SAND2023-07988C

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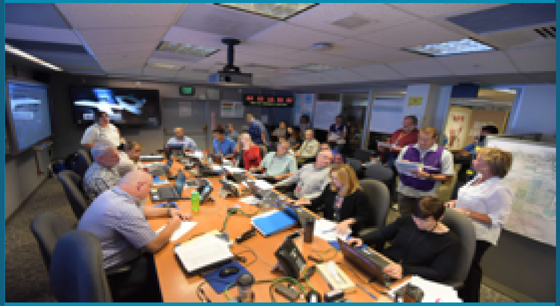
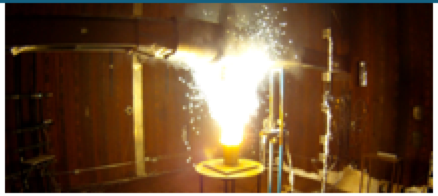
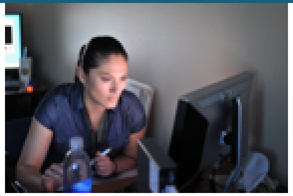
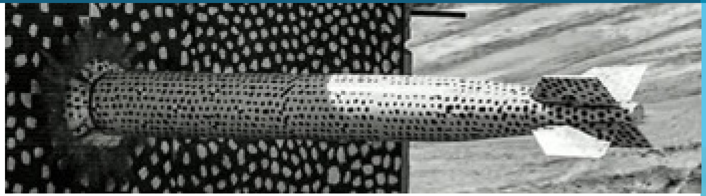
A multigrid method for generalized (or extended) magnetohydrodynamics









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What is generalized or extended magnetohydrodynamics (GMHD)

Equations of interest

$$\partial_t \mathbf{D} - \operatorname{curl} \mathbf{H} + \mathbf{J} = 0, \qquad \partial_t \mathbf{B} + \operatorname{curl} \mathbf{E} = 0, \qquad \mathbf{D} = \epsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H},$$

$$\tau \partial_t \mathbf{J} + \mathbf{J} - \boldsymbol{\beta} \times \mathbf{J} - \sigma \mathbf{E} = 0$$

Generalized Ohm's law replaces classical Ohm's law,

$$\mathbf{J} = \sigma \mathbf{E}.$$

$$\boldsymbol{\beta} = \frac{\mathrm{e}\tau}{m_e} \mathbf{B}, \qquad \sigma = \frac{\mathrm{e}^2 n_e}{m_e} \tau.$$

E is the electric field, **H** is the magnetic field, **D** is the displacement field, **J** is the current density, **B** is the magnetic flux density (magnetic induction), $|\boldsymbol{\beta}|$ is the so-called Hall parameter, σ is the electrical conductivity, μ is the magnetic permeability, ϵ is the electric permittivity, e denotes the unit electric charge, n_e the electron number density, m_e the electron mass, and τ the ion-electron relaxation time.

Classical Ohm’s law vs. Generalized Ohm’s Law

Hall term

$$\mathbf{J} = \sigma \mathbf{E}$$

vs.

$$\tau \partial_t \mathbf{J} + \mathbf{J} - \boldsymbol{\beta} \times \mathbf{J} - \sigma \mathbf{E} = 0$$

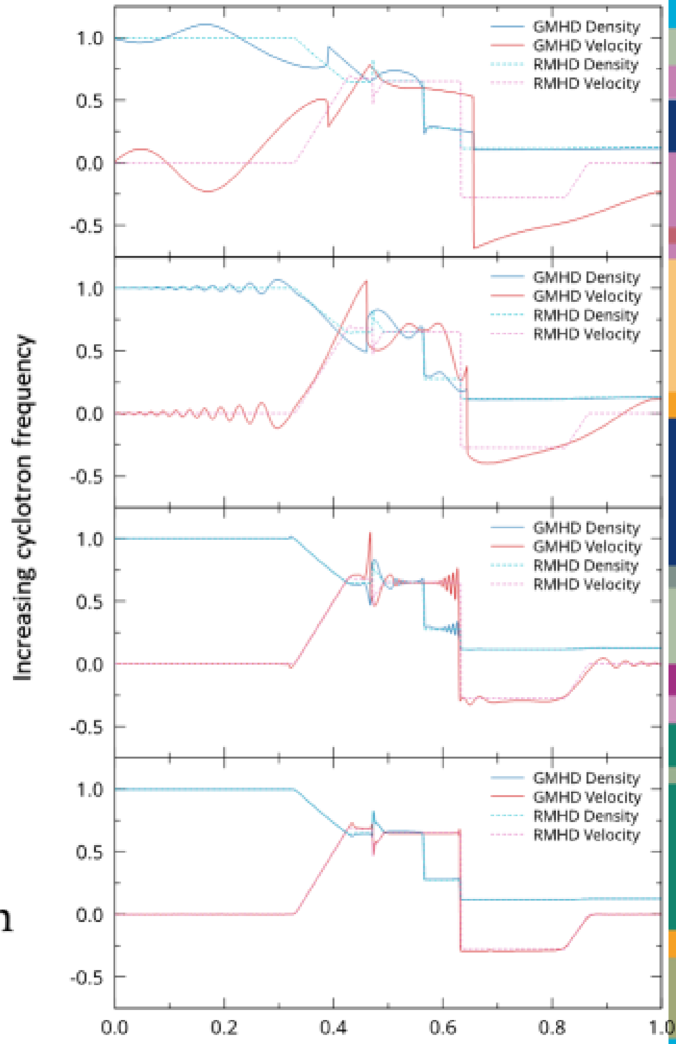
assumes certain quantities vary only slightly between electron-ion collisions and ions are stationary

