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TORO II – A Finite Element Computer Program for Nonlinear Quasi-Static Problems in Electromagnetics

Part I – Theoretical Background

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TORO II - A FINITE ELEMENT COMPUTER PROGRAM FOR NONLINEAR QUASI-STATIC PROBLEMS IN ELECTROMAGNETICS

PART I - THEORETICAL BACKGROUND

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ABSTRACT

The theoretical and numerical background for the finite element computer program, TORO II, is presented in detail. TORO II is designed for the multi-dimensional analysis of nonlinear, electromagnetic field problems described by the quasi-static form of Maxwell's equations. A general description of the boundary value problems treated by the program is presented. The finite element formulation and the associated numerical methods used in TORO II are also outlined. Instructions for use of the code are documented in SAND96-0903; examples of problems analyzed with the code are also provided in the user's manual.

Acknowledgments

The significant contributions of several individuals to this code development effort must be acknowledged. The formulation and capabilities included in the code were heavily influenced by the expertise of A. J. Russo (9114). Extensive testing and benchmarking of the code, as well as generation of several example problems, was performed by P. L. Hopkins (9112). R. E. Hogan (9113) developed the pointer scheme for the sparse matrix solver used in TORO II.

Contents

1 Introduction

The simulation of electric and magnetic fields is of significant interest for a variety of engineering applications. Obviously, traditional electrical engineering design and analysis problems are primary candidates for the use of such numerical simulation capability. However, the increasingly widespread need to perform multidiscplinary and systems analyses has expanded the requirement for electromagnetic field computations beyond the strict electrical device applications to less traditional application areas.

The purpose for developing the finite element code described here, TORO, was to provide an electromagnetic (EM) field simulation capability that could be easily coupled to other types of applied mechanics codes. Though the capabilities of TORO could be exercised on standard electrical engineering problems the primary intent of the code was to provide the appropriate EM forcing functions for coupled mechanics problems involving solid, fluid and/or thermal analyses. TORO was designed to address a limited class of EM problems, including static and quasi-static fields, with a user-oriented input format that was easy to learn and remember. The code structure follows previously developed finite element codes for fluid and thermal applications [1,2,3].

The present document describes the theoretical and numerical background for the TORO II program. This volume is intended as a background document for the user's manual found in [4]. Potential users of TORO II are encouraged to become familiar with the present report before using the program.

In the following section the initial-boundary value problems treated by TORO II are described. Section 3 presents a brief description of the finite element method (FEM) and its application to the current problem. Sections 4 and 5 outline the computational techniques that are involved in forming the individual element equations and the equation solution procedures needed for the various static and dynamic problems. The last section outlines the auxiliary calculation procedures found in the code to perform flux, force and volume heating computations.

2 Formulation of the Basic Equations

TORO II was primarily developed for the solution of multi-dimensional, nonlinear, static and quasi-static, electromagnetic (EM) field problems. Within this broad definition are included problems in electrostatics, magnetostatics and time dependent diffusion problems such as eddy currents. Specifically excluded from the current formulation are EM fields involving wave propagation. The results from TORO II simulations are designed to be used directly as forces and volume heat sources in other mechanics code applications.

In the following section the equations describing the basic electromagnetic field problem will be outlined along with the limiting assumptions used in developing TORO II. A subsequent section will discuss all relevant boundary and material interface conditions for the EM problem as expressed in both fundamental (primitive) variables and in the potential variables used in the numerical formulation. The questions associated with a gauge condition for the potential variables are also outlined. The theoretical development in each section will treat the general three-dimensional problem; the two-dimensional, planar case follows in a straightforward manner. Details of the development for axisymmetric geometries are covered in an Appendix. Also, most of the equation derivation will be for the eddy current problem with the static field problems developed as subsets of the more general case.

2.1 Maxwell's Equations

The appropriate mathematical description of electromagnetic phenomena in a conducting material region, Ω_C, is given by Maxwell's equations. In rational MKSA notation these equations are expressed as [5,6,7]

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{4}$$

where the vector field variables are the electric field intensity, $\mathbf{E}$, the magnetic field intensity, $\mathbf{H}$, the magnetic flux density, $\mathbf{B}$, the electric flux (displacement) density, $\mathbf{D}$, the conduction current density, $\mathbf{J}$ and the scalar source charge density, ρ. Typically, equation (1) is referred to as Faraday's law, equation (2) is Ampere's law (as modified by Maxwell) and equation (4) is Coulomb's law. A continuity condition on the current density is also defined by

$$\nabla \cdot \mathbf{J} = \frac{\partial \rho}{\partial t} \tag{5}$$

Note that only three of the above five equations are independent; the combinations (1), (2) and (4) or (1), (2) and (5) form valid descriptions of the fields. To complete the formulation, the constitutive relations for the material are required. The fluxes are related to the field variables by

$$\mathbf{D} = \epsilon_0 \epsilon_r \cdot \mathbf{E} = \epsilon \cdot \mathbf{E} \tag{6}$$

$$\mathbf{B} = \mu_0 \mu_r \cdot \mathbf{H} = \mu \cdot \mathbf{H} \quad or \quad \mathbf{H} = \nu_0 \nu_r \cdot \mathbf{B} = \nu \cdot \mathbf{B} \tag{7}$$

$$\mathbf{J} = \sigma \cdot \mathbf{E} + \sigma \cdot (\mathbf{u} \times \mathbf{B}) \tag{8}$$

where ϵ is the permittivity of the material, μ is the permeability, ν is the reluctivity and σ is the conductivity. In equations (6) and (7) the subscripts r and 0 on the material coefficients represent relative and base (or free space) values, repsectively. Note also that in general the material properties are tensorial in nature and may be field dependent and anisotropic; some properties may also exhibit hysteris effects. In equation (8) the second term gives the current induced by the motion of a conductor with velocity $\mathbf{u}$ in the presence of a magnetic field. For many problems of interest this term is identically zero or relatively small. The above equations have been written for the case of a conducting medium; the system can be simplified for the case of free space and this will be done explicitly in a later section.

2.2 Electromagnetic Forces and Volume Heating

The coupling of electromagnetic fields with other types of problems in continuum mechanics occurs through the dependence of material properties on EM field quantities and the production of EM induced body forces and volumetric energy production. The Lorentz or body force in a charged conductor due to the presence of electric and magnetic fields is given by

$$\mathbf{F_B} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{9}$$

where in the general case the current is defined by equation (8). The first term on the right-hand-side of equation (9) is the electric field contribution to the Lorentz force; the magnetic $\mathbf{J} \times \mathbf{B}$ is usually of more interest in applied mechanics problems. The energy generation or Joule heating in a conductor is described by

$$Q_J = \mathbf{J} \cdot \mathbf{E} \tag{10}$$

which takes on a more familiar form if the simplified form ($\mathbf{u} = 0$) of (8) is used to produce

$$Q_J = \sigma^{-1} \mathbf{J}^2 \tag{11}$$

The above forces and heat source are of primary importance in fluid, thermal and structural mechanics and form part of the primary output of the TORO II code.

2.3 Quasi-static Approximation

For good conductors, the conduction current, $\mathbf{J}$, is large compared to the displacement current, $\mathbf{D}$, for most frequencies of interest. Neglecting the displacement current allows Ampere's law (2) to be simplified and Coulomb's law (4) to be omitted (Coulomb's law is essential to the electrostatic case as shown later). Also, the continuity relation is simpified since the divergence of $\nabla \times \mathbf{H}$ is zero by an identity. The omission of the displacement current is termed a quasi-static approximation since the propagation of electromagnetic waves is precluded. Under this assumption Maxwell's equations for a conducting region become

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{12}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{13}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{14}$$

The continuity condition is

$$\nabla \cdot \mathbf{J} = 0 \tag{15}$$

The required constitutive relations are again

$$\mathbf{B} = \mu \cdot \mathbf{H} \quad or \quad \mathbf{H} = \nu \cdot \mathbf{B} \tag{16}$$

$$\mathbf{J} = \sigma \cdot \mathbf{E} + \sigma \cdot (\mathbf{u} \times \mathbf{B}) \quad or \quad \mathbf{J} = \sigma \cdot \mathbf{E} \tag{17}$$

The above EM field description was developed for a conductive material region without regard for the possible presence of conductors with specified source currents or electrically nonconductive (dielectric) materials. The general problem of interest may include multiple material types and is shown schematically in Figure 1. For purposes of identification, conducting regions are labeled Ω_C, conducting regions with a specfied source current, $\mathbf{J_s}$, are indicated by Ω_J and dielectric (electrically insulated or free space) regions are labeled Ω_D. The equation set in (12)-(15) may be made specific for each of these particular regions.

