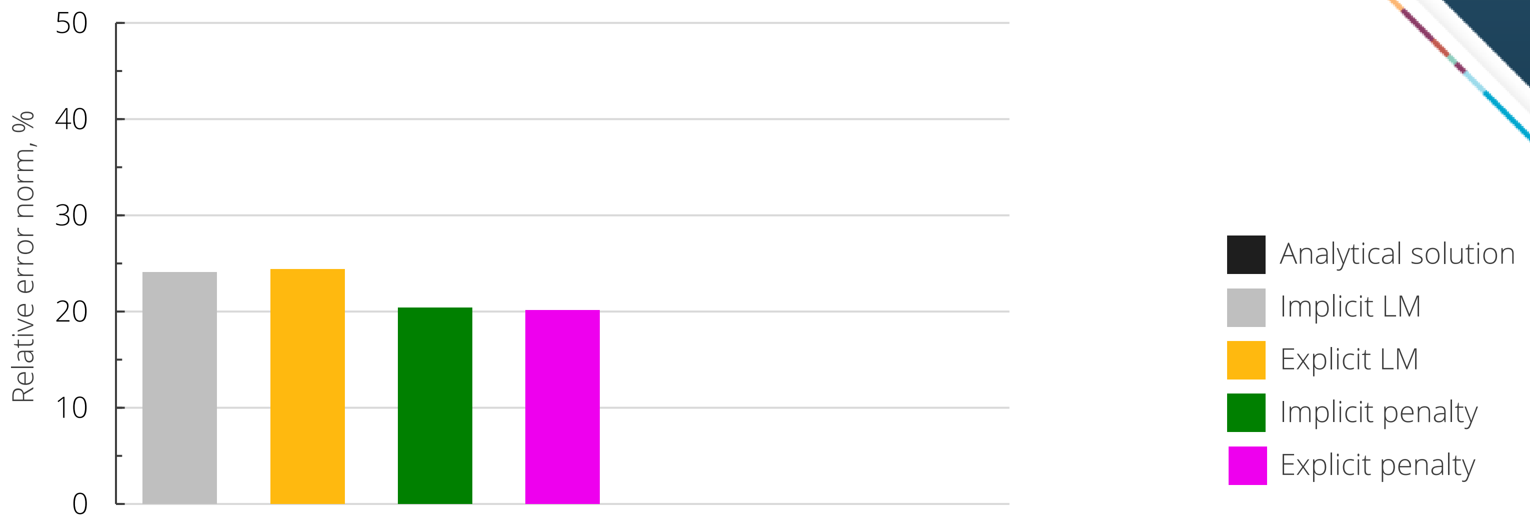




- **Lagrange Multiplier methods:** artificial oscillations during the contact phase and after the release

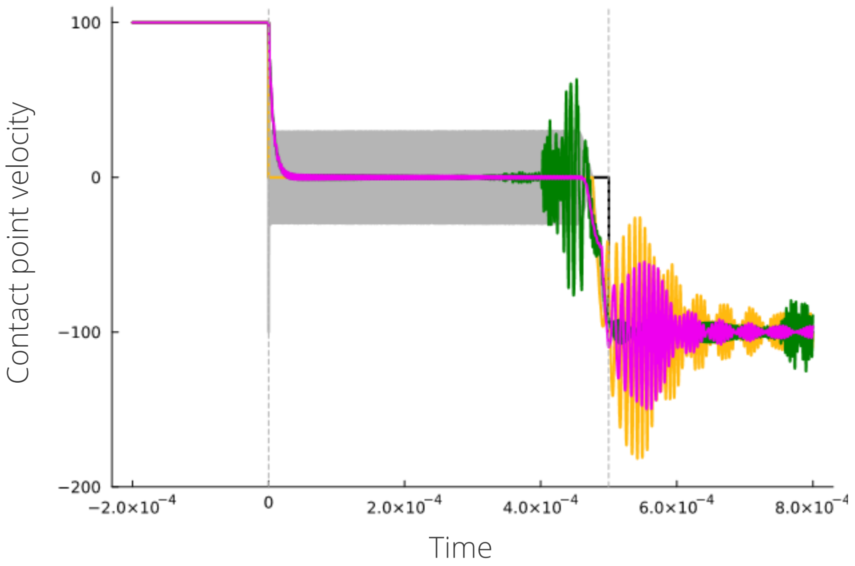
Contact point velocity

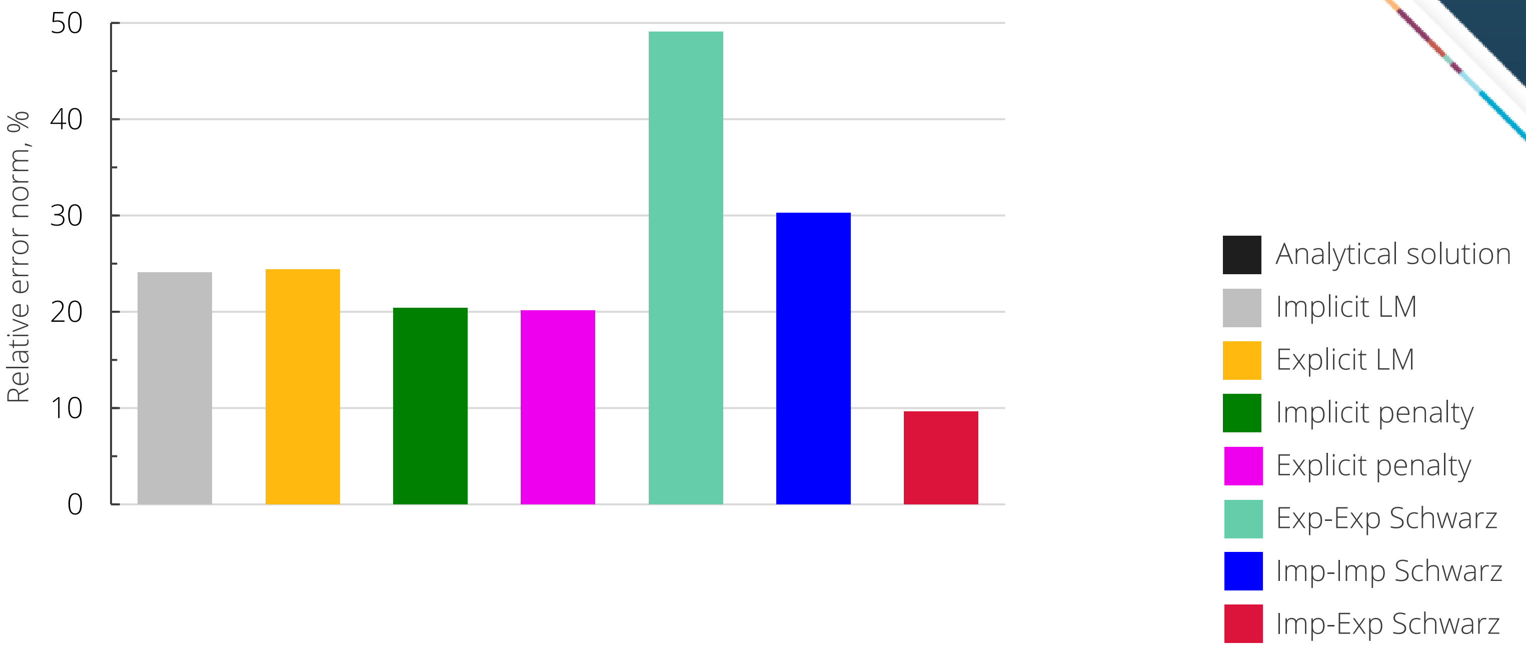


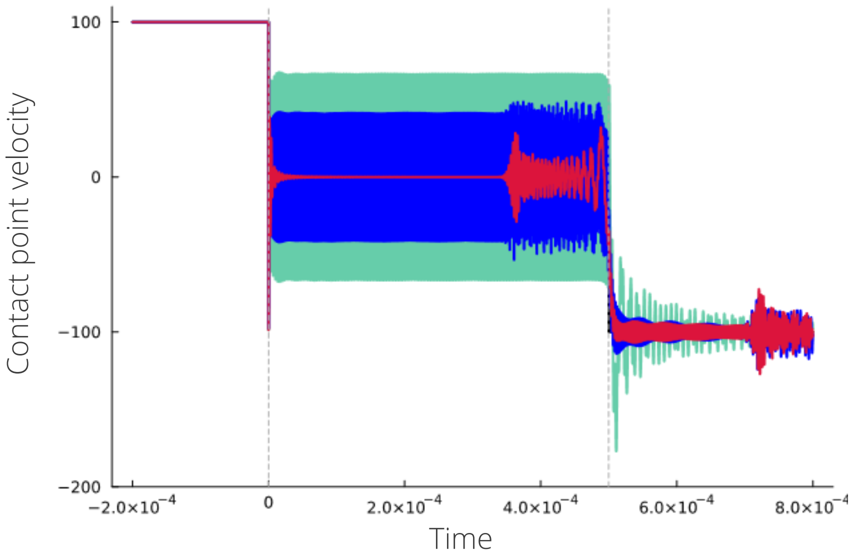


- **Lagrange Multiplier methods:** artificial oscillations during the contact phase and after the release
- **Penalty methods**: artificial oscillations during the contact phase and after the release