incorporates accelerations, important for plasmas (e.g., high current pulsed power accelerators at high plasma densities). Allows for asymmetric coupling between current & magnetic fields.

Recall,

$$\boldsymbol{\beta} = \frac{e\tau}{m_e}\mathbf{B},$$

…. and $\boldsymbol{v} \times \boldsymbol{v} = 0$ … so $\boldsymbol{\beta} \times \mathbf{J}$ has null space



Generalized Ohm’s law can be derived from two-fluid approximation. Other MHD extensions also incorporate additional physics (Nerst, Ettingshausen, Leduc-Righi), which may incur new linear solver challenges.

Formulation

Recall, GMHD equations include unknowns **E**, **J**, **B**, **D**, **H**.

D and **H** easily eliminated. While **E**, **J**, and **B** are coupled, we can often use **B** at a prior time step to advance **E** & **J**, so main focus is on **E-J** system.

We consider discrete 2x2 block linear system which avoids element projections within Ohm's law to produce matrix *more amenable to solver.*

$$\begin{bmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{E}^{n+1} \\ \mathbf{J}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix},$$

Hall term

$$\mathbb{A}_{11}(\widehat{\mathbf{E}}_i, \widehat{\mathbf{E}}_j) = \frac{\epsilon}{\Delta t} \int_\Omega \widehat{\mathbf{E}}_i \cdot \widehat{\mathbf{E}}_j \, d\Omega + \frac{\Delta t}{\mu} \int_\Omega \operatorname{curl} \widehat{\mathbf{E}}_i \cdot \operatorname{curl} \widehat{\mathbf{E}}_j \, d\Omega$$

$$\mathbb{A}_{12}(\widehat{\mathbf{E}}_i, \widehat{\mathbf{J}}_j) = \int_\Omega \widehat{\mathbf{E}}_i \cdot \widehat{\mathbf{J}}_j \, d\Omega$$

$$\mathbb{A}_{21}(\widehat{\mathbf{J}}_i, \widehat{\mathbf{E}}_j) = -\int_\Omega \sigma^{n+1} \left(\widehat{\mathbf{J}}_i \cdot \widehat{\mathbf{E}}_j \right) d\Omega$$

$$\mathbb{A}_{22}(\widehat{\mathbf{J}}_i, \widehat{\mathbf{J}}_j) = \int_\Omega \left(\left(1 + \frac{\tau}{\Delta t}\right) \left(\widehat{\mathbf{J}}_i \cdot \widehat{\mathbf{J}}_j \right) - \frac{e\tau}{m_e} \left[\widehat{\mathbf{J}}_i \cdot \left(\mathbf{B}^n \times \widehat{\mathbf{J}}_j \right) \right] \right) d\Omega$$

edge basis functions

Many AMG solvers developed f $\mathbb{A}_{11}$, e.g. auxiliary space methods.

GMHD Formulation Alternatives

We have considered a few alternative formulations and discretizations (all based on edge elements). Formulation still remains a research topic. Among those considered, 2x2 system presented on previous page is attractive in terms of solution quality and amenability to linear solvers

One alternative formulation

$$\int \left(\frac{\epsilon}{\Delta t} + \underline{\chi}_\beta \sigma^{n+1} \Pi_P\right) \mathbf{E}^{n+1} \cdot \mathbf{\Psi}\, d\Omega + \int \frac{\Delta t}{\mu} \operatorname{curl} \mathbf{E}^{n+1} \cdot \operatorname{curl} \mathbf{\Psi}\, d\Omega$$

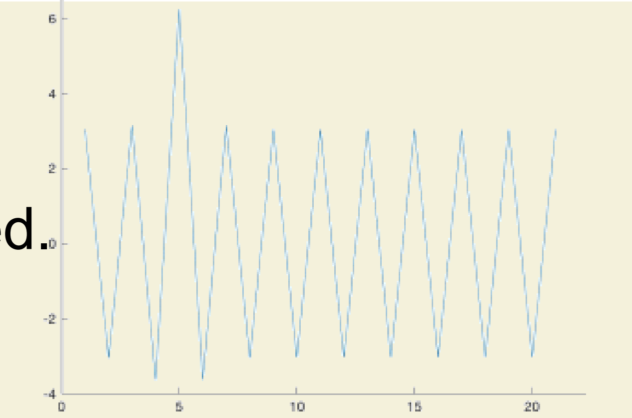
when Hall term dominates, $\approx \frac{1}{3}$ of spectrum is nearly 0

solve for $\mathbf{E}^{n+1}$

where $\underline{\chi}_\beta^{-1} = \left(1 + \frac{\tau}{\Delta t}\right)\underline{\mathbf{I}} - \underline{\boldsymbol{\beta}}_P^n$ & $\boldsymbol{\beta}_P^n = \frac{e\tau}{m_e} \Pi_P(\mathbf{B}^n)$

A continuous Schur complement elimination that can be efficiently formed.