Conduction Region, Ω_C

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{J} = 0$$

Conduction Region with Source Current, Ω_J

$$\nabla \times \mathbf{H} = \mathbf{J_s}$$

$$\nabla \cdot \mathbf{B} = 0$$

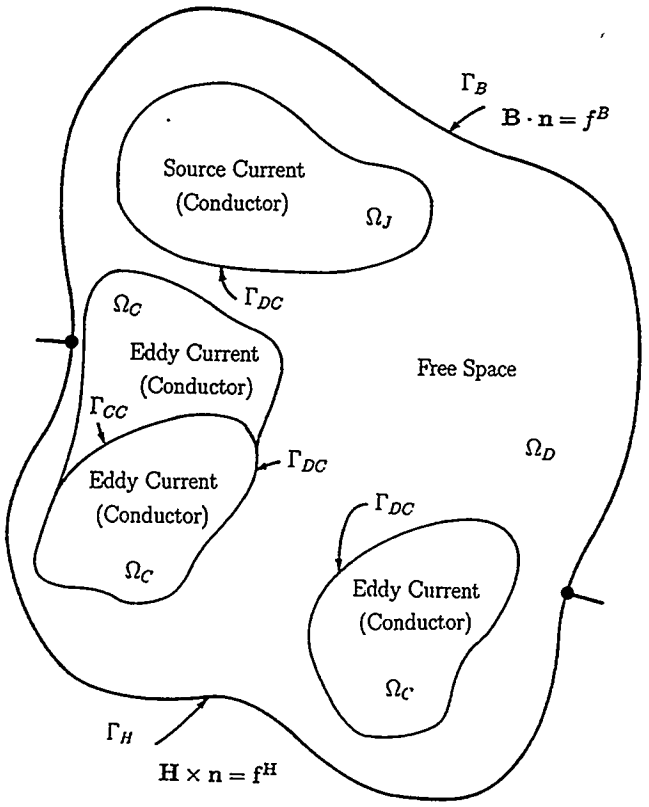

Figure 1: Schematic of regions for general electromagnetic field problem.

Dielectric/Free Space Region, Ω_D

$$\nabla \times \mathbf{H} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

Equations (16) and (17) are applicable in all three regions as stated.

It is clear from the above specification that the magnetic fields are required in all of the material regions. The electric field must be found only in conductors subject to induced (eddy) currents; the electric fields in other regions are either known, assumed or do not exist. This variation in the number of equations between regions is significant to the selection of appropriate computational variables and a computational procedure.

The general field equations and constitutive relations outlined above form the theoretical basis for the TORO II code; the force and heat source definitions in (9) and (10) are also included in this formulation. To complete the initial-boundary value specification, appropriate boundary conditions and conditions at material interfaces must be defined. Following a discussion of these conditions, the above equation set will be rewritten in terms of variables that are more suitable for computation.

2.4 Boundary and Interface Conditions

Boundary and interface conditions for the quasi-static, electromagnetic field problem are most easily described by reference to the generic domain shown in Figure 1. The region Ω is composed of a number of different materials ($\Omega = \Omega_C \cup \Omega_J \cup \Omega_D$), several of which are illustrated in Figure 1. The boundary or interface between two conductors is denoted by Γ_{CC} while the boundary between a dielectric or free space region and a conductor is labeled Γ_{DC}. Note that since the equations for Ω_J and Ω_D are the same except for a source function, no specific designation for an interface between these regions is required. The external boundary of the entire domain Ω is defined by Γ which may be composed of one or more well defined segments. A two-dimensional representation of the region is used for simplicity.

The electromagnetic problem requires that on the exterior free space boundary either the magnetic flux or the magnetic field be specified at all points of the boundary, Γ. In equation form these conditions are given by

$$\mathbf{B} \cdot \mathbf{n} = f^B(s_i, t) \qquad on \quad \Gamma_B \tag{18}$$

$$\mathbf{H} \times \mathbf{n} = \mathbf{f}^{\mathbf{H}}(s_i, t) \qquad on \quad \Gamma_H \tag{19}$$

In equations (18) and (19) the f^B and $\mathbf{f}^{\mathbf{H}}$ functions are specified values of the known boundary magnetic flux and magnetic field. Also, $\mathbf{n}$ is the outward unit normal to the boundary Γ, s_i are coordinates defined on the boundary, t is the time and $\Gamma = \Gamma_B \cup \Gamma_H$. The functions f^B and f^H are generally simple expressions for most boundaries of practical interest. When a conductor forms part of the external boundary the above conditions are augmented with a condition on the current flux or the electric field. That is

$$\mathbf{J} \cdot \mathbf{n} = f^J(s_i, t) \qquad on \quad \Gamma_J \tag{20}$$

$$\mathbf{E} \times \mathbf{n} = \mathbf{f}^{\mathbf{E}}(s_i, t) \qquad on \quad \Gamma_E \tag{21}$$

Along a material interface, such as Γ_{CC} or Γ_{DC} , the usual assumption is that the normal component of the magnetic flux is continuous and the tangential component of the magnetic field is discontinuous by an amount equal to the surface current, $\mathbf{J_s}$. These conditions are specified by

$$(\mathbf{B_2} - \mathbf{B_1}) \cdot \mathbf{n} = 0 \qquad on \quad \Gamma_{CC}, \Gamma_{DC} \tag{22}$$

$$(\mathbf{H_2} - \mathbf{H_1}) \times \mathbf{n} = \mathbf{J_s} \qquad on \quad \Gamma_{CC}, \Gamma_{DC} \tag{23}$$

where the subscripts 1 and 2 indicate variables evaluated on either side of the interface. In many cases the surface current is not important and may be neglected.

The divergence and curl relations for the electric field also provide two conditions at a material interface. In this case the normal component of the current density is continuous and the tangential components of the electric field are continuous. That is

$$(\mathbf{J_2} - \mathbf{J_1}) \cdot \mathbf{n} = 0 \qquad on \quad \Gamma_{CC} \tag{24}$$

$$(\mathbf{E_2} - \mathbf{E_1}) \times \mathbf{n} = 0 \qquad on \quad \Gamma_{CC} \tag{25}$$

for the boundary between two conductors and

$$\mathbf{J_2} \cdot \mathbf{n} = 0 \qquad on \quad \Gamma_{DC} \tag{26}$$

$$\mathbf{E_2} \times \mathbf{n} = 0 \qquad on \quad \Gamma_{DC} \tag{27}$$

for the boundary between a very good conductor and a dielectric where the subscript 2 refers to the conducting region.

2.5 Electromagnetic Potentials

For many static and quasi-static applications it is usual to introduce a set of potential functions to represent the electric and magnetic field variables and reduce the number of partial differential equations requiring solution. Two basic systems of potentials may be considered: a) the electric scalar potential, V and a magnetic vector potential, $\mathbf{A}$ and b) the electric vector potential, $\mathbf{T}$ and the scalar magnetic potential, ψ. The $\mathbf{T}, \psi$ formulation is of limited value for general analyses since there are significant difficulties in representing multiply-connected domains. Though the $\mathbf{A}, V$ formulation generally leads to a larger number of differential equations, it is preferred for numerical computation due to its complete generality. The $\mathbf{A}, V$ potentials form the basis for the TORO II code.

From the condition $\nabla \cdot \mathbf{B} = 0$, it follows that $\mathbf{B}$ is derivable from a vector potential. By definition then

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{28}$$

where $\mathbf{A}$ is the magnetic vector potential. In addition, from Faraday's law (12)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial (\nabla \times \mathbf{A})}{\partial t}$$

or rearranging

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \tag{29}$$

For a scalar V, the vector identity $\nabla \times \nabla V = 0$ holds and allows the following definition to be derived from equation (29)

$$-\nabla V = \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}$$

or

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{30}$$

where V is the electric scalar potential.The definitions in (28) and (30) may be used in the appropriate forms of Ampere's law and the current continuity equation to produce the needed field equations for $\mathbf{A}$ and V in conductors and in free space regions.