Contact point velocity

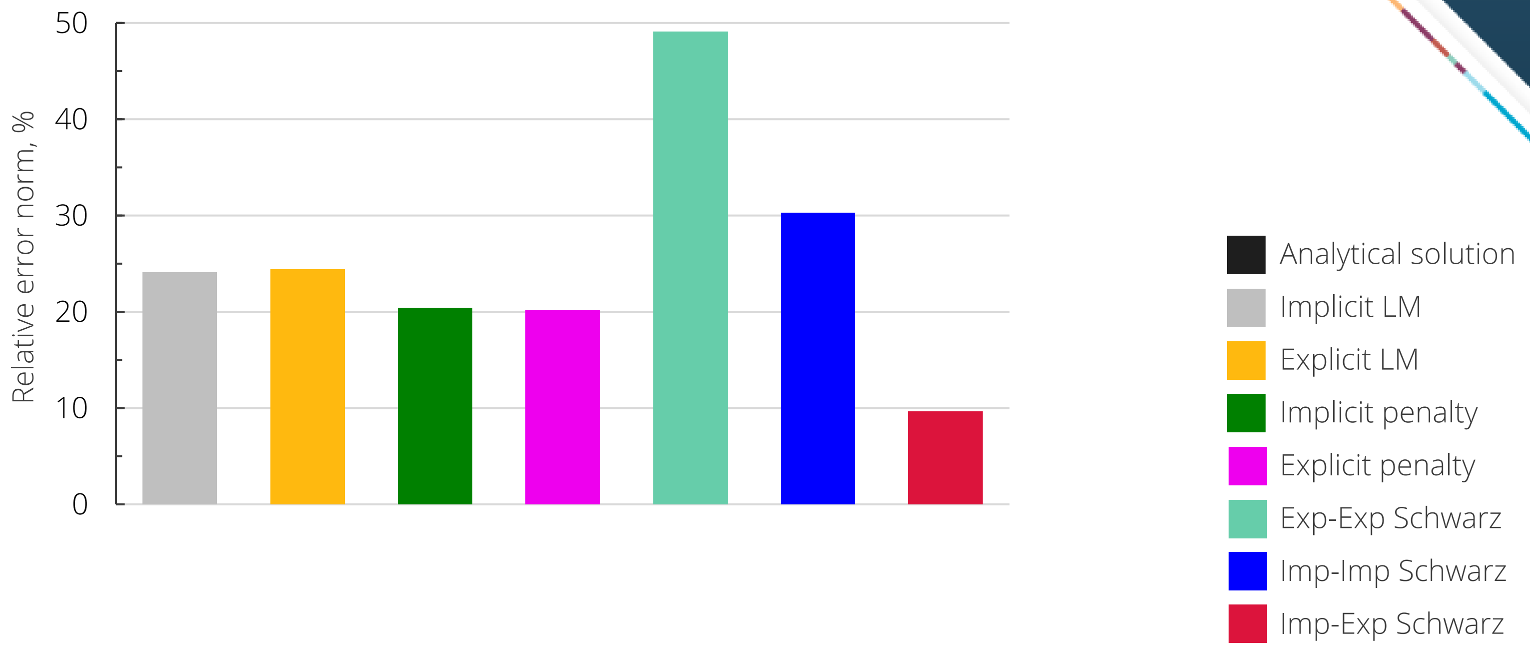


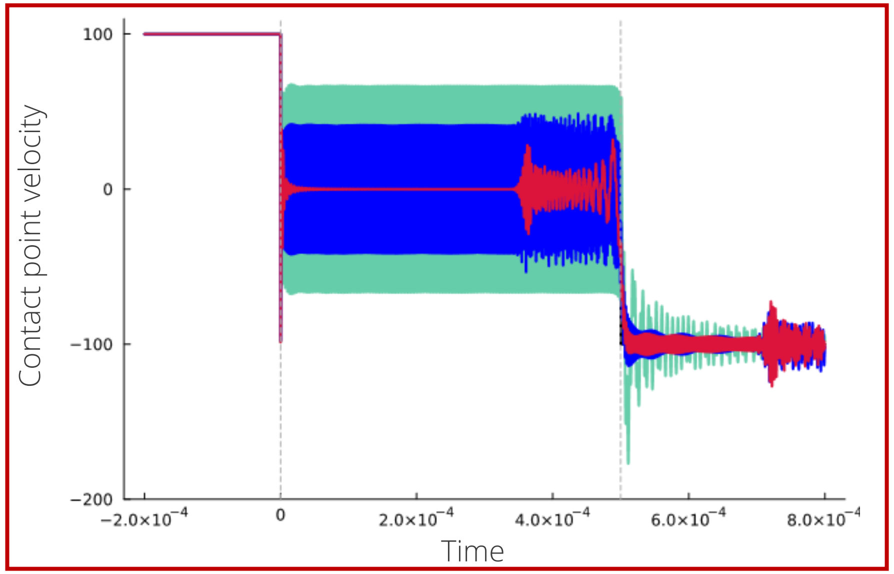




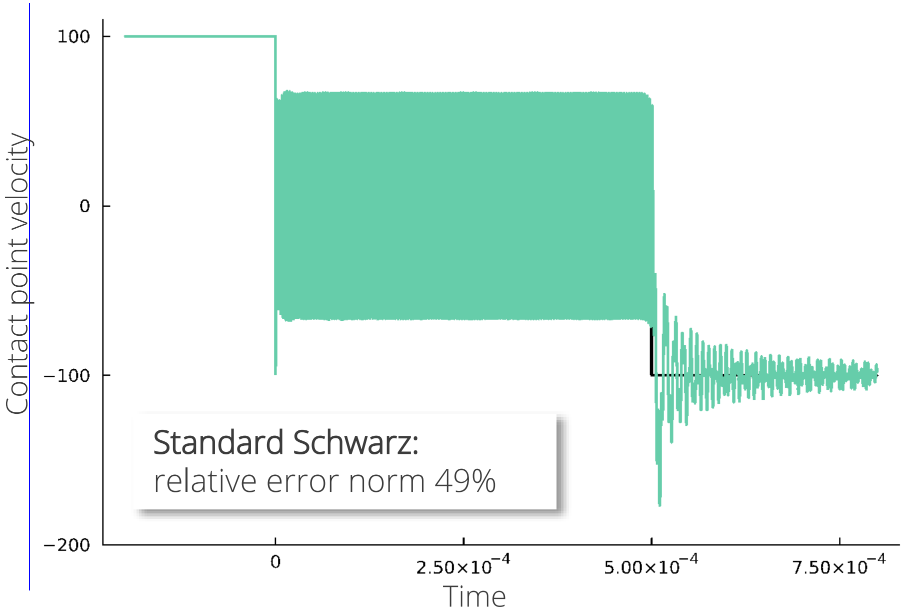
- **Lagrange Multiplier methods:** artificial oscillations during the contact phase and after the release
- **Penalty methods**: artificial oscillations during the contact phase and after the release
- **Schwarz methods**: artificial oscillations during the contact phase and after the release

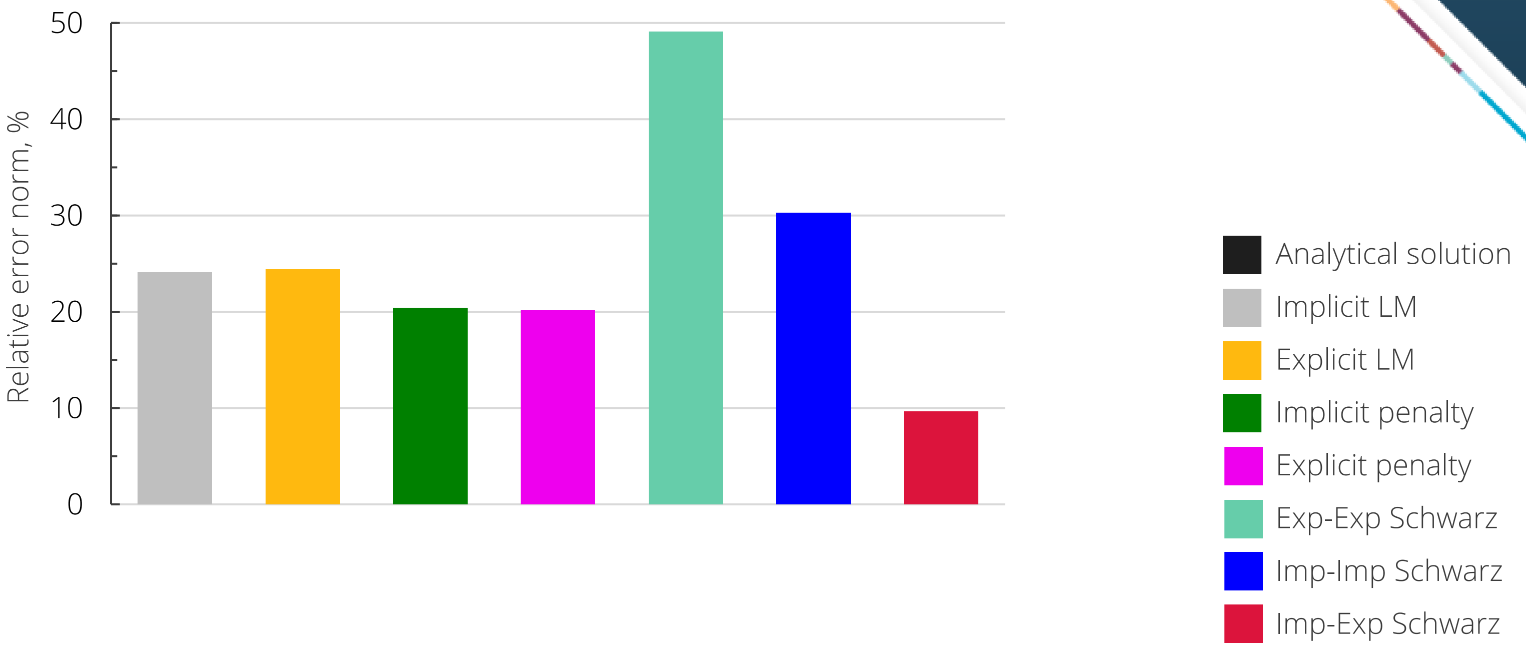
Techniques for reducing artificial oscillations

