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GLOBAL SENSITIVITY ANALYSIS IN MONTE CARLO RADIATION TRANSPORT

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OVERVIEW

Variance-based global sensitivity analysis

Quantify variance in model output and attribute it to different uncertainty sources

Stochastic computational models

Identical inputs produce different results

Variance deconvolution

Separate variance contributions from stochastic solver and uncertainty sources

BACKGROUND – GLOBAL SENSITIVITY ANALYSIS

- Consider $Q = Q(\xi_1, \xi_2, \xi_3)$, function of 3 uncertainty sources
- Statistics with respect to ξ
  - $\mathbb{E}_\xi[Q]$, mean
  - $\mathbb{V}ar_\xi[Q]$, variance

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- Fix ξ_1 as constant → conditional mean $\mathbb{E}_{\xi_2,\xi_3}[Q|\xi_1 = constant]$

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- Effect of ξ_1 → first-order sensitivity index (SI) $\dfrac{\mathbb{V}_1}{\mathbb{V}_\xi[Q]} \stackrel{\text{def}}{=} S_1$

BACKGROUND – SENSITIVITY INDICES (SI)

General case

$$Q = Q(\xi_1, \xi_2, \dots, \xi_d)$$

Consider ξ_i and $\xi_{\sim i}$

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Fractional contribution of ξ_i

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Total sensitivity index (SI)

Fractional contribution of ξ_i and all of its interactions

not i

$$T_i = 1 - \frac{\mathbb{V}_{\xi_{\sim i}}\left[\mathbb{E}_{\xi_i}[Q|\xi_{\sim i}]\right]}{\mathbb{V}ar_{\xi}[Q]} = 1 - \frac{\mathbb{V}_{\sim i}}{\mathbb{V}_{\xi}[Q]}$$

BACKGROUND – SALTELLI METHOD (GSA PRIMER, 2008)

Sampling-based estimation for S_i and T_i

1. Given $Q = Q(\xi_1, \xi_2, \dots, \xi_d)$, generate a $(N_\xi, 2d)$ matrix of independent input samples and define matrices A and B, each containing half of the sample

$$A = \begin{bmatrix} \xi_1^{(1)} & \cdots & \xi_i^{(1)} & \cdots & \xi_d^{(1)} \\ \vdots & & \ddots & & \vdots \\ \xi_1^{(N_\xi)} & \cdots & \xi_i^{(N_\xi)} & \cdots & \xi_d^{(N_\xi)} \end{bmatrix}, \qquad B = \begin{bmatrix} \xi_{d+1}^{(1)} & \cdots & \xi_{d+i}^{(1)} & \cdots & \xi_{2d}^{(1)} \\ \vdots & & \ddots & & \vdots \\ \xi_{d+1}^{(N_\xi)} & \cdots & \xi_{d+i}^{(N_\xi)} & \cdots & \xi_{2d}^{(N_\xi)} \end{bmatrix}$$

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2. For all d inputs, create matrix C_i using all columns of B except the ith column, which is from A

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3. Compute model output vector y of dimension $(N_\xi, 1)$ for A, B, and all C_i

$$y_A = \begin{bmatrix} Q\left(\xi_1^{(1)}, \ldots, \xi_i^{(1)}, \ldots, \xi_d^{(1)}\right) \\ \vdots \\ Q\left(\xi_1^{(N_\xi)}, \ldots, \xi_i^{(N_\xi)}, \ldots, \xi_d^{(N_\xi)}\right) \end{bmatrix}$$

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