The discrete near null space includes non-physical oscillatory modes complicating
linear solve.



null space mode along a grid line across domain for model problem

Null space associated with Hall term

Notice that there are no derivatives in the Hall term associated with the null space

$$\tau\partial_t \mathbf{J} + \mathbf{J} - \underbrace{\boldsymbol{\beta} \times \mathbf{J}}_{\text{no derivatives}} - \sigma\mathbf{E} = 0$$

A basis for the discrete null space has a local character. Though connected to continuous null space, $\mathbf{J}(x,y,z) = \boldsymbol{\beta}(x,y,z)$, the nature of discretization basis functions plays a role.

We try to address both curl-curl null space and Hall term null space using Arnold-Falk-Winther style smoothers.

Patch Smoothers Background

An overlapping Schwarz method with n_p patches

$$M^{-1} = W \sum_{k=1}^{n_p} V_k^T A_k^{-1} V_k$$

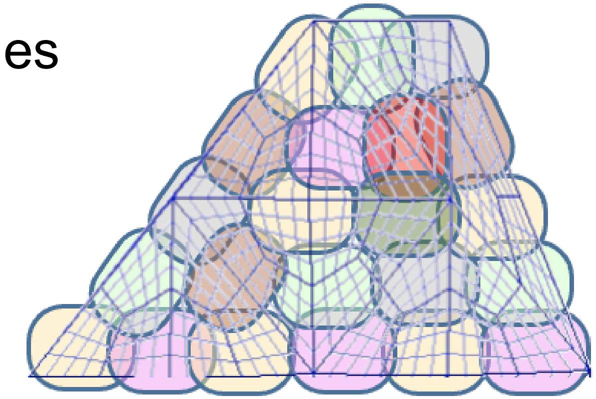
where V_k a is a Boolean restriction matrix to k^{th} patch ,

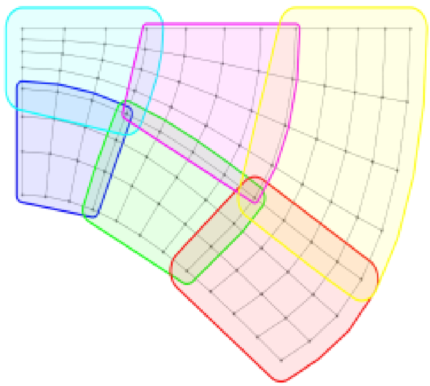
$$W = \left(\sum_{\ell=1}^{n_p} V_\ell^T V_\ell \right)^{-1}$$

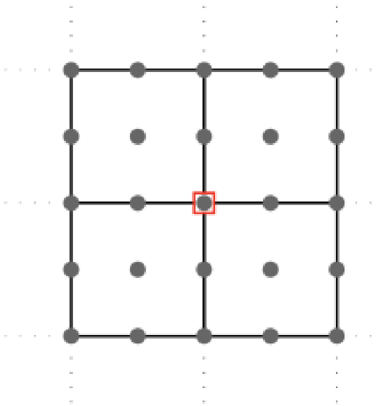
and

$$A_k = V_k A V_k^T, \quad k = 1, \ldots, n_p$$

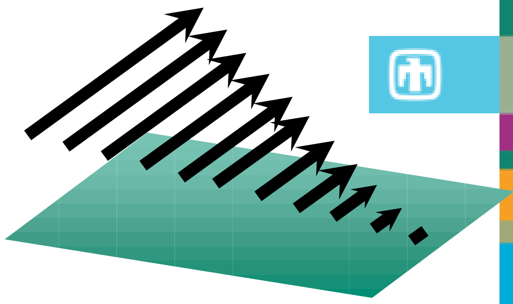
examples with patches







Edge Elements and Arnold-Falk-Winther (AFW) Patches



1 edge basis function

$$\nabla \times \boldsymbol{\mu}^{-1} \nabla \times \boldsymbol{E} \quad + \quad \boldsymbol{\sigma} \boldsymbol{E} = \boldsymbol{f}$$