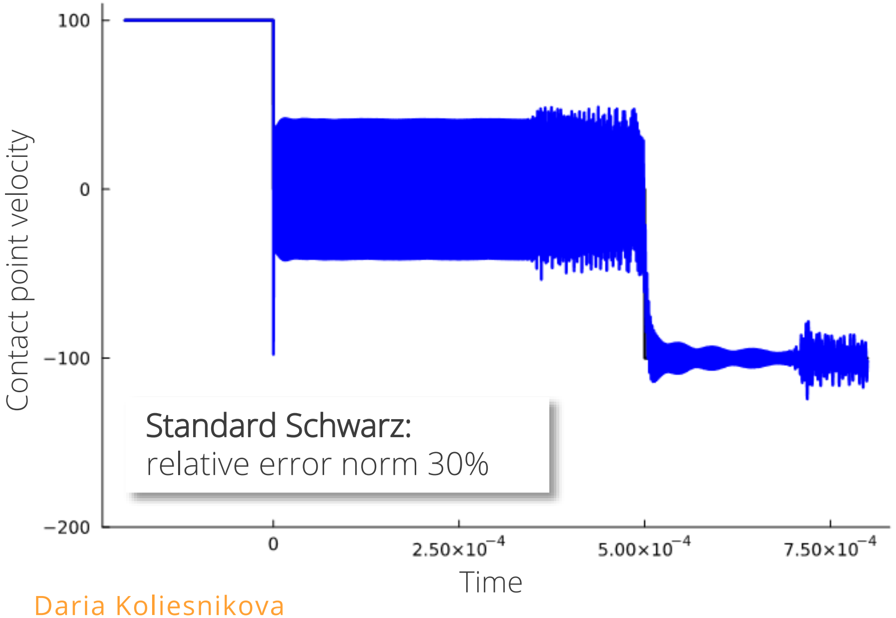




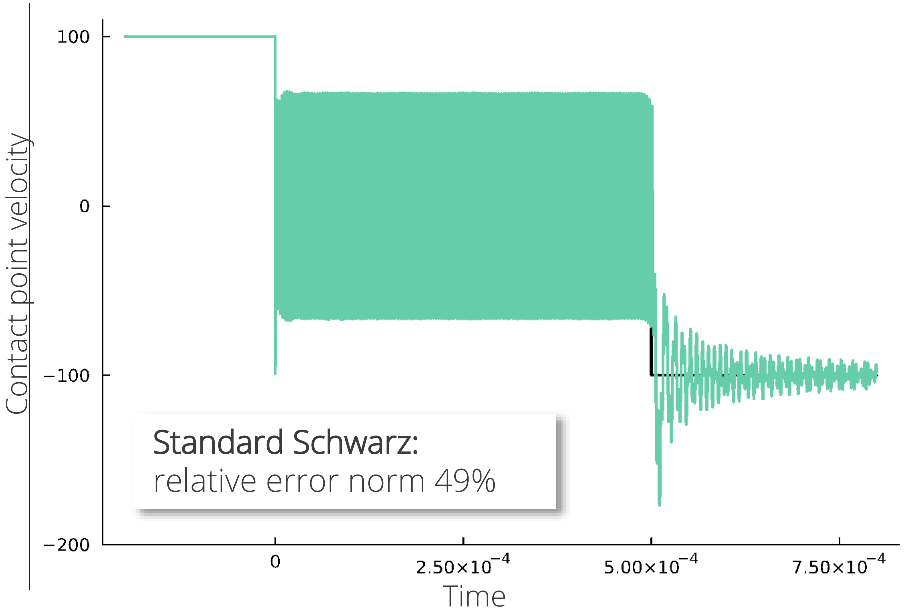
Techniques for reducing artificial oscillations