Using the constitutive relation (16b) and the definition in (28), Ampere's law in equation (13) can be written as

$$\nabla \times (\nu \cdot \nabla \times \mathbf{A}) = \mathbf{J} = \sigma \cdot \mathbf{E}$$

where Ohm's law for a stationary conductor (17b) has also been used on the right-hand-side. Substituting the definition in (30) then produces

$$\nabla \times (\nu \cdot \nabla \times \mathbf{A}) = -\sigma \cdot \frac{\partial \mathbf{A}}{\partial t} - \sigma \cdot \nabla V \tag{31}$$

In addition, the current continuity equation (15) can be written in terms of $\mathbf{A}$ and V by use of (17b) and (30). That is,

$$\nabla \cdot \left(-\sigma \cdot \nabla V - \sigma \cdot \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \tag{32}$$

Equations (31) and (32) describe the general, quasi-static electromagnetic field problem of interest in terms of the magnetic vector potential, $\mathbf{A}$ and electric scalar potential V.

The general problem outlined above may now be specialized to particular material regions and types of current specifications. For conduction regions without source currents, both equations (31) and (32) are generally required to describe both the electric and magnetic fields. Conduction regions that have specified currents require some alteration of (31)-(32) depending on the form of the current source [8]. Electric currents described by distributions of the electric potential require no alteration to the equation set since this specification would appear as a boundary condition on the variable V. However, when current densities are specified, these are identified with Ohm's law and the last term on the right-hand-side of (31) is rewritten to produce

$$\nabla \times (\nu \cdot \nabla \times \mathbf{A}) = -\sigma \cdot \frac{\partial \mathbf{A}}{\partial t} + \mathbf{J_s}. \tag{33}$$

Also, equation (32) is no longer required in the source region since the imposed current density is assumed to be divergence free. This formulation for imposed current densities is not universal. In some developments the entire right-hand-side of (31) is identified as the source current. This has the effect of eliminating any temporal variation of the $\mathbf{A}$ field from the current source. The formulation in TORO II retains the distinction between $\mathbf{J_s}$ and the induced currents $-\sigma \cdot (\partial \mathbf{A}/\partial t)$; a specified current density is assumed to mean that $\mathbf{J_s}$ is known within the conductor. The equations needed for free space or dielectric regions are simply those of (31) with σ set to zero; equation (32) is not required as no electric fields are considered. Finally, note that simplification of the equations for all regions is possible for problems of reduced dimensionality. If the geometry is two-dimensional (planar or axisymmetric) and the currents and potential gradients are oriented orthogonal to the plane, then equation (32) is not required and (31) reduces to a single equation for the remaining (axial) component of the magnetic potential.

2.6 Boundary and Interface Conditions for Potentials

The boundary conditions for the general, quasi-static EM problem expressed in terms of potentials follow directly from the boundary conditions written in terms of the primitive variables (see Section 2.4). On an external boundary where the normal component of the magnetic flux is specified, the relevant condition is

$$\mathbf{B}\cdot\mathbf{n} = (\nabla\times\mathbf{A})\cdot\mathbf{n} = f^{B}(s_i,t) \qquad on \quad \Gamma_B \tag{34}$$

An external boundary with specified tangential components of the magnetic field is transformed to

$$\mathbf{H}\times\mathbf{n} = (\nu\cdot\mathbf{B})\times\mathbf{n} = (\nu\cdot\nabla\times\mathbf{A})\times\mathbf{n} = \mathbf{f}^{\mathbf{H}}(s_i,t) \qquad on \quad \Gamma_H \tag{35}$$

For a conductor that has an external boundary

$$\mathbf{J}\cdot\mathbf{n} = \sigma\cdot\mathbf{E}\cdot\mathbf{n} = \left(-\sigma\cdot\frac{\partial\mathbf{A}}{\partial t} - \sigma\cdot\nabla V\right)\cdot\mathbf{n} = f^{J}(s_i,t) \qquad on \quad \Gamma_J \tag{36}$$

Along a material interface similar conditions must be enforced. For the magnetic field the normal component of the **B** field is continuous which implies that the curl of the magnetic potential is continuous

$$\cdot\ (\mathbf{B_2}-\mathbf{B_1})\cdot\mathbf{n} = (\nabla\times\mathbf{A_1}-\nabla\times\mathbf{A_2})\cdot\mathbf{n} = 0 \qquad on \quad \Gamma_{CC},\Gamma_{DC} \tag{37}$$

Also, the tangential components of the H field are discontinuous which leads to

$$(\mathbf{H_2}-\mathbf{H_1})\times\mathbf{n} = (\nu_1\cdot\nabla\times\mathbf{A_1}-\nu_2\cdot\nabla\times\mathbf{A_2})\times\mathbf{n} = \mathbf{J_s} \qquad on \quad \Gamma_{CC},\Gamma_{DC} \tag{38}$$

For the electrical problem, the general condition at a material interface is given by

$$(\mathbf{J_2}-\mathbf{J_1})\cdot\mathbf{n} = (\sigma_2\cdot\mathbf{E_2}-\sigma_1\cdot\mathbf{E_1})\cdot\mathbf{n} =$$

$$\left(-\sigma_2\cdot\frac{\partial\mathbf{A_2}}{\partial t} - \sigma_2\cdot\nabla V_2 + \sigma_1\cdot\frac{\partial\mathbf{A_1}}{\partial t} + \sigma_1\cdot\nabla V_1\right)\cdot\mathbf{n} = 0 \qquad on \quad \Gamma_{CC} \tag{39}$$

Equation (39) is simplified when the boundary is between a conductor and a nonconductor.

2.7 Gauge Conditions

The quasi-static form of Maxwell's equations are given by (31)-(33) in terms of the magnetic and electric potentials. The original or primitive variable form of Maxwell's equations (12)-(14) can be shown to provide unique solutions for the **B** and **E** fields when appropriate boundary conditions are specified. However, with the introduction of

the potential variables, uniqueness of the solution is not retained – equations (31) and (33) define the curl of $\mathbf{A}$, but $\mathbf{A}$ itself is only defined up to the gradient of an arbitrary scalar function. Typically this arbitrariness in $\mathbf{A}$ is resolved by defining the divergence of $\mathbf{A}$ and supplying appropriate boundary conditions for $\mathbf{A}$ (rather than boundary conditions for the curl of $\mathbf{A}$). The incorporation of a $\nabla \cdot \mathbf{A}$ constraint, termed gauging, may be accomplished in any of several ways. The Coulomb gauge is one particular choice that has found extensive use in numerical simulation methods. In this case the magnetic vector potential is made unique by the constraint

$$\nabla \cdot \mathbf{A} = 0 \tag{40}$$

Other choices for the gauge condition, such as the Lorentz gauge, select a nonhomogeneous form for (40). Note that in some cases of reduced dimensionality, the explicit use of (40) is not required since $\mathbf{A}$ is automatically divergence free.

The actual implementation of (40) in a numerical method may take any of several forms, including modification of the field equations, penalty methods, projection methods and the construction of divergence free basis functions [7,8,9,10]. All of these techniques incur a computational penalty in terms of either additional work or the modification of the equation system to a less desirable form. The finite element scheme in TORO II is based on a projection method which is outlined here. Assume that $\mathbf{A}^*$ is a solution to the unconstrained field equations given by (31)-(33). If the unique value of $\mathbf{A}$ can differ from the nonunique value by the gradient of an arbitrary scalar, then define

$$\mathbf{A} = \mathbf{A}^* + \nabla S \tag{41}$$

Substituting (41) into the gauge condition (40) provides

$$\nabla \cdot \nabla S = -\nabla \cdot \mathbf{A}^* \tag{42}$$

With appropriate boundary conditions equation (42) can be used to determine S and from (41) the unique, divergence-free potential, $\mathbf{A}$.

In some applications a unique value of the magnetic potential is not required and the above computation may be elminated. In particular, if the magnetic field is time independent, then all field quantities are related to the curl of $\mathbf{A}$ and the scalar function S is not required. However, for the time dependent case, the electric field $\mathbf{E}$ is related to the time derivative of $\mathbf{A}$ in equation (30) and the $\mathbf{A}$ field must be unique.