Edge element discretization

$$S_1 \qquad + \qquad M_1 \quad = \; \ldots$$

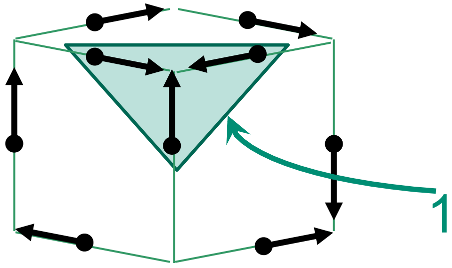
<u>Null(curl,curl)</u>

continuous : $\{\nabla\varphi \mid \varphi \text{ scalar}\}$

discrete : $S_1 D_0 = \emptyset$

$$D_0(\boldsymbol{e}, \boldsymbol{v}) = \begin{cases} +1 & \text{if } \boldsymbol{v} = \text{head}(\boldsymbol{e}) \\ -1 & \text{if } \boldsymbol{v} = \text{tail}(\boldsymbol{e}) \end{cases}$$

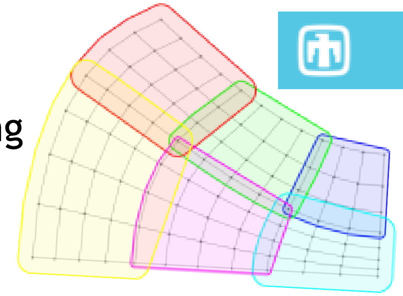
k^{th} AFW patch takes all edge dofs adjacent to k^{th} vertex, which corresponds to nonzeros in k^{th} column of D_0



1 patch on 1x1x1 element mesh

Patch Compression

patch relaxation: overlapping Schwarz with small domains



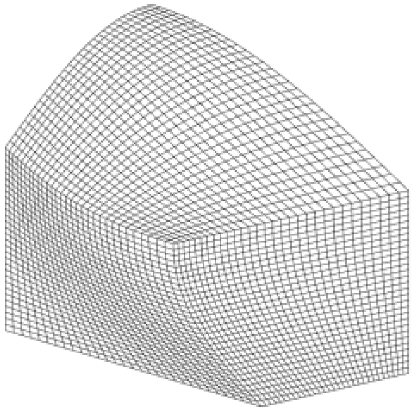
$$M^{-1} = W \sum_{k=1}^{n_p} V_k^T A_k^{-1} V_k \approx W \sum_{k=1}^{n_p} V_k^T A_{(k)}^{-1} V_k$$

$\cong A_k^{-1}$

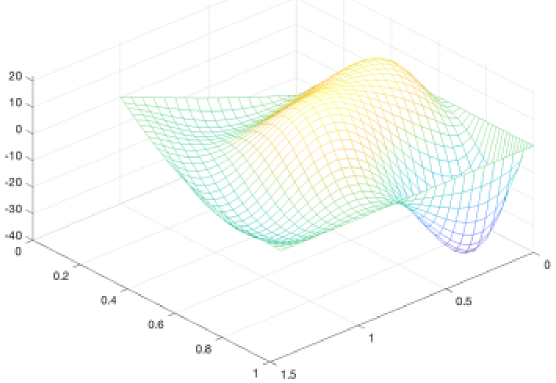
<u>compression example</u>

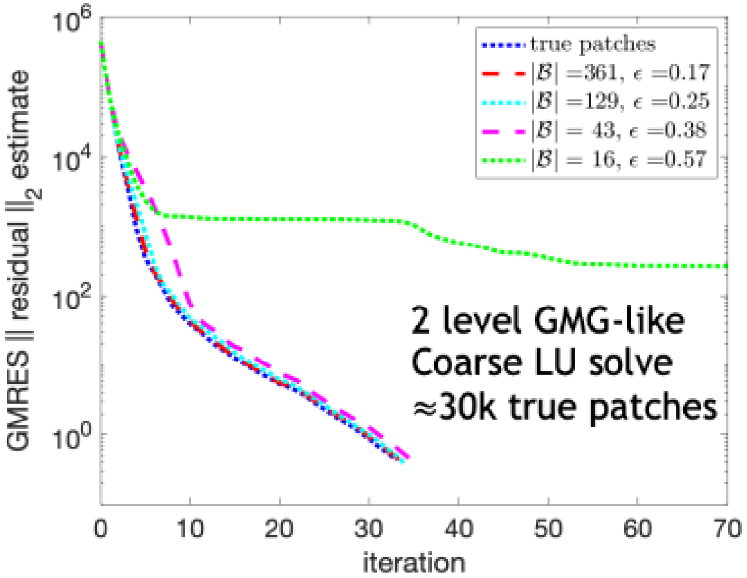
- approx. & patches using
- approx. & patches using

(curl v, curl u) + .0001 (v , u) = f



u for z-oriented edges in xy plane





Often able to effectively solve linear systems with < 5% of true patches !