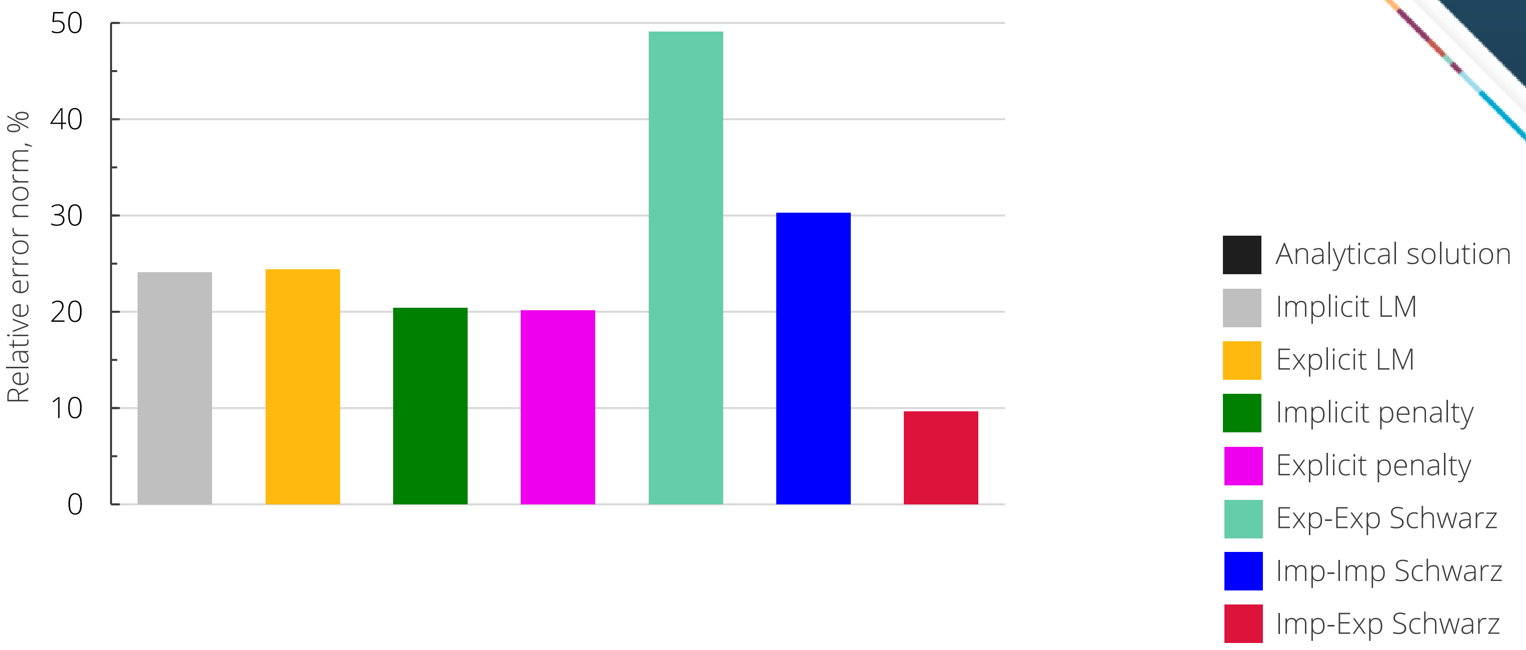


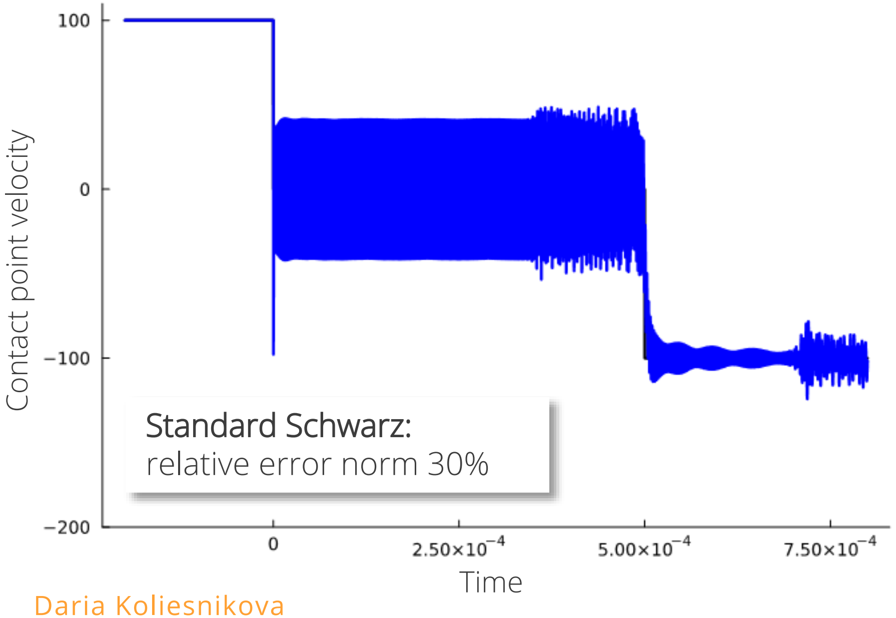




Techniques for reducing artificial oscillations

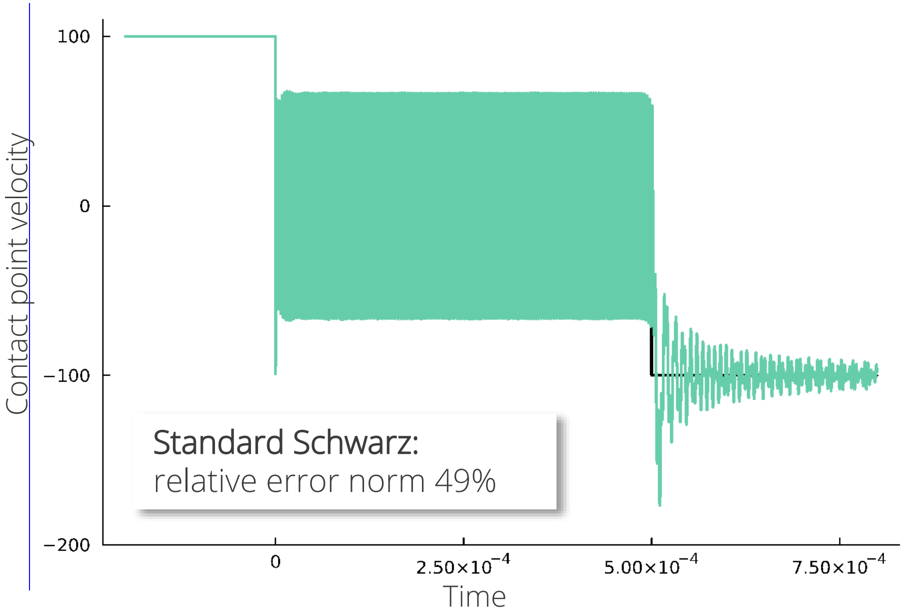


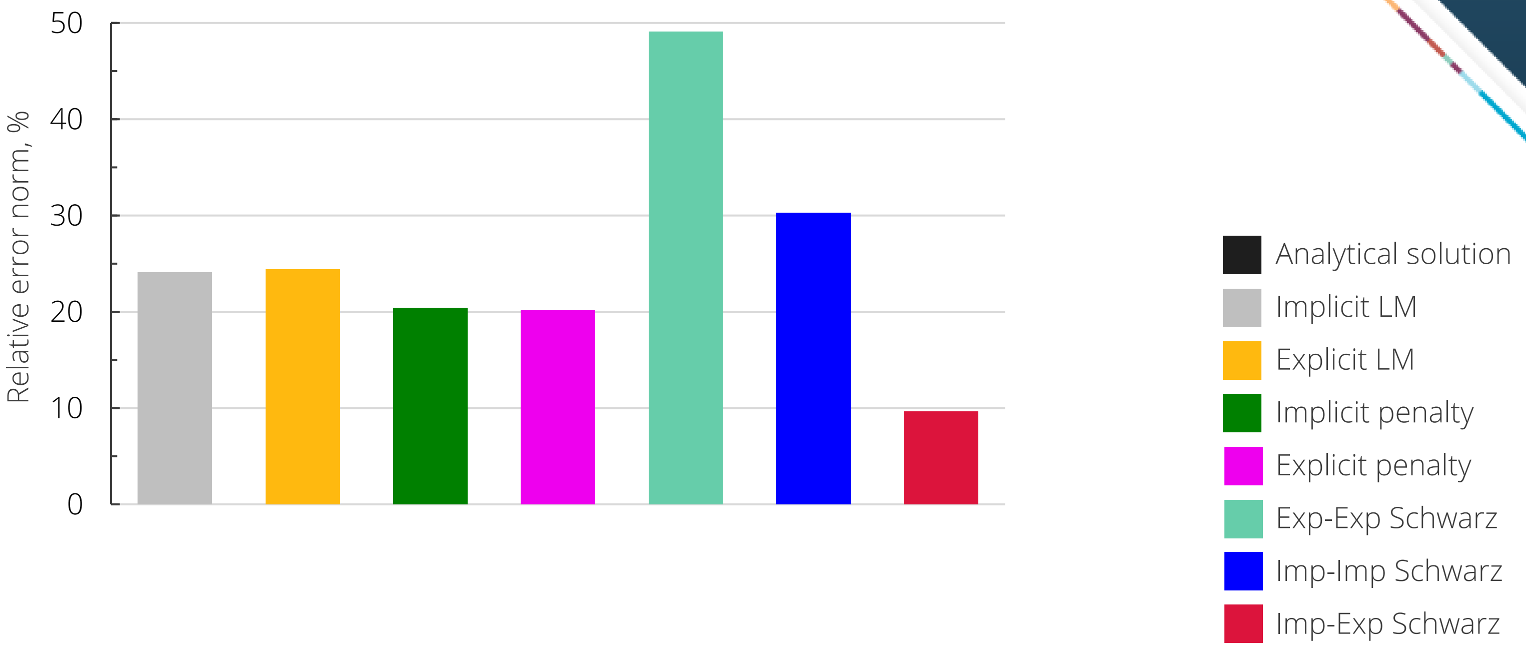


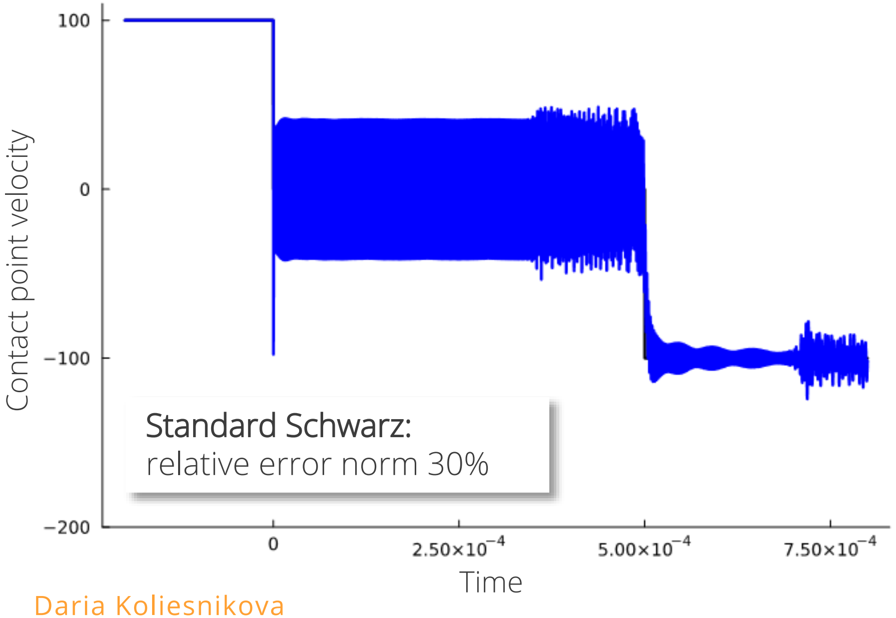


- **Contact enforcement methods:** contact laws in terms of the velocity *[M. Jean (1999) | J.J. Moreau. (1999)]* or position *[L. Paoli at al. (2001, 2002)]*
- **Mass redistribution methods:** reconstruct the mass matrix with zero mass assigned to nodes on the contact boundaries, keep unchanged the invariants of original mass matrix *[H. Khenous at al. (2008) | C. Hager et al. (2008)]*
- **Time integrators introducing numerical dissipation:** modified variants of the Newmark-beta scheme *[A. Chaudhary et al. (1986) | J. Chung et al. (1994) | B. Tchamwa et al. (1999) | T. C. Fung et al. (2003)]*
- **Stabilization methods:** make the inertia on the contact boundary vanish *[C. Kane et al. (1999) | P. Deuflhard et al. (2008) | D. Doyen et al. (2009)]*