2.8 Time-Harmonic Problems

The general time-dependent equations given in (31)-(33) are applicable to any type of time varying field problem. However, in the often encountered, special case of a single frequency, time-harmonic excitation (*e.g.*, alternating current) the equations may be

simplified through use of a phasor representation. Let any specified current densities be represented as a time harmonic excitation and assume that the electric scalar and magnetic vector potentials have a time-harmonic form that can be represented as

$$\mathbf{J_s} = \mathbf{J_{s_0}}\, e^{i\omega t} = (\mathbf{J_s^R} + i\,\mathbf{J_s^I})\, e^{i\omega t}$$

$$V = V_0\, e^{i\omega t} = (V^R + i\,V^I)\, e^{i\omega t} \quad ; \quad \mathbf{A} = \mathbf{A_0} e^{i\omega t} = (\mathbf{A^R} + i\,\mathbf{A^I})\, e^{i\omega t} \tag{43}$$

where ω is the circular frequency ($= 2\pi f$, f is the imposed AC frequency in Hz), $i = \sqrt{-1}$ and t is the time. The superscripts R and I denote the real and imaginary components of a variable. Then substituting (43) into (31) and (32) and eliminating the common exponential factor produces

$$\nabla \times (\nu \cdot \nabla \times \mathbf{A_0}) + i\omega\sigma\mathbf{A_0} + \sigma \cdot \nabla V_0 = 0 \tag{44}$$

$$\nabla \cdot (\sigma \cdot \nabla V_0 + i\omega\sigma \cdot \mathbf{A_0}) = 0 \tag{45}$$

and for conduction regions with source currents (33) produces

$$\nabla \times (\nu \cdot \nabla \times \mathbf{A_0}) + i\omega\sigma\mathbf{A_0} = \mathbf{J_{s_0}} \tag{46}$$

These complex equations now describe the amplitudes for the potentials. Note that the boundary conditions for V and $\mathbf{A}$ must also be expressed in terms of the harmonic approximation given in (43). Implicit in the use of the phasor representation is the assumption that material properties are independent of the temporal behavior of the electromagnetic fields. This is a particularly stringent requirement that is violated when considering high field applications such as induction heating or any type of ferromagnetic material.

2.9 Static Field Problems

Within the general framework established above, a number of simpler static problems may also be defined. Each of these problem classes may have importance in the context of coupling with other mechanics problems or as standalone analyses in electromagnetics. As subclasses of the general formulation they may be solved with many of the same numerical techniques as the eddy current problem and are available as options in the TORO II code.

2.9.1 Electrostatics

The electrostatic problem is described by Coulomb's law (4) and the definition of the electric potential. Combining equation (4) with the steady form of (30) and using the constitutive relation in (6) produces

$$\nabla \cdot (\epsilon \cdot \nabla V) = -\rho \tag{47}$$

where ρ is the spatial distribution of electric charge. Boundary conditions for electrostatics generally involve specification of the potential, V, or the definition of the flux normal to the boundary (*i.e.*, the normal derivative of the potential). When the spatial variation of the potential has been determined the electric field $\mathbf{E}$ may be found from equation (30) and the current from the static form of (8). Also, the Joule heating could be derived from equation (11).

2.9.2 Steady Current Flow

For time-independent problems the system in (31)-(33) becomes decoupled and the current continuity condition (32) may be written as

$$\nabla \cdot (-\sigma \cdot \nabla V) = 0 \tag{48}$$

with

$$\mathbf{J} = -\sigma \cdot \nabla V \tag{49}$$

The equations in (48)-(49) describe steady electric currents within a conductor. Boundary conditions on the system would normally include specification of the electric potential (voltage) over part of the boundary and/or the current flux normal to the boundary, *i.e.* $\frac{\partial V}{\partial n}$. Once the current distribution has been found then the Joule heating could be recovered from the definition in equation (11).

2.9.3 Magnetostatics

Ampere's law (31) or (33) for the time-independent case becomes

$$\nabla \times (\nu \cdot \nabla \times \mathbf{A}) = -\sigma \cdot \nabla V = \mathbf{J_s} \tag{50}$$

This describes the magnetic field due to specified current distributions $\mathbf{J_s}$. Note that the conduction currents could be specified directly or computed from the steady current flow equation (48) for the electric potential. When the magnetic potential is known, then the magnetic field $\mathbf{B}$ may be computed from its definition in (28). In addition, the Lorentz forces can be found from equation (9).

3 Finite Element Equations

The spatial discretization of the boundary value problem outlined in Section 2 by use of the finite element method may be approached by either of two procedures. One popular approach consists of rewriting the boundary value problem in a variational form for use with the finite element approximation. An equivalent method uses the Galerkin form of the method of weighted residuals to create an integral form of the basic field equations. This latter method is employed here. In Section 3.1 the general time-dependent case is considered with the time-harmonic problem outlined in Section 3.2. The gauge condition and the various static equations will be treated in subsequent sections.

3.1 EM Field Equations - Time Dependent

To simplify the ensuing derivation only the EM equations for a conducting region will be considered initially. The free space equations will be considered separately in the following section. The partial differential equations of concern are listed in (31)-(33).

3.1.1 Conduction Region, Ω_C

Let the region of interest, Ω_C, be divided into a number of simply shaped regions called finite elements, as shown in Figure 2. Within each element, a set of nodal points are established at which the dependent variables (*i.e.*, $\mathbf{A}, V$) are evaluated. The variations of the magnetic and electric potential within each element are represented by continuous interpolation functions associated with each node. For purposes of developing the equations for the nodal point coefficients, an individual element may be separated from the assembled system. The global equation set is reassembled by use of the appropriate continuity conditions between nodal point variables in adjoining elements.

Within each element, the components of the magnetic potential and the electric potentials are approximated by expansions of the following forms

$$A_i(x_i, t) = \Phi^T(x_i)\mathbf{A_i}(t) \tag{51}$$

$$V(x_i, t) = \Psi^T(x_i)\mathbf{V}(t) \tag{52}$$

where the $\mathbf{A_i}$ and $\mathbf{V}$ are vectors of nodal point unknowns, Φ and Ψ are vectors of interpolation (basis) functions and superscript T denotes a vector transpose. For the general three-dimensional case the subscript i runs from 1 to 3.

Substitution of the definitions in (51) and (52) into the EM field equations in (31) and (32) yields a set of residual equations, due to the approximate nature of equations (51) and (52). In functional form the components of Ampere's law are

$$f_{A_1}(\Phi, \Psi, \mathbf{A}, \mathbf{V}) = R_{A_1}$$

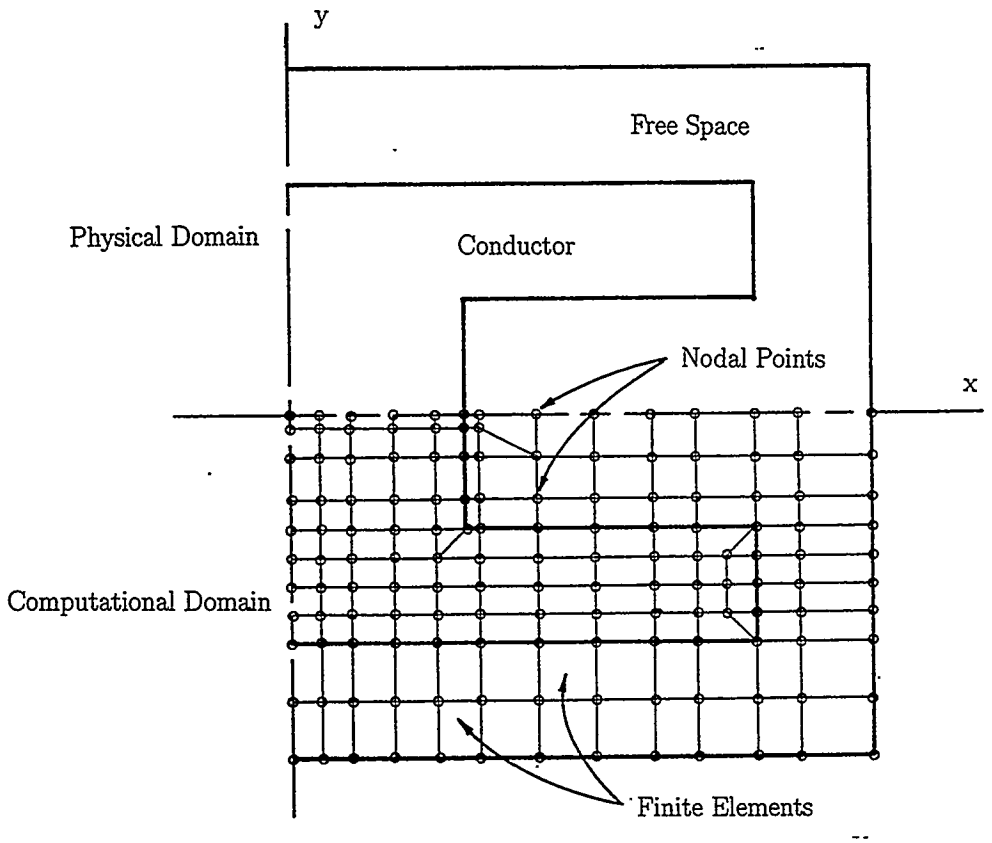

Figure 2: Finite element discretization of a region.

$$f_{A_2}(\Phi, \Psi, \mathbf{A}, \mathbf{V}) = R_{A_2} \tag{53}$$
$$f_{A_3}(\Phi, \Psi, \mathbf{A}, \mathbf{V}) = R_{A_3}$$

and the current continuity relation is

$$f_V(\Phi, \Psi, \mathbf{A}, \mathbf{V}) = R_V \tag{54}$$

where the R's denote the residuals (errors) for each equation.