Graham Harper talk: Friday (01170(3/3):5D@E708

GMHD generalization of AFW patches

$$\begin{bmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{J} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix},$$

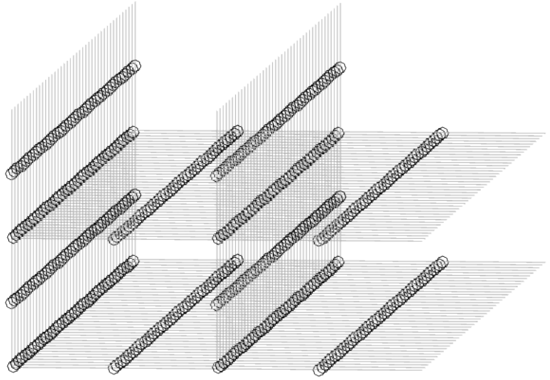
Define AFW for $\mathbb{A}_{11}$ and then take co-located unknowns associa $\mathbb{A}_{22}$ with

Notice that when applying patch algorithm across MG hierarchy we need a notion of nodes or D_0 on all levels.

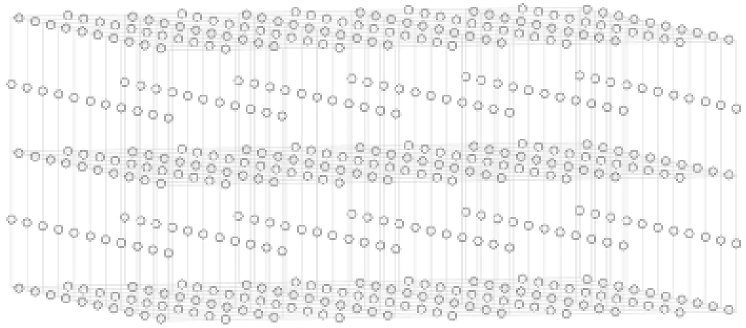
We plan to use patches in conjunction with a variant of the Rietzinger/Schoberl AMG algorithm for (curl,curl) operators as this algorithm naturally develops a notion of nodes, edges, and D_0 on all levels.

Test Problems

Shock tube *n* x *2* x *2* elements x *2* elements



Scaled Current Sheet *2n* x *n*



Edge element basis functions centered at ‘o’ locations

Dirichlet BCs in long direction

periodic BCs in other directions

$$\sigma = 10^6,\ m_e = 10^{-4}/(1+10^{-4}),$$
$$\mathrm{e} = 10^4/(1+10^{-4}),\ \mu = 1, \text{ and } \epsilon = 10^{-4}$$

$$n_e = \begin{cases} 1.0, & 0.0 \le x < 0.5, \\ 0.125, & 0.5 < x \le 1.0, \end{cases} \quad \mathbf{B} = \begin{cases} (0.75, +1.0, 0.0), & 0.0 \le x < 0.5, \\ (0.75, -1.0, 0.0), & 0.5 < x \le 1.0, \end{cases}$$