Techniques for reducing artificial oscillations

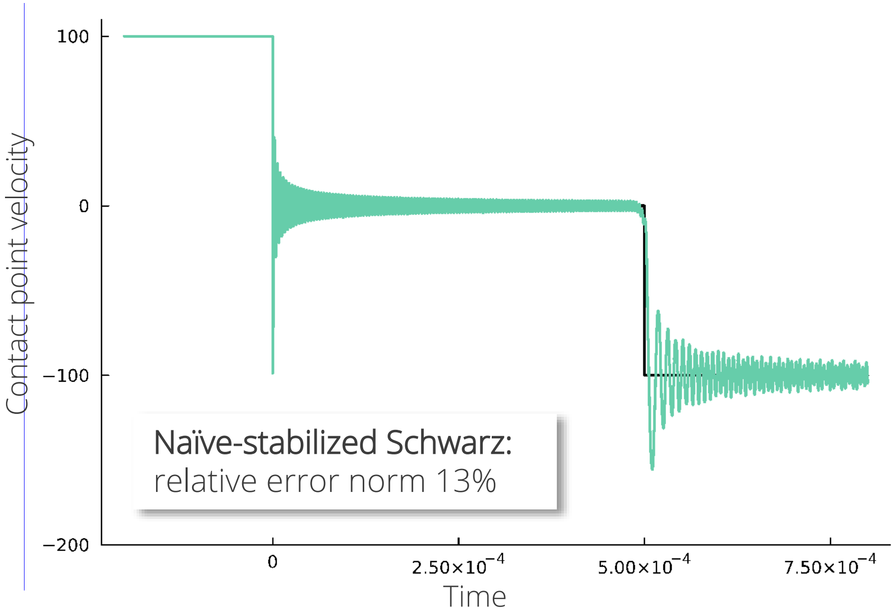


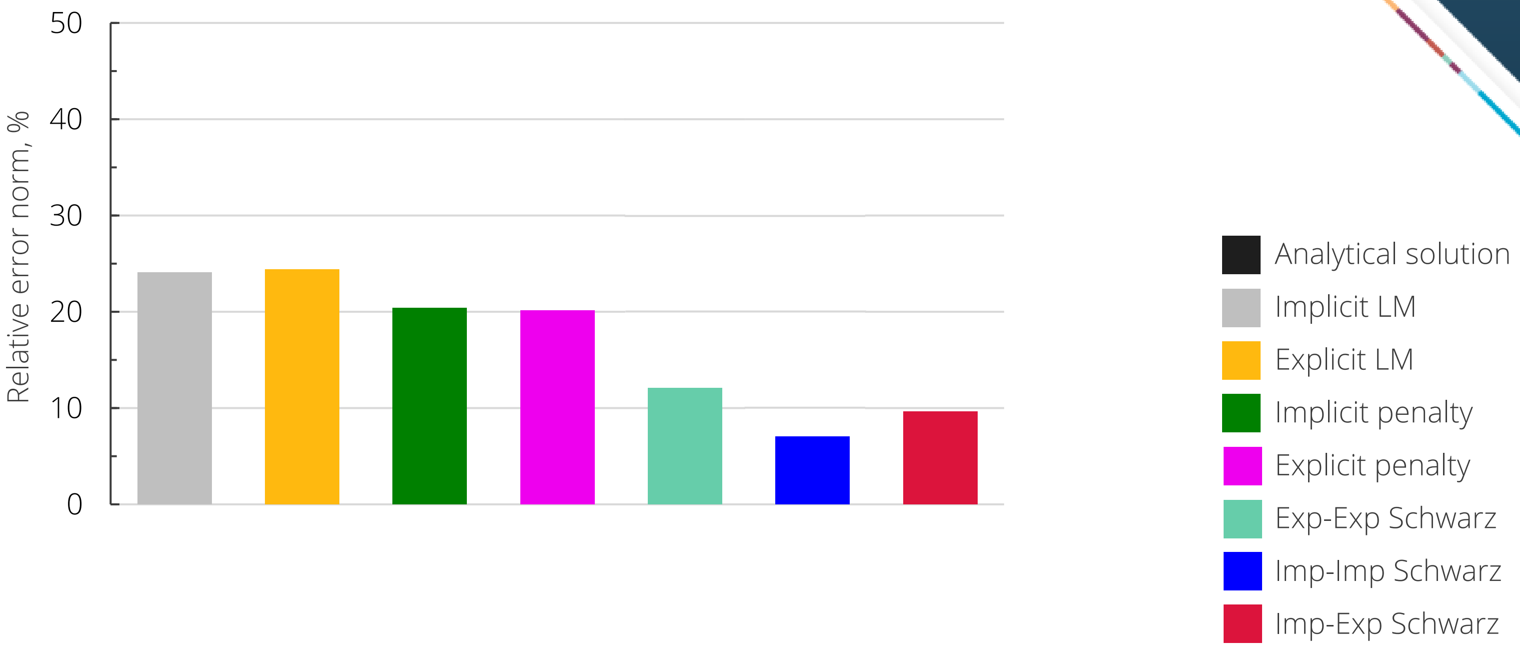


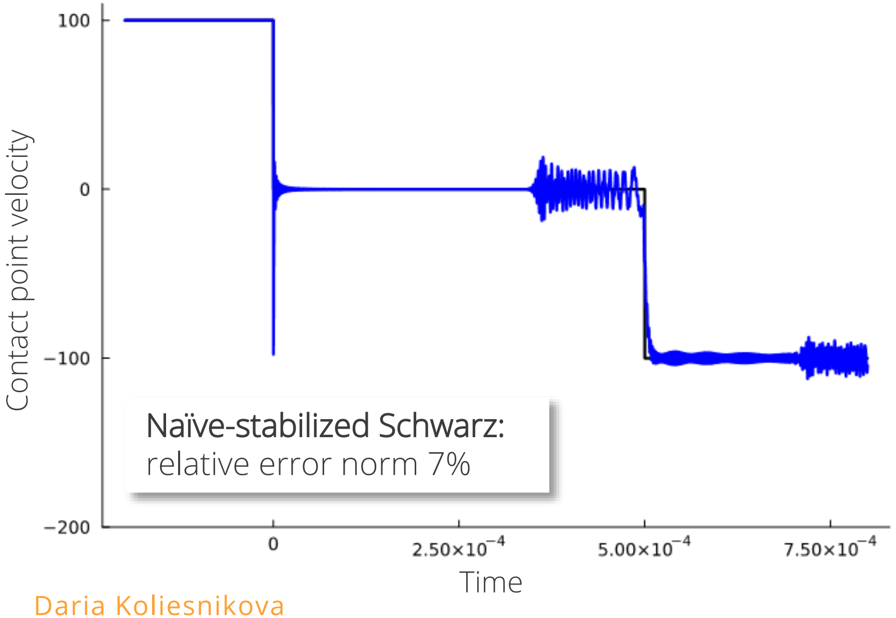


- **Contact enforcement methods:** contact laws in terms of the velocity *[M. Jean (1999) | J.J. Moreau. (1999)]* or position *[L. Paoli at al. (2001, 2002)]*
- **Mass redistribution methods:** reconstruct the mass matrix with zero mass assigned to nodes on the contact boundaries, keep unchanged the invariants of original mass matrix *[H. Khenous at al. (2008) | C. Hager et al. (2008)]*
- **Time integrators introducing numerical dissipation:** modified variants of the Newmark-beta scheme *[A. Chaudhary et al. (1986) | J. Chung et al. (1994) | B. Tchamwa et al. (1999) | T. C. Fung et al. (2003)]*
- **Stabilization methods:** make the inertia on the contact boundary vanish *[C. Kane et al. (1999) | P. Deuflhard et al. (2008) | D. Doyen et al. (2009)]*
  - ➢ naïve-stabilized approach: setting the contact accelerations to zero

Techniques for reducing artificial oscillations







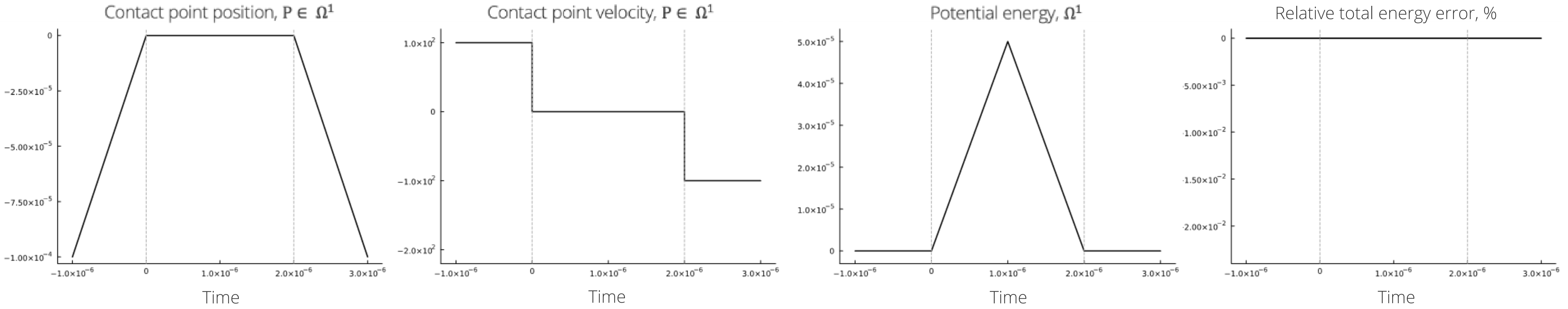
- Significant reduction of artificial oscillations
- Global relative error smaller compared to the conventional methods
- Original Schwarz method's accuracy and energy conservation properties preserved

3D benchmark

Schwarz contact method

3D benchmark

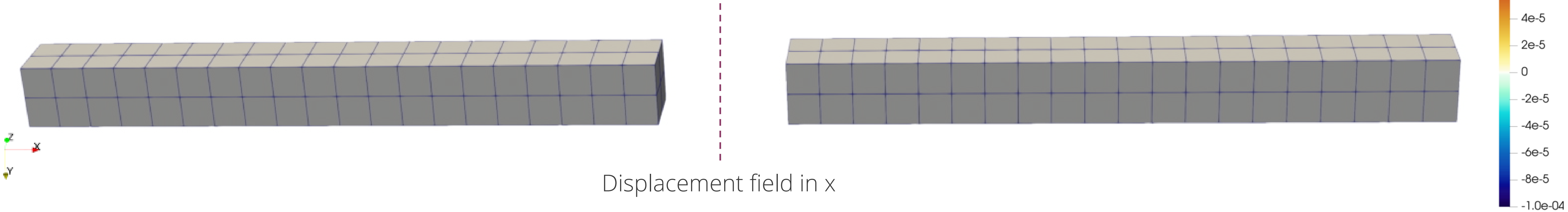
Analytical solution



3D benchmark

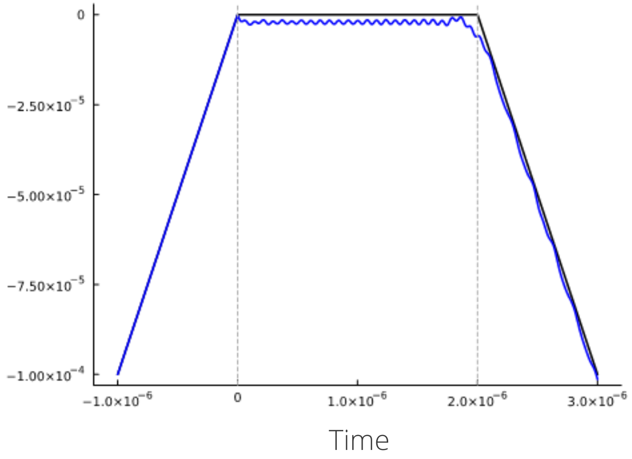
Analytical solution										
	Elements type		Mesh size		Number of nodes		Time step		Average number of Schwarz iterations
	Ω^1	Ω^2	Ω^1	Ω^2	Ω^1	Ω^2	Ω^1	Ω^2	
Imp-Imp Schwarz	HEX8	HEX8	$1/2 \cdot 10^{-4}$		189		$1 \cdot 10^{-8}$		7