The Galerkin method guarantees the orthogonality of the residual vectors to the space spanned by the interpolation functions. This orthogonality is expressed for each equation by the inner product,

$$\begin{aligned} \langle \Phi, f_{A_1} \rangle &= \langle \Phi, R_{A_1} \rangle = 0 \\ \langle \Phi, f_{A_2} \rangle &= \langle \Phi, R_{A_2} \rangle = 0 \\ \langle \Phi, f_{A_3} \rangle &= \langle \Phi, R_{A_3} \rangle = 0 \\ \langle \Psi, f_V \rangle &= \langle \Psi, R_V \rangle = 0 \end{aligned} \tag{55}$$

where $\langle a, b \rangle$ denotes the inner product defined by

$$\langle a, b \rangle = \int_{\Omega_e} a \cdot b \, d\Omega \tag{56}$$

where Ω_e is the volume of the element.

Carrying out the above operations explicitly for the electromagnetic equations yields the following systems of matrix equations,

$$\mathbf{M}\dot{\mathbf{A}} + \mathbf{K}\mathbf{A} + \mathbf{N}\mathbf{V} = \mathbf{F_A} \tag{57}$$

and

$$\mathbf{N}^T\dot{\mathbf{A}} + \mathbf{L}\mathbf{V} = \mathbf{F_V} \tag{58}$$

where the superposed dot indicates a time derivative and equation (57) represents the three component equations for the magnetic potential. Details of the derivation of the above equation set are given in Appendix A. The matrix equations in (57) and (58) may be combined into a matrix equation of the following form

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{N}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{A}} \\ \dot{\mathbf{V}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{N} \\ \mathbf{0} & \mathbf{L} \end{bmatrix} \begin{Bmatrix} \mathbf{A} \\ \mathbf{V} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F_A} \\ \mathbf{F_V} \end{Bmatrix} \tag{59}$$

Unfortunately, the above system is unsymmetric and is of an unusual form from the standpoint of time integration. To restore symmetry, the following definition is employed

$$\mathbf{V} \equiv \frac{\partial \mathbf{v}}{\partial t} = \dot{\mathbf{v}} \tag{60}$$

Using (60) equations (57)-(58) can be rewritten as

$$\mathbf{M}\dot{\mathbf{A}} + \mathbf{K}\mathbf{A} + \mathbf{N}\dot{\mathbf{v}} = \mathbf{F_A} \tag{61}$$

and

$$\mathbf{N}^T\dot{\mathbf{A}} + \mathbf{L}\dot{\mathbf{v}} = \mathbf{F_v} \tag{62}$$

and the assembled form in (59) becomes

$$\begin{bmatrix} \mathbf{M} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{L} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{A}} \\ \dot{\mathbf{v}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{A} \\ \mathbf{v} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F_A} \\ \mathbf{F_V} \end{Bmatrix} \tag{63}$$

A general symbolic form for the system in (63) can be written as

$$\overline{\mathbf{M}}\dot{\mathbf{U}} + \overline{\mathbf{K}}\mathbf{U} = \overline{\mathbf{F}} \tag{64}$$

where

$$\mathbf{U}^T = \{\mathbf{A}_1^\mathbf{T}, \mathbf{A}_2^\mathbf{T}, \mathbf{A}_3^\mathbf{T}, \mathbf{v}^\mathbf{T}\}$$

The matrix equations given in (61)-(64) represent the discrete analog of the field equations for an individual element. The discrete representation of the entire conducting region of interest is obtained through an assemblage of elements such that interelement continuity of the approximate magnetic and electrical potentials are enforced. This continuity requirement is met through the summation of equations for nodes common to adjacent elements, the so-called direct stiffness method [11,12]. The result of such an assembly process is a system of equations that is of the same form as given in (61)-(64).

3.1.2 Dielectric and Conduction Regions with Source Currents, Ω_D, Ω_J

The above derivation was carried out specifically for conducting (eddy current) regions where both the magnetic potential and the electric potential were required. However, the other regions of interest, such as specified current regions or dielectric regions, are subsets of the above formulation and the required equations may be obtained directly from (61)-(64). As noted previously, for the case of a conductor with a current specified through a potential, the general equation set is the same as shown in (61)-(64). For the case of a conductor with a specified current density, equation (62) need not be considered and the relevant matrix equation is a modified form of (61). That is

$$\mathbf{M}\dot{\mathbf{A}} + \mathbf{K}\mathbf{A} = \mathbf{F_A} + \mathbf{F_J} \tag{65}$$

where the specified current is contained in the force vector $\mathbf{F_J}$. Similarly, the equations for a dielectric region are as given by (65) but without the time derivative term and the source current. That is

$$\mathbf{K}\mathbf{A} = \mathbf{F_A} \tag{66}$$

Note that when specified current and dielectric regions occur in a general eddy current problem, equations (65) and (66) do not contribute to the electric potential equations; the proper interface and boundary conditions for the electric potential in a conductor allow this variable to be eliminated in the nonconducting regions. Details of the development of (65)-(66) are presented in Appendix A.

3.2 EM Field Equations - Time-Harmonic

The time-dependent EM equations in discretized form may be specialized for the case of time-harmonic fields. As was done in the continuum case in equations (43), let the nodal point dependent variables be represented as

$$\mathbf{V} = \mathbf{V_0} e^{i\omega t} \quad ; \quad \dot{\mathbf{v}} = \mathbf{V} = \mathbf{V_0} e^{i\omega t} = \mathbf{v_o} i\omega e^{i\omega t}$$

$$\mathbf{A} = \mathbf{A_0} e^{i\omega t}$$

Substituting these definitions into (61)-(64) leads to

$$i\omega\mathbf{M}\mathbf{A_0} + \mathbf{K}\mathbf{A_0} + i\omega\mathbf{N}\mathbf{v_0} = \mathbf{F_{A_0}} \tag{67}$$

and

$$i\omega\mathbf{N}^T\mathbf{A_0} + i\omega\mathbf{L}\mathbf{v_0} = \mathbf{F_{v_0}} \tag{68}$$

Corresponding to the matrix form in (63), equations (67) and (68) may be written as

$$\begin{bmatrix} i\omega\mathbf{M} & i\omega\mathbf{N} \\ i\omega\mathbf{N}^T & i\omega\mathbf{L} \end{bmatrix} \begin{Bmatrix} \mathbf{A_0} \\ \mathbf{v_0} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{A_0} \\ \mathbf{v_0} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F_A} \\ \mathbf{F_v} \end{Bmatrix} \tag{69}$$