Hall term dominates

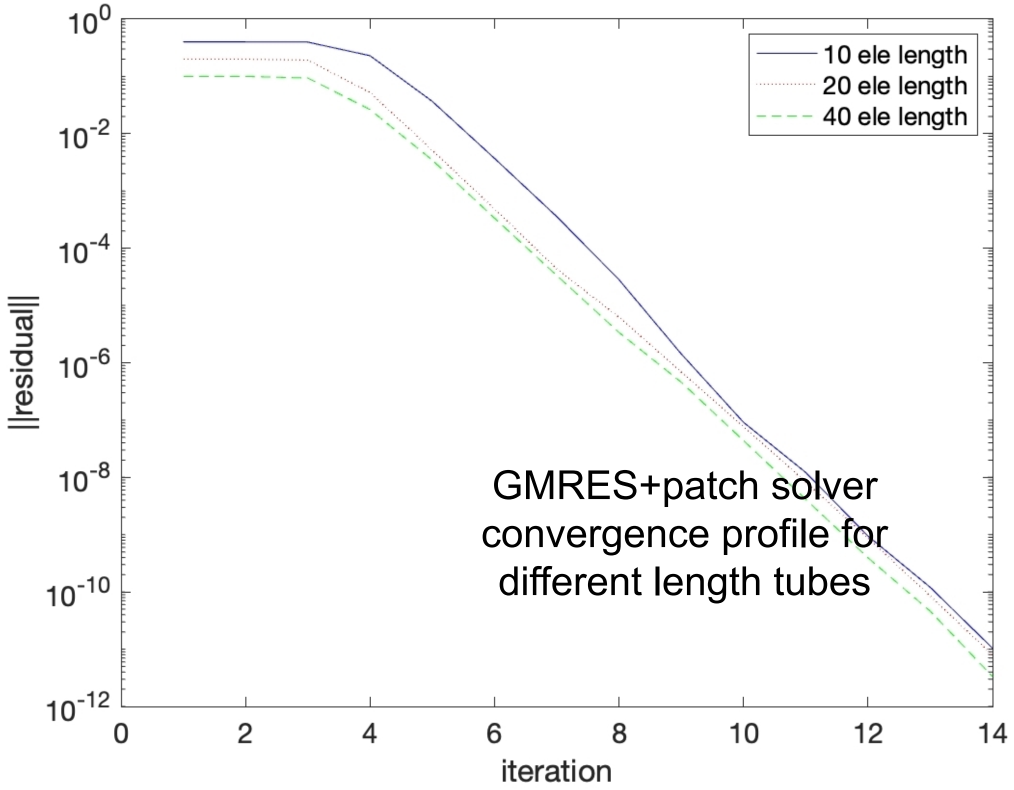
Dirichlet BCs in middle direction

$$\sigma = 10^6,\ \mathrm{e} = 1,\ \mu = 1,$$
$$\epsilon = 1/c^2\ c = \delta m_i^{-1/2},$$
$$m_e = 4.16 \times 10^{-2}/(1 + 4 \times 10^{-2}),$$
$$m_i = 1.04/(1 + 4 \times 10^{-2}),$$
$$\delta \text{ either } 10^2 \text{ or } 10^6.$$

δ

Hall term dominates for δ small

Shock tube results using geometric MG



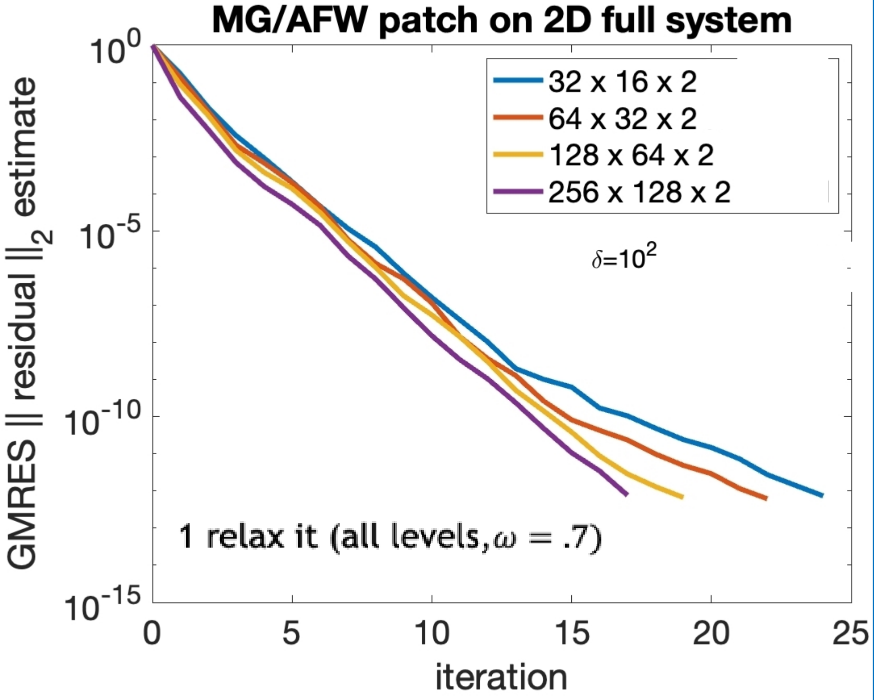
Able to solve problem where we previously struggled to converge