Contact boundaries Γ^1 and Γ^2

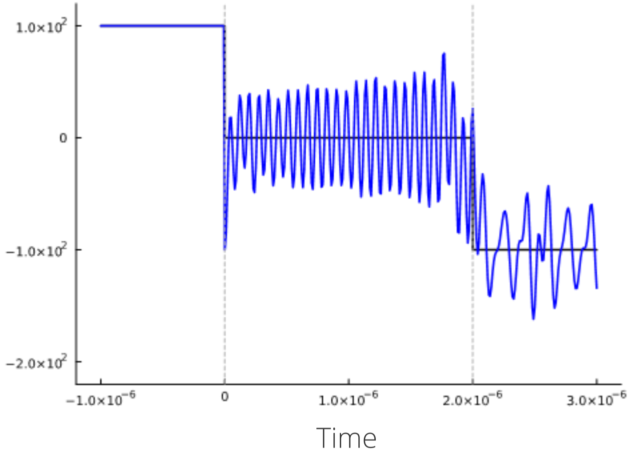


Displacement field in x

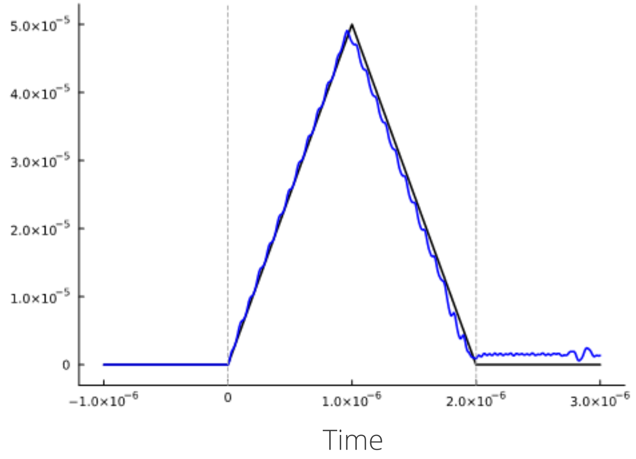
Contact point position, $P \in \Omega^1$



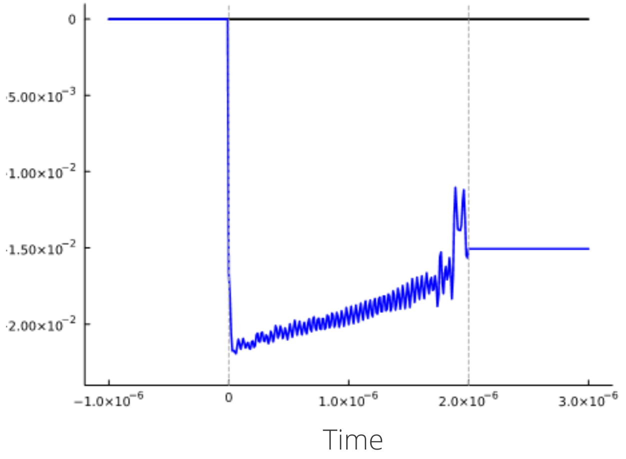
Contact point velocity, $P \in \Omega^1$



Potential energy, Ω^1



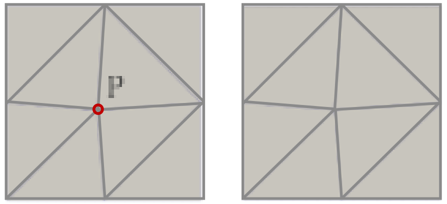
Relative total energy error, %

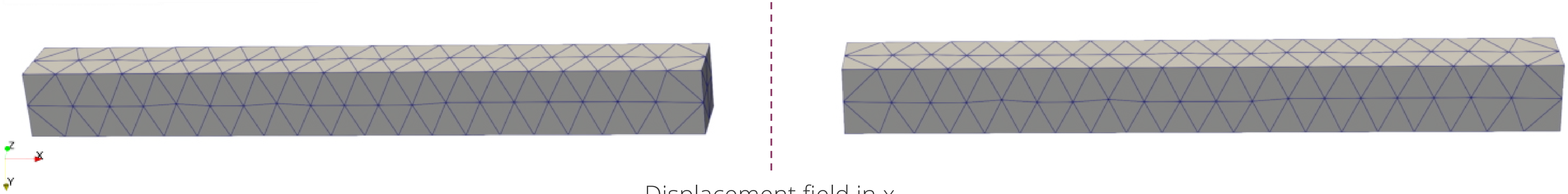


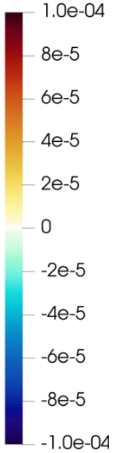
3D benchmark

Analytical solution	Elements type		Mesh size		Number of nodes		Time step		Average number of Schwarz iterations
	Ω^1	Ω^2	Ω^1	Ω^2	Ω^1	Ω^2	Ω^1	Ω^2	
Imp-Imp Schwarz	HEX8	HEX8	$1/2 \cdot 10^{-4}$		189		$1 \cdot 10^{-8}$		7
Exp-Exp Schwarz	TETRA4	TETRA4	$1/2 \cdot 10^{-4}$		199		$1 \cdot 10^{-9}$		6