Equation (69) is a complex matrix equation for the complex potentials V_0 and $\mathbf{A_0}$ and could be solved in this form using complex arithmetic. Since the matrix solvers utilized in TORO II only use real arithmetic, the system in (69) is split into two matrix problems by using the definitions $V_0 = (V^R + iV^I)$ and $\mathbf{A_0} = (\mathbf{A^R} + i\,\mathbf{A^I})$. When substituted into (69) and separated into real and imaginary equations the system becomes

$$\begin{bmatrix} -\omega\mathbf{M} & -\omega\mathbf{N} \\ -\omega\mathbf{N}^T & -\omega\mathbf{L} \end{bmatrix} \begin{Bmatrix} \mathbf{A^I} \\ \mathbf{v^I} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{A^R} \\ \mathbf{v^R} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F_A^R} \\ \mathbf{F_v^I} \end{Bmatrix} \tag{70}$$

for the real equation and

$$\begin{bmatrix} \omega\mathbf{M} & \omega\mathbf{N} \\ \omega\mathbf{N}^T & \omega\mathbf{L} \end{bmatrix} \begin{Bmatrix} \mathbf{A^R} \\ \mathbf{v^R} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{A^I} \\ \mathbf{v^I} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F_A^I} \\ \mathbf{F_v^I} \end{Bmatrix} \tag{71}$$

for the imaginary equation. These may be combined into a single matrix equation for the real and imaginary parts of each unknown potential

$$\begin{bmatrix} \mathbf{K} & \mathbf{0} & -\omega\mathbf{M} & -\omega\mathbf{N} \\ \mathbf{0} & \mathbf{0} & -\omega\mathbf{N}^T & -\omega\mathbf{L} \\ \omega\mathbf{M} & \omega\mathbf{N} & \mathbf{K} & \mathbf{0} \\ \omega\mathbf{N}^T & \omega\mathbf{L} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{A^R} \\ \mathbf{v^R} \\ \mathbf{A^I} \\ \mathbf{v^I} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F_A^R} \\ \mathbf{F_v^R} \\ \mathbf{F_A^I} \\ \mathbf{F_v^I} \end{Bmatrix} \tag{72}$$

Note that for a general three-dimensional problem equation (72) has eight unknowns per node and the system is unsymmetric. As described previously, for dielectric regions or conductors with specified current densities, equation (72) is simplified by the omission of the electric potential unknowns. Known complex current densities appear in the definition of the force vectors $\mathbf{F_A}$.

3.3 Gauge Condition

In circumstances where the gauge condition on the magnetic potential is required, the elliptic equation (42) for the scalar variable S must be transformed into a computational form. As in the previous section the Galerkin form of the method of weighted residuals may be used. The scalar correction variable S is represented by a finite element expansion of the form

$$S(x_i, t) = \Theta^T(x_i)\mathbf{S}(t) \tag{73}$$

Substitution of (73) and (57) into weighted integral form of the gauge condition in (42) leads to the matrix system given by

$$\mathbf{K_s S} = \mathbf{F_S} + \mathbf{F_{A^*}} \tag{74}$$

where the vector $\mathbf{F_S}$ contains any boundary conditions on $\mathbf{S}$ and the vector $\mathbf{F_{A^*}}$ contains the divergence of the unconstrained magnetic potential. Details of the derivation of (74) are given in Appendix A. Note that $\mathbf{K_s}$ is altered when the problem is time-harmonic.

3.4 Static Field Equations

The derivation of the finite element equations for the static field problems defined in Section 2.9 follows the same procedure used for the general eddy current formulation. The individual cases are considered explicitly in Appendix A. The relevant matrix equations are cataloged here for reference. Due to the complete similarity between the following static cases and the general problem, all of the numerical techniques and solution methods used for the eddy current problem will be applicable to the simplified cases. In the remainder of this text no further explicit mention will be made regarding the static field equations though they are all included in the TORO II code.

3.4.1 Electrostatics

The matrix equation for electrostatic applications is a scalar system for the electric potential given by

$$\mathbf{K}_\epsilon \mathbf{V} = \mathbf{F}_\rho + \mathbf{F}_\mathbf{V} \tag{75}$$

where $\mathbf{F}_\rho$ contains the volume charge density and $\mathbf{F}_\mathbf{V}$ incorporates the boundary conditions on the electric potential, $\mathbf{V}$. Compared to the current flow matrix $\mathbf{L}$ in (58) the electrostatic problem produces a slightly modified finite element matrix due to the presence of a different material property, the permittivity.

3.4.2 Steady Current Flow

The steady current flow problem is a small subset of the eddy current problem and the finite element equation follows directly from that formulation. That is

$$\mathbf{L}\mathbf{V} = \mathbf{F}_\mathbf{V} \tag{76}$$

where boundary conditions on the electric potential are contained in $\mathbf{F}_\mathbf{V}$.

3.4.3 Magnetostatics

The magnetostatic problem is also a subset of the general eddy current formulation and has two forms depending on the specification of the electric current field. If the electric field is defined using the electric potential then a steady current flow problem is solved first with the relevant equation being (76)

$$\mathbf{L}\mathbf{V} = \mathbf{F}_\mathbf{V}$$

Following this solution, electric currents are computed from the gradient of $\mathbf{V}$ and used to define the force vector $\mathbf{F_J}$ in a reduced form of (65)

$$\mathbf{KA} = \mathbf{F_A} + \mathbf{F_J} \tag{77}$$

for the magnetic vector potential. If the current densities are given initially then the electric potential equation is neglected and (77) is solved directly. Note that a gauge condition is not required for this type of problem.

4 Elements and Element Matrix Construction

The formulation of the equations for an individual element, as indicated by equations (A24) and (A34) from Appendix A, requires the specification of the shape function vectors for the approximation of the magnetic potential, electric potential and the scalar correction. The form of the shape functions depend on the particular element being used; TORO II employs two basic elements for two-dimensional analyses and three element types in the three-dimensional case. The interpolation functions for each of these elements are described below. For each element type both linear and quadratic interpolation is available; the higher-order functions are generally of the "serendipity" type [12,13] and avoid the use of nodes located in the interior of the element. Other element types, such as the higher-order Lagrange elements, could be added to TORO II with minor code modifications.

4.1 Triangular Elements (2D)

The triangular elements used in the two-dimensional formulations of TORO II consist of a straight-sided, three-node element and a six-node element as shown in Figure 3. The linear interpolation function for the three-node element is given by

$$\mathbf{\Phi_l} = \mathbf{\Psi_l} = \mathbf{\Theta_l} = \left\{ \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array} \right\} \tag{78}$$

and the corresponding quadratic function for the six-node element is

$$\mathbf{\Phi_q} = \mathbf{\Psi_q} = \mathbf{\Theta_q} = \left\{ \begin{array}{c} L_1(2L_1 - 1) \\ L_2(2L_2 - 1) \\ L_3(2L_3 - 1) \\ 4L_1L_2 \\ 4L_2L_3 \\ 4L_3L_1 \end{array} \right\} \tag{79}$$

The ordering of the functions in (78) and (79) corresponds to the ordering of the nodes shown in Figure 3. The shape functions are expressed in terms of the area or natural coordinates, L_i, for a triangle [11,12,13] which range from 0 to 1, and are related by the auxiliary condition $L_1 + L_2 + L_3 = 1$ (*i.e.*, there are only two independent area coordinates).

When the element interpolation functions are written in terms of the area coordinates, the relationship between the physical coordinates x, y (or r, z in the axisymmetric case) and the element coordinates is obtained from the parametric mapping concept originally developed by Ergatoudis, *et al* [14]. That is, the coordinate transformation is given by

$$x = \mathbf{\Upsilon}^T\mathbf{x} \quad ; \quad y = \mathbf{\Upsilon}^T\mathbf{y} \tag{80}$$

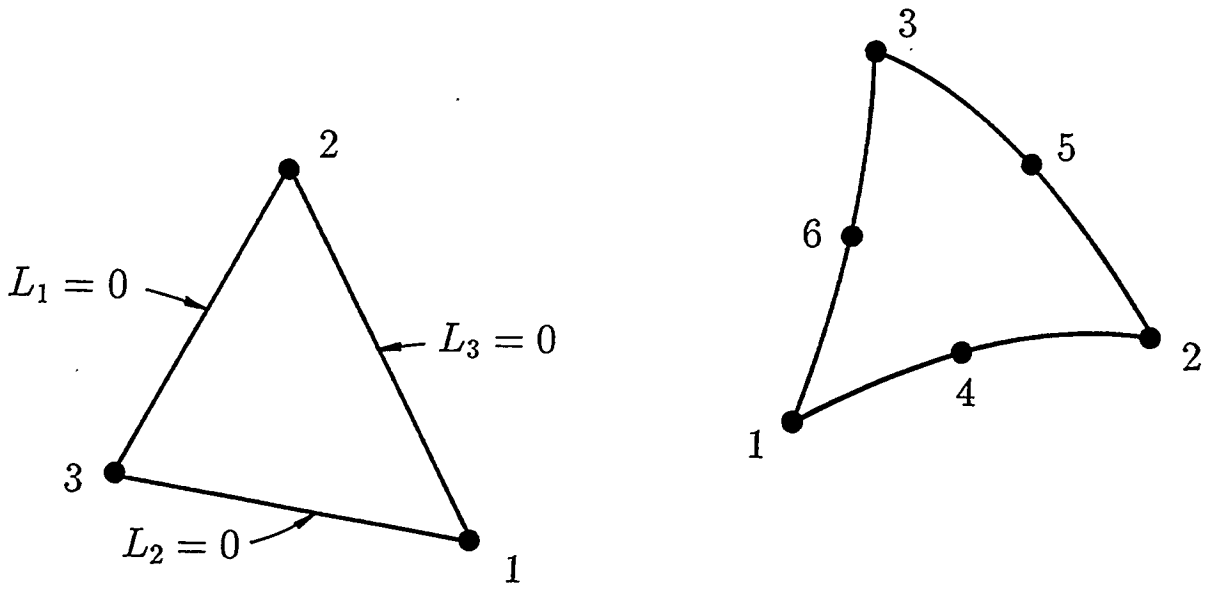

Figure 3: Two-dimensional triangular elements.