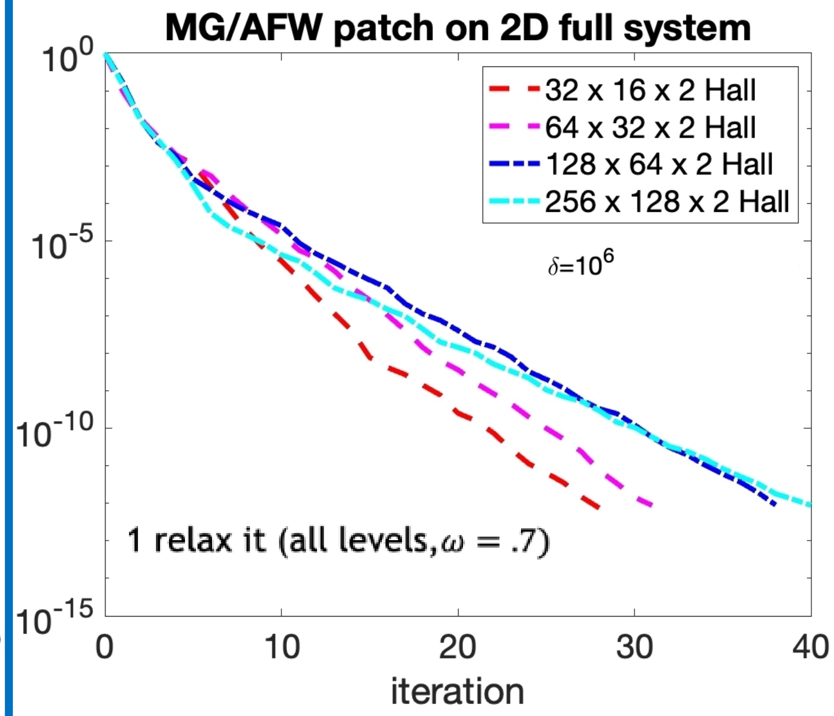
No MG needed as Hall term dominates

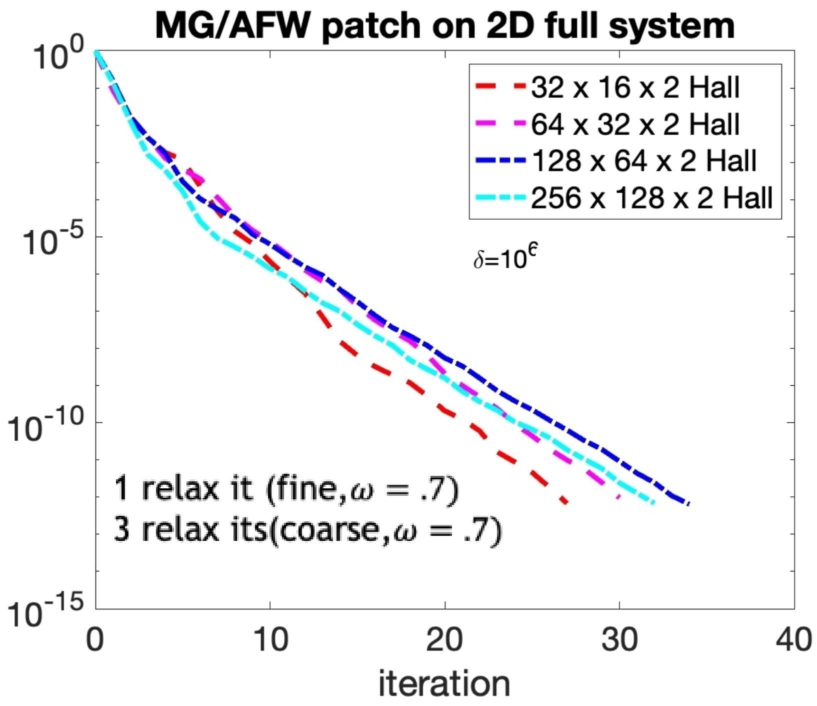
Each block of 2 x 2 system is scaled by a scalar to improve matrix conditioning. Still, cond(A) = 10^7.

ALEGRA (https://www.sandia.gov/alegra/) shock hydrodynamics and multiphysics code used to generate matrices for both test problems.

Current sheet results using geometric MG







Hall term dominates

nificant

Hall & (curl,curl) terms both

Cond(A) = 4.7e10

for 64 x 32 x 2 mesh

Cond(A) = 7.4e11

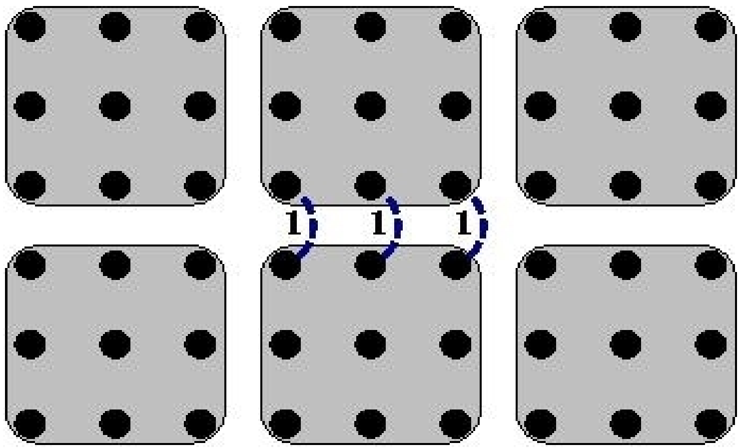
for 64 x 32 x 2 mesh

AMG strategy (just getting started)

Extend variant of Rietzinger/Schoberl AMG, as it has notion of nodes/edges on all levels

$\Rightarrow$ $\mathcal{D}_0$ can be used to define AFW patch smoother on all levels.

R/S interpolation basis function



<u>key idea</u> enforce commuting relationship so hierarchy satisfies a de Rham complex