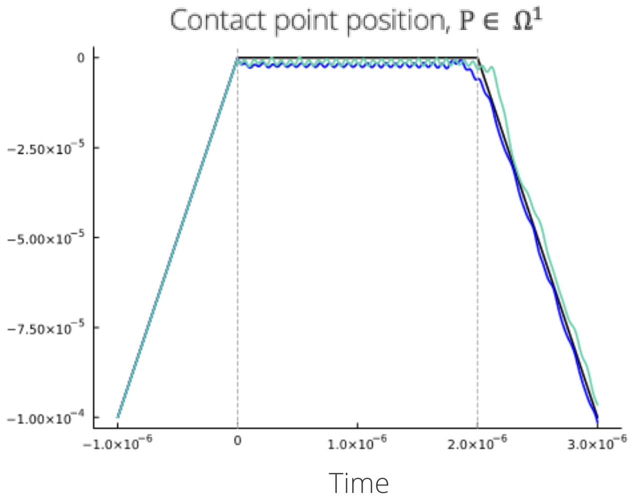
Contact boundaries Γ^1 and Γ^2

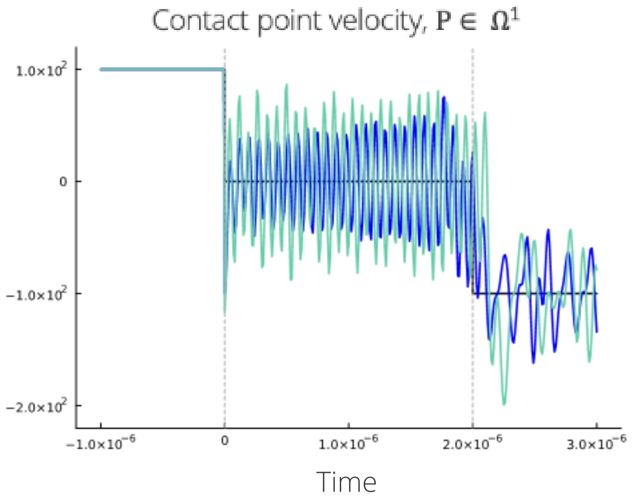


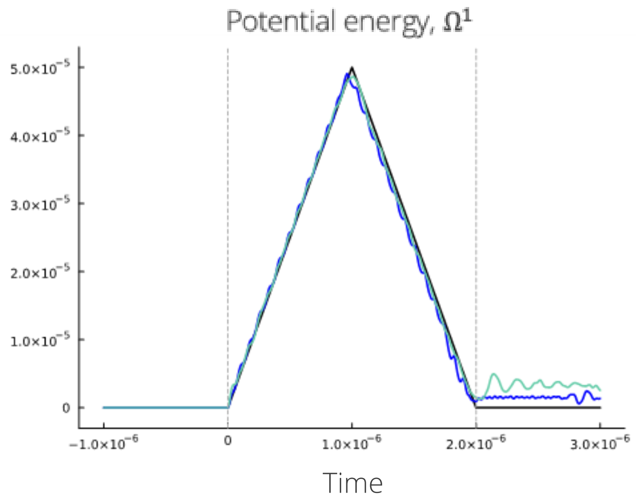


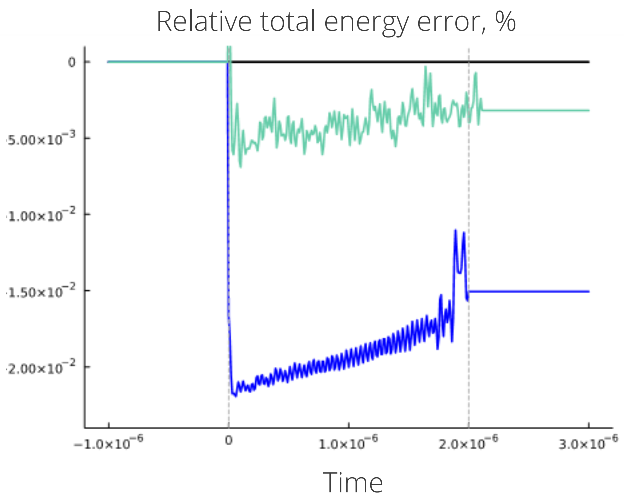


Displacement field in x



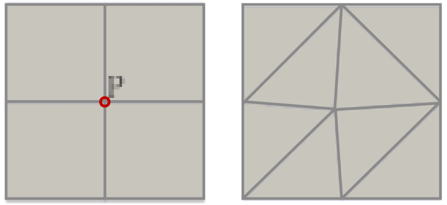




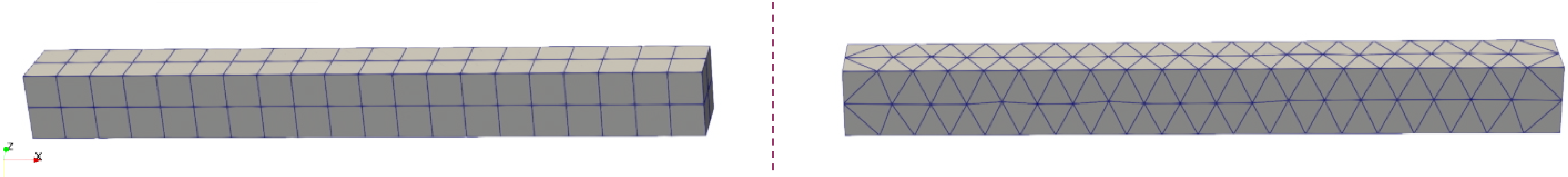


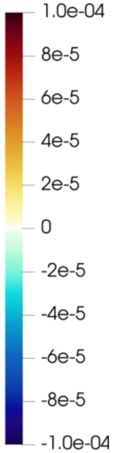
3D benchmark

Contact boundaries Γ^1 and Γ^2



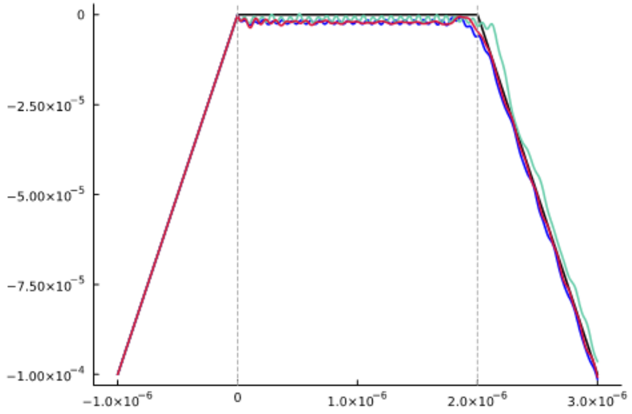
Analytical solution	Elements type		Mesh size		Number of nodes		Time step		Average number of Schwarz iterations
	Ω^1	Ω^2	Ω^1	Ω^2	Ω^1	Ω^2	Ω^1	Ω^2	
Imp-Imp Schwarz	HEX8	HEX8	$1/2 \cdot 10^{-4}$		189		$1 \cdot 10^{-8}$		7
Exp-Exp Schwarz	TETRA4	TETRA4	$1/2 \cdot 10^{-4}$		199		$1 \cdot 10^{-9}$		6
Imp-Exp Schwarz	HEX8	TETRA4	$1/2 \cdot 10^{-4}$		189	199	$1/2 \cdot 10^{-8}$	$1 \cdot 10^{-9}$	6





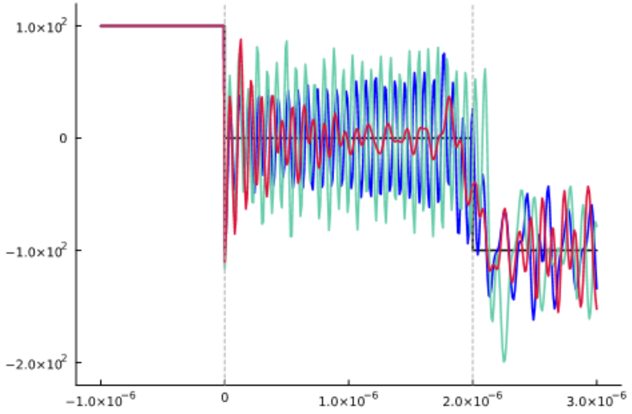
Displacement field in x

Contact point position, $P \in \Omega^1$



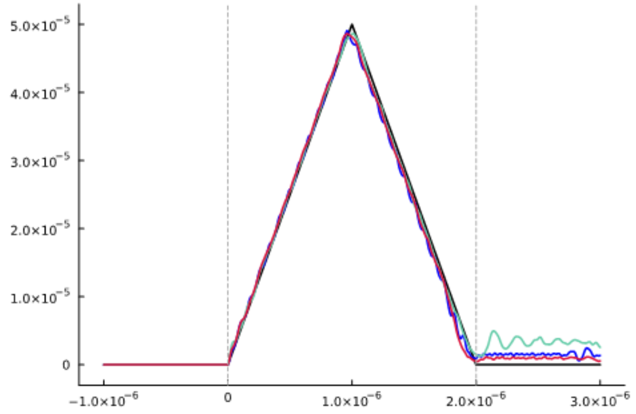
Time

Contact point velocity, $P \in \Omega^1$



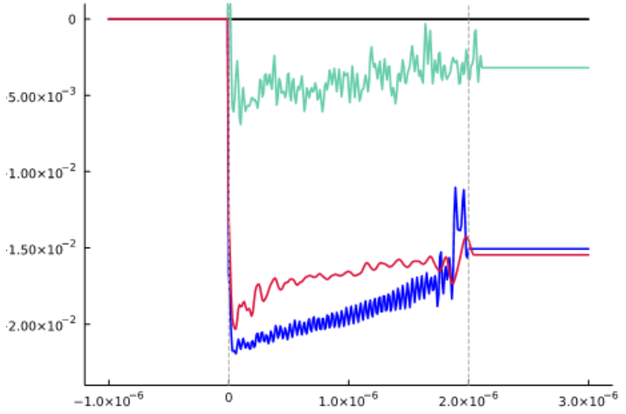
Time

Potential energy, Ω^1



Time

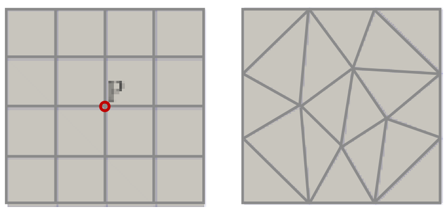
Relative total energy error, %



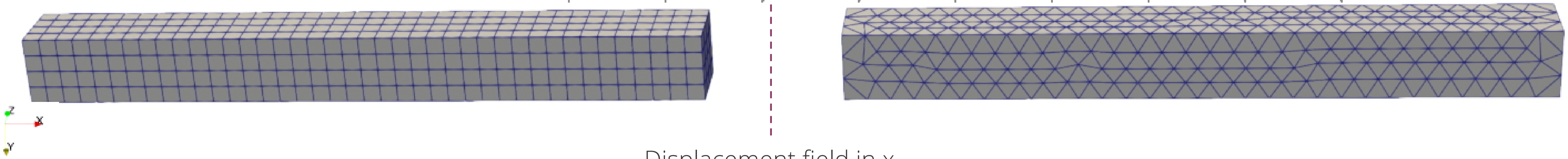
Time

3D benchmark

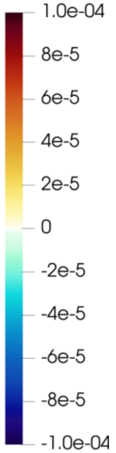
Contact boundaries Γ^1 and Γ^2



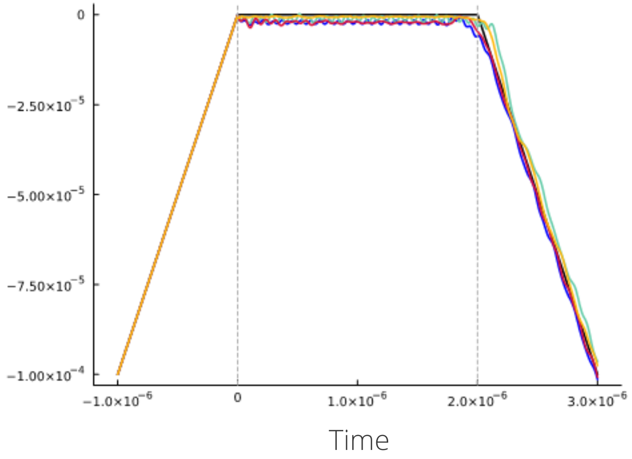
Analytical solution	Elements type Ω^1	Elements type Ω^2	Mesh size Ω^1	Mesh size Ω^2	Number of nodes Ω^1	Number of nodes Ω^2	Time step Ω^1	Time step Ω^2	Average number of Schwarz iterations
Imp-Imp Schwarz	HEX8	HEX8	$1/2 \cdot 10^{-4}$		189		$1 \cdot 10^{-8}$		7
Exp-Exp Schwarz	TETRA4	TETRA4	$1/2 \cdot 10^{-4}$		199		$1 \cdot 10^{-9}$		6
Imp-Exp Schwarz	HEX8	TETRA4	$1/2 \cdot 10^{-4}$		189	199	$1/2 \cdot 10^{-8}$	$1 \cdot 10^{-9}$	6
Exp-Imp Schwarz	HEX8	TETRA4	$1/4 \cdot 10^{-4}$	$1/3 \cdot 10^{-4}$	1025	745	$1 \cdot 10^{-9}$	$1/2 \cdot 10^{-8}$	8