where Υ is a vector of interpolation functions on the triangle and the $\mathbf{x}, \mathbf{y}$ are vectors of coordinates describing the geometry of the element (generally, nodal point coordinates). The transformation given in (80) is quite general and allows for the description of curved-sided elements. In the present case, if $\Upsilon = \Phi_\mathbf{l} = \Psi_\mathbf{l} = \Theta_\mathbf{l}$, a linear interpolation of the element boundary is possible. When $\Upsilon = \Phi_\mathbf{q} = \Psi_\mathbf{q} = \Theta_\mathbf{q}$, a quadratic interpolation of the element geometry is allowed. Note that when the functions defining the element geometry are the same as those defining the dependent variable the element is termed isoparametric; a geometric description which is lower order than the dependent variable is defined as subparametric. TORO II directly supports only isoparametric elements; straight-sided, higher order elements can be utilized by appropriately locating mid-edge nodes.

4.2 Quadrilateral Elements (2D)

Two types of quadrilateral elements are used TORO II – a four-node and an eight-node element. For the linear, four node element the interpolation functions are given by

$$\Phi_\mathbf{l} = \Psi_\mathbf{l} = \Theta_\mathbf{l} = \left\{ \begin{array}{l} 1/4(1-s)(1-t) \\ 1/4(1+s)(1-t) \\ 1/4(1+s)(1+t) \\ 1/4(1-s)(1+t) \end{array} \right\} \tag{81}$$

The ordering of the functions in (81) corresponds to the nodal point ordering shown in Figure 4. The interpolation functions are written in terms of the normalized or natural coordinates for the element, s, t, which vary from -1 to $+1$ as shown in the figure.

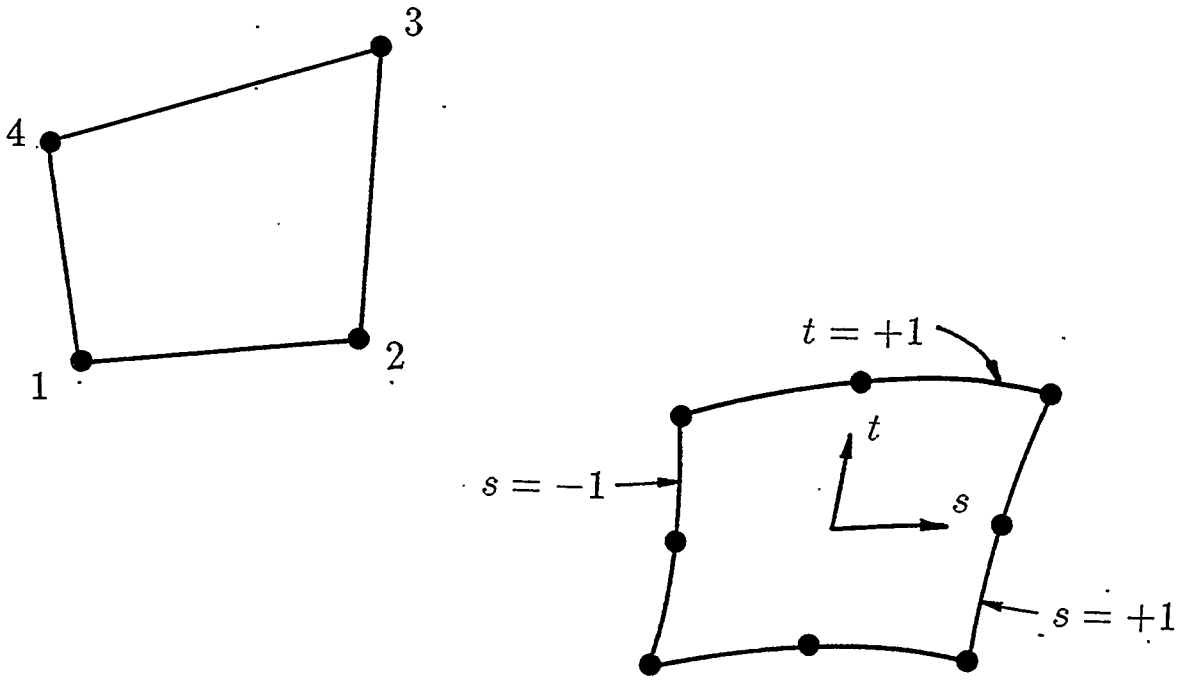

Figure 4: Two-dimensional quadrilateral elements.

The eight-node element uses the biquadratic, "serendipity" functions [13] given by

$$\Phi_{\mathbf{q}} = \Psi_{\mathbf{q}} = \Theta_{\mathbf{q}} = \left\{ \begin{array}{c} 1/4(1-s)(1-t)(-s-t-1) \\ 1/4(1+s)(1-t)(s-t-1) \\ 1/4(1+s)(1+t)(s+t-1) \\ 1/4(1-s)(1+t)(-s+t-1) \\ 1/2(1-s^2)(1-t) \\ 1/2(1+s)(1-t^2) \\ 1/2(1-s^2)(1+t) \\ 1/2(1-s)(1-t^2) \end{array} \right\} \tag{82}$$

The parametric mapping concept described for the triangular element is also available for use with the quadrilaterals. Therefore, to relate the global coordinates x, y (or r, z) to the local s, t system, let

$$x = \Upsilon^T \mathbf{x} \quad ; \quad y = \Upsilon^T \mathbf{y} \tag{83}$$

where Υ may be either a linear or quadratic ("serendipity") interpolation function. Again, TORO II only supports the formulation of isoparametric quadrilaterals, though subparametric elements can be emloyed through the proper location of mid-edge nodes.

4.3 Hexahedral Elements (3D)

The hexahedral or brick elements available in TORO II for three-dimensional analyses consist of a straight-edged, linear, eight-node element and a curved-sided, quadratic,

