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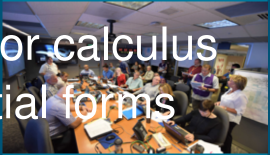
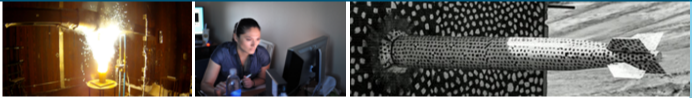
Progress towards a discrete exterior calculus for (vector) bundle-valued differential forms





Presented by:

Chris Eldred (SNL)

Sandia National Laboratories

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A "new" framework for continuum mechanics

A powerful framework for building continuum mechanics models is a **geometric mechanics formulation** (variational, Hamiltonian, metriplectic, etc.) written using **exterior calculus** (differential forms):

$$\delta \int \mathscr{L}[\mathbf{x}] = 0 \qquad \frac{\partial \mathbf{x}}{\partial t} = \mathbb{J}(\mathbf{x})\frac{\delta \mathscr{H}}{\delta \mathbf{x}} + \mathbb{M}(\mathbf{x})\frac{\delta \mathscr{S}}{\delta \mathbf{x}}$$

- **Coordinate** and **orientation independent** description valid on **arbitrary manifolds**
- Exposes key features such as **conservation laws** and **involution constraints**
- Basis for building **structure-preserving** numerical methods

*Not new, but there have been recent developments regarding treatment of momentum/stress/etc. in these approaches

Structure-preserving discretizations

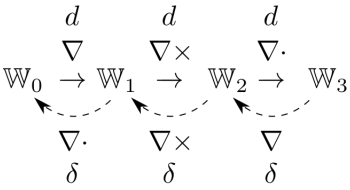
Structure-preserving = discretize the framework above using **mimetic discretizations** in a way that keeps the key features of continuous GM formulations

- mimetic = discrete version of (exterior) calculus with discrete analogues of key (exterior) calculus identities such as
  - annihilation/exact sequence: $\nabla \cdot \nabla \times = 0$, $\nabla \times \nabla = 0$
  - integration by parts: $\int_\Omega a \nabla \cdot \mathbf{b} + \int_\Omega \nabla a \cdot \mathbf{b} = \int_\Omega \nabla \cdot (a\mathbf{b}) = \int_{\partial\Omega} a\mathbf{b} \cdot \hat{\mathbf{n}}$
  - deRham cohomology: harmonic spaces have the correct size
- Discretize objects from GM formulations (such as $\mathscr{H}[\mathbf{x}]$ and $\mathbb{J}(\mathbf{x})$) instead of equations of motion

Types of mimetic (spatial) discretizations

Single deRham Complex

standard exterior calculus

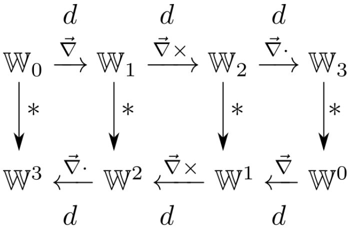


$$(da^k, b^{k+1}) = (a^k, \delta b^{k+1})$$

compatible Galerkin methods (ex. finite element exterior calculus, compatible isogeometric methods, mimetic Galerkin differences)

Double deRham Complex

split exterior calculus



discrete exterior calculus (DEC), split compatible Galerkin methods

This approach is well-understood for **scalar-valued differential forms** (SVDFs): ex. mass density ρ, specific entropy η, velocity **v**, electromagnetic fields **D** and **B**

Exterior calculus of (vector) bundle-valued differential forms

Momentum (and stress) are not SVDFs, instead they are **(vector)-bundle valued differential forms** (BVDFs)!

BVDFs: $\mathbf{x}_E^k \in \Lambda^k(E)$ and $\tilde{\mathbf{x}}_E^k \in \tilde{\Lambda}^k(E)$: smooth section of the tensor product bundle of vector bundle E with the kth exterior power of the cotangent bundle $T^\star$

Vector bundle: A vector space $V(x)$ attached to each point x of a $\mathscr{M}$

Can define an exterior calculus for BVDFs that mirrors the one used for SVDFs:

- (covariant) exterior derivative d_X, Hodge star $\tilde{\star}$, topological pairing $\langle\langle,\rangle\rangle_\chi$, inner product $\langle,\rangle_\chi$, Trace $\mathbb{T}$, Inclusion, $\mathbb{I}$, flat $\flat_1$, sharp $\sharp_1$, Lie derivative, L interior product i
- BVDF exterior calculus reduces to SVDF exterior calculus when $E = \mathbb{R}$ or $E = \Psi$

Aside: measures of fluid flow

Four commonly used measures of fluid flow*:

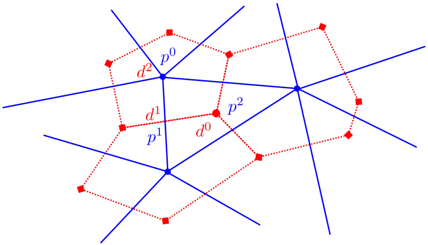
Object	Geometric Mechanics Relations	Name
$\mathbf{u}_{\mathrm{T}}^{0}$	$\mathscr{L}[\mathbf{u}_{\mathrm{T}}^{0}, a]$, $\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}_{\mathrm{T}\star}^{\mathrm{n}}}$	convective velocity
$\tilde{\mathbf{m}}_{\mathrm{T}\star}^{\mathrm{n}}$	$\frac{\delta \mathscr{L}}{\delta \mathbf{u}_{\mathrm{T}}^{0}}$, $\mathrm{H}[\mathbf{u}_{\mathrm{T}}^{0}, a]$	momentum
v^{1}	"$\frac{\tilde{\mathbf{m}}_{\mathrm{T}\star}^{\mathrm{n}}}{D}$", $\mathscr{H}[\mathrm{v}^{1}, a]$	velocity
$\tilde{F}^{n-1}$	$\frac{\delta \mathscr{H}}{\delta \mathrm{v}^{1}}$	mass flux