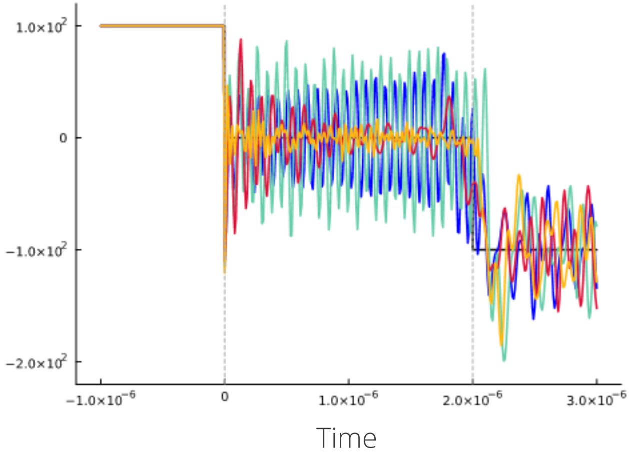
Displacement field in x



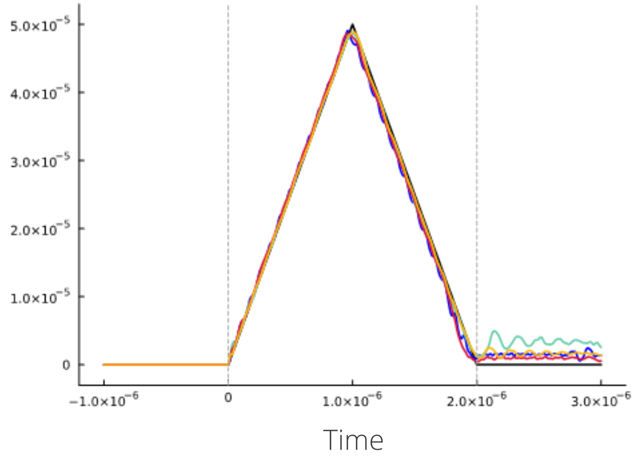
Contact point position, $P \in \Omega^1$



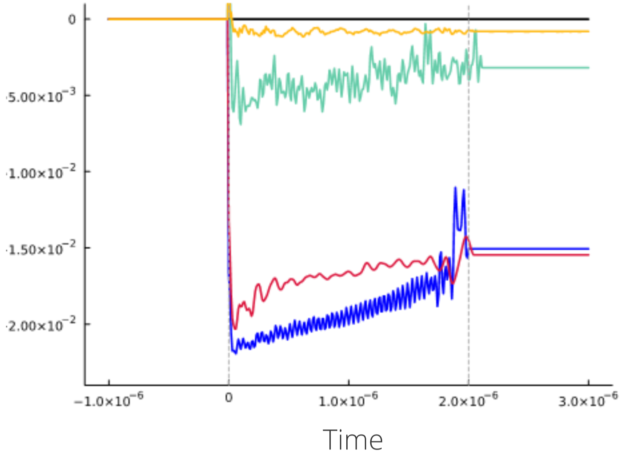
Contact point velocity, $P \in \Omega^1$



Potential energy, Ω^1



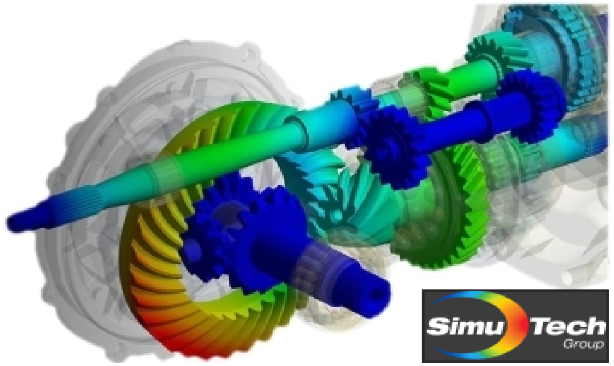
Relative total energy error, %

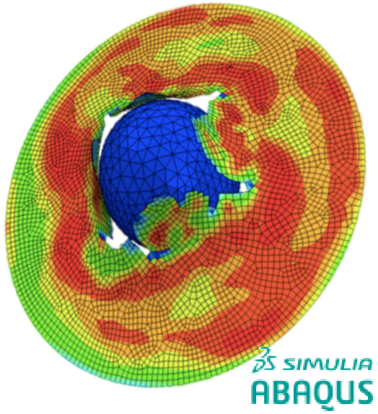


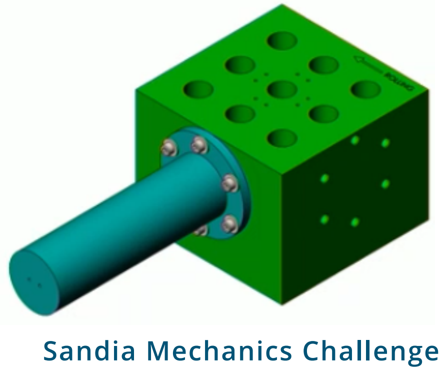
Summary and perspectives

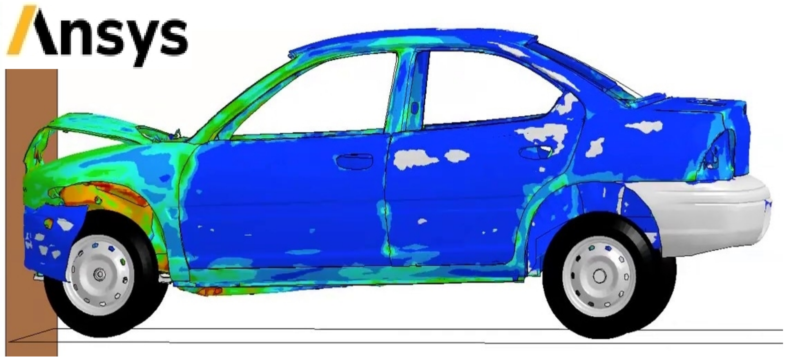
Summary

- Schwarz contact method: a promising alternative to conventional contact methods
  - Accurate predictions of quantities of interest
  - Remarkable energy conservation
  - Different integrators, times steps, mesh resolutions and mesh types !







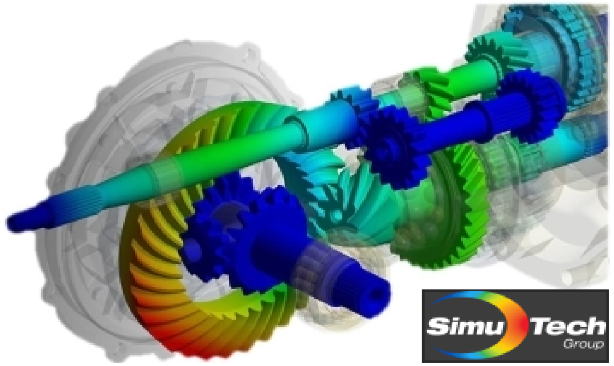


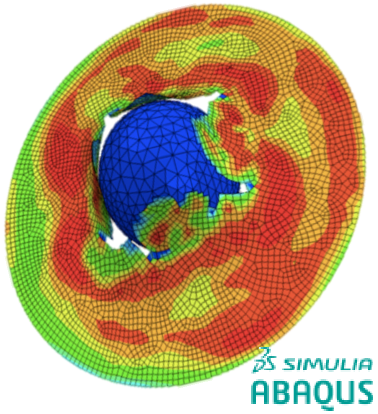
Summary

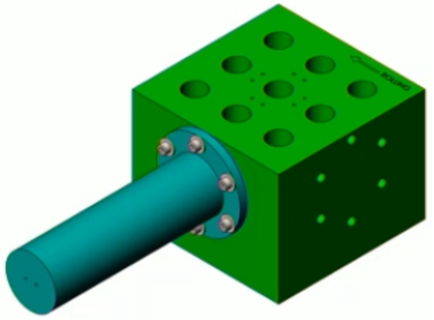
- Schwarz contact method: a promising alternative to conventional contact methods
  - Accurate predictions of quantities of interest
  - Remarkable energy conservation
  - Different integrators, times steps, mesh resolutions and mesh types !

Ongoing/future work

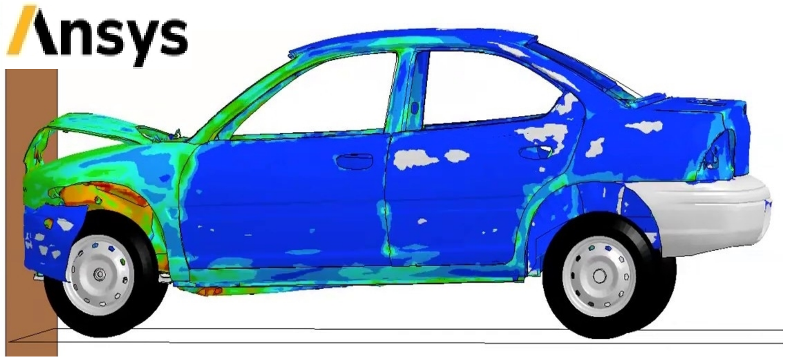
- Julia prototype for multidimensional Schwarz methods: https://github.com/lxmota/norma
  - Schwarz-based contact method
  - Schwarz-based coupling/multiphysics algorithm (overlapping and non-overlapping approaches)
- Adding friction, rolling, sliding conditions, …
- Implementation in the Sandia's production codes (Sierra/SM, Albany)







Sandia Mechanics Challenge



Sandia National Laboratories

Exceptional service in the national interest

Thank you for your attention!


Daria Koliesnikova[1]


Alejandro Mota[1]


Irina Tezaur[2]

Sandia National Laboratories

[1] Mechanics of materials department

[2] Quantitative modeling and analysis department

17th U. S. National Congress on Computational Mechanics

Albuquerque, New Mexico, July 23-27, 2023