$P_e D_{0,H} = D_{0,h} P_n$

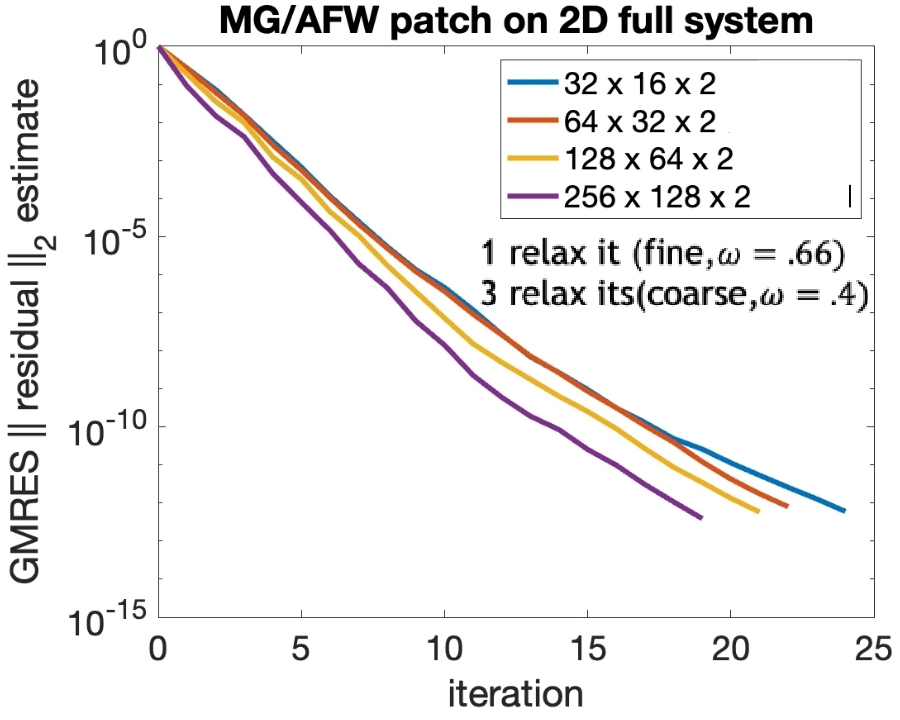
P_n: piecewise constant

can use smoothed aggregation AMG idea to improve interpolant

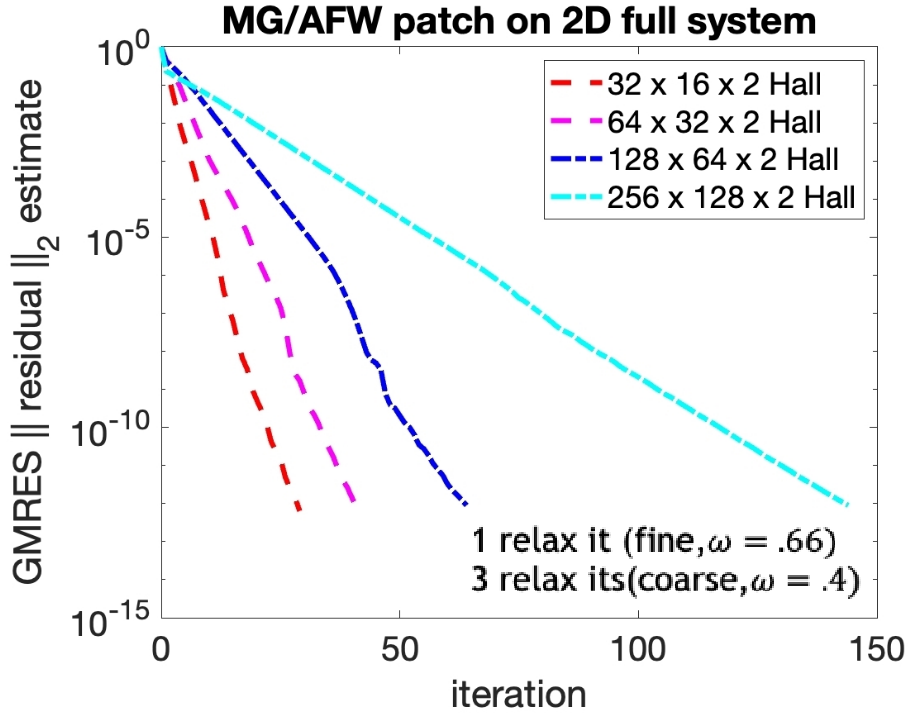
$$\mathcal{P}_e = (I - \rho D^{-1} \mathcal{S} + \beta \mathcal{D}_{0,h} (\mathcal{D}_{0,h})^T) P_e , \qquad D = diag(\mathcal{S})$$

$$\mathcal{P}_e \mathcal{D}_{0,H} = \mathcal{D}_{0,h} (I + \beta (\mathcal{D}_{0,h})^T \mathcal{D}_{0,h}) P_n = \mathcal{D}_{0,h} (I + \beta (\mathcal{D}_{0,h})^T \mathcal{D}_{0,h}) P_n = \mathcal{D}_{0,h} \mathcal{P}_n$$

1st AMG (quick) attempt



dominate Hall term



Hall & (curl,curl) both significant

Good results when Hall term dominates, but currently not scalable when (curl,curl) term significant. R/S AMG strugglin $\mathbb{A}_{11}$ with just solve. Perhaps, better to look at adapting patch scheme t $\mathbb{A}_{11}$ ux-style AMG for . , which is quite ill-conditioned for this problem

Summary

- GMHD linear solvers include Hall term that introduces a 2nd null space term (in addition to the null space of curl-curl operator).
  - Depending on regime, curl-curl term might dominate or Hall term might dominate or both.
- Questions remain as to how best to incorporate general Ohm's law and discretize.
- A scalable geometric multigrid solver can be devised by extending Arnold-Falk-Winther patch relaxation ideas.
- Rietzinger/Schoberl AMG ideas are natural with AFW patches, though other AMG extensions are possible. Still working on AMG adaptations.