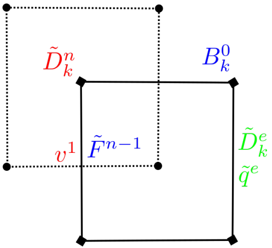
- Look the "same" in vector calculus in $\mathbb{R}^3$, (very) distinct in exterior calculus: source of much confusion; ex. compressible Euler has $\mathbf{m} = \rho \mathbf{v}$, $\mathbf{F} = \rho \mathbf{u}$, $\mathbf{u} = \mathbf{v}$
- All play a key role in geometric mechanics formulations: $\tilde{\mathbf{m}}_{\mathrm{T}\star}^{\mathrm{n}}$ and $\mathbf{u}_{\mathrm{T}}^{0}$ in Euler-Poincaré and Lie-Poisson formulations; v^{1} and $\tilde{F}^{n-1}$ in Kelvin-Noether and Curl-Form formulations
- This fits with the discussion in Tonti2013/Tonti2014 about the dual nature of velocity, see for example FLU3 (=SVDFs) vs. FLU6 (=BVDFs) in Tonti2014

*These are the four vector proxies in exterior calculus. Also have "pseudovector proxies": $\tilde{\mathbf{x}}_{T}^{0}$, $\mathbf{x}_{T\star}^{n}$, $\tilde{x}^{1}$ and x^{n-1}, not discussed here

Key ideas of SVDF DEC I

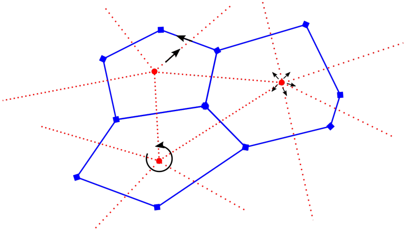
- Two grids in duality (straight and twisted , 1-1 relationship between k and $(n-k)$ cells on opposite grids
- Discrete differential forms are **integrated values** over **geometric entities**, 1-1 relationship between k-form and $(n-k)$-forms on opposite grids (**Hodge star**)

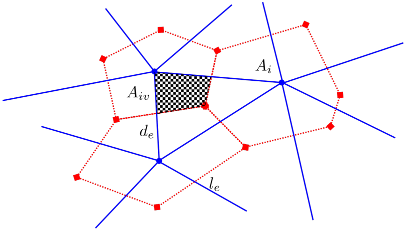




Key ideas of SVDF DEC II

- Operators are either **topological** (exterior derivative, wedge product) or **metric** (Hodge star, interior product), discretize accordingly + separately



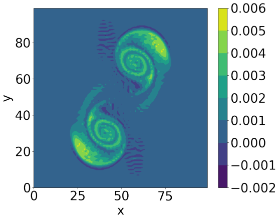


- Key: **topological operators** give most of the **desirable properties** (ex. conservation, involutions, no spurious computational modes), **metric operators** give **accuracy**

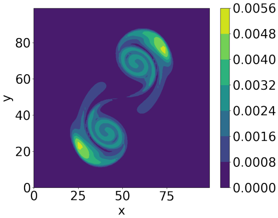
Recent SVDF DEC developments

SVDF DEC is now competitive with compatible Galerkin methods:

- **consistent** treatment of **arbitrary boundaries** and **boundary conditions**
- **structure-preserving high-order, oscillation-limiting, positive-definite** (SPHOOL-PD) transport operators



"Usual" DEC transport operator



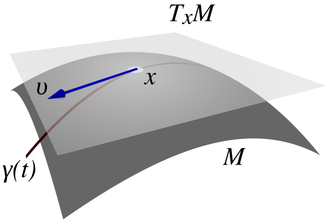
SPHOOL-PD DEC transport operator

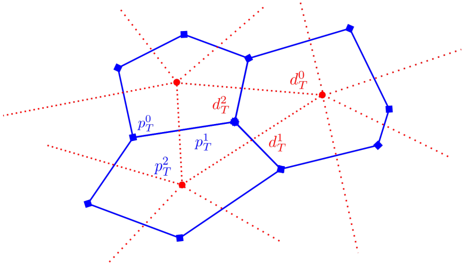
- **higher-order Hodge stars** (work in progress on unstructured grids)

BVDF DEC

Extended key operators of SVDF discrete exterior calculus to (vector) bundle-valued differential forms

- Borrows heavily from SVDF DEC, reduces to it for the case of $E = \mathbb{R}$ or $E = \Psi$
- Focused on fundamental exterior calculus operators: d_X, $\tilde{\star}$, $\dot{\wedge}$, $\langle , \rangle_\chi$, $\langle\langle , \rangle\rangle_\chi$
  - For $\mathbb{R}^3$, where tangent and cotangent bundles are flat: trivial connection and metric, global basis





Future Work

- Transport operators for arbitrary SVDFs/BVDFs i.e. Lie derivatives $\mathrm{L}_{\mathbf{u}_T^0}$, interior products i; and associated raising/lowering operators: $\mathbb{T}, \mathbb{I}, \flat_1, \sharp_1, \flat, \sharp$

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- Application to momentum-based formulations of fluids, especially charged fluid models
- Extension to arbitrary manifolds i.e. non-flat bundles: will require a discrete connection X_E and metric $\mathbf{g}_E$

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Questions?

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