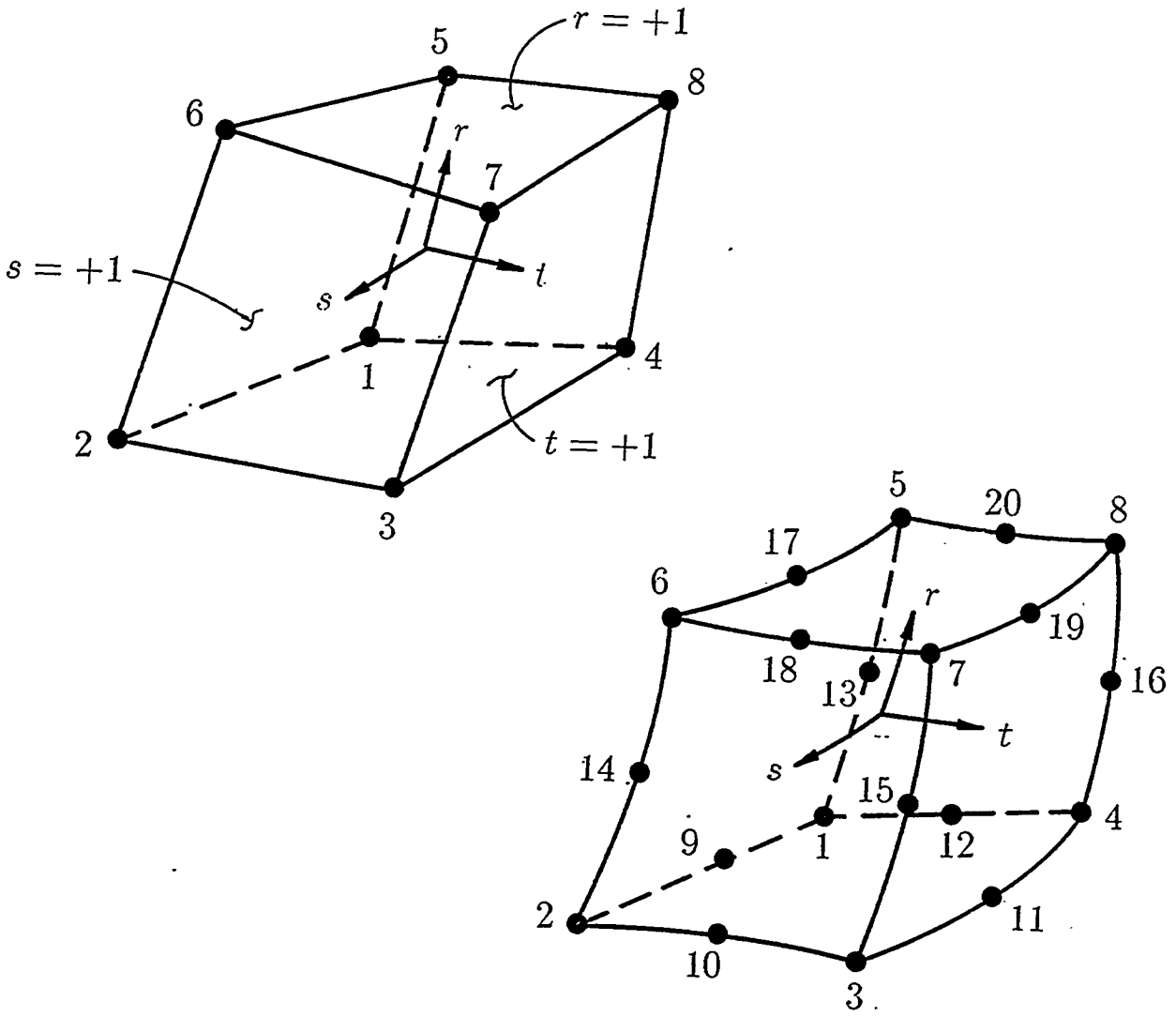

Figure 5: Three-dimensional brick elements.

twenty-node element as shown in Figure 5. The linear element has shape functions given by

$$\Phi_1 = \Psi_1 = \Theta_1 = \left\{ \begin{array}{l} 1/8(1-s)(1-t)(1-r) \\ 1/8(1+s)(1-t)(1-r) \\ 1/8(1+s)(1+t)(1-r) \\ 1/8(1-s)(1+t)(1-r) \\ 1/8(1-s)(1-t)(1+r) \\ 1/8(1+s)(1-t)(1+r) \\ 1/8(1+s)(1+t)(1+r) \\ 1/8(1-s)(1+t)(1+r) \end{array} \right\} \tag{84}$$

The quadratic shape functions for the twenty-node element are given by

$$\Phi_{\mathbf{q}} = \Psi_{\mathbf{q}} = \Theta_{\mathbf{q}} = \left\{ \begin{array}{c} 1/8(1-s)(1-t)(1-r)(-s-t-r-2) \\ 1/8(1+s)(1-t)(1-r)(s-t-r-2) \\ 1/8(1+s)(1+t)(1-r)(s+t-r-2) \\ 1/8(1-s)(1+t)(1-r)(-s+t-r-2) \\ 1/8(1-s)(1-t)(1+r)(-s-t+r-2) \\ 1/8(1+s)(1-t)(1+r)(s-t+r-2) \\ 1/8(1+s)(1+t)(1+r)(s+t+r-2) \\ 1/8(1-s)(1+t)(1+r)(-s+t+r-2) \\ 1/4(1-s^2)(1-t)(1-r) \\ 1/4(1+s)(1-t^2)(1-r) \\ 1/4(1-s^2)(1+t)(1-r) \\ 1/4(1-s)(1-t^2)(1-r) \\ 1/4(1-s)(1-t)(1-r^2) \\ 1/4(1+s)(1-t)(1-r^2) \\ 1/4(1+s)(1+t)(1-r^2) \\ 1/4(1-s)(1+t)(1-r^2) \\ 1/4(1-s^2)(1-t)(1+r) \\ 1/4(1+s)(1-t^2)(1+r) \\ 1/4(1-s^2)(1+t)(1+r) \\ 1/4(1-s)(1-t^2)(1+r) \end{array} \right\} \tag{85}$$

The functions in (84) and (85) are ordered according to the nodal point ordering shown in Figure 5 and are written in terms of the local, normalized coordinates, s, t, r, which range from -1 to $+1$. TORO II allows only the isoparametric form of each three-dimensional element where

$$x = \Upsilon^T \mathbf{x} \quad ; \quad y = \Upsilon^T \mathbf{y} \quad ; \quad z = \Upsilon^T \mathbf{z} \tag{86}$$

and Υ takes on the appropriate linear or quadratic form.

4.4 Prism Elements (3D)

TORO II employs a linear and quadratic version of a triangular prism or wedge element. The linear, straight-sided, six-node prism and curved-sided, fifteen-node quadratic element are shown in Figure 6. The shape functions for these elements are given by

$$\Phi_{\mathbf{l}} = \Psi_{\mathbf{l}} = \Theta_{\mathbf{l}} = \left\{ \begin{array}{c} 1/2L_1(1-r) \\ 1/2L_2(1-r) \\ 1/2L_3(1-r) \\ 1/2L_1(1+r) \\ 1/2L_2(1+r) \\ 1/2L_3(1+r) \end{array} \right\} \tag{87}$$

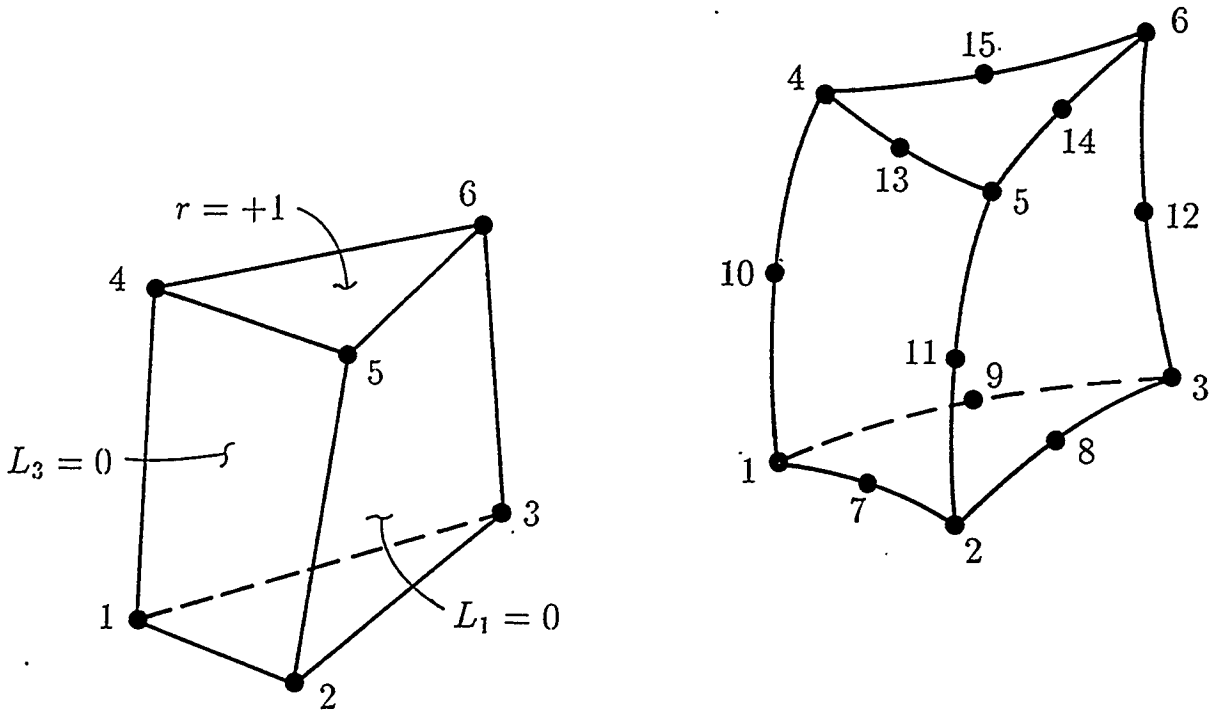

Figure 6: Three-dimensional prism elements.

and

$$\Phi_{\mathbf{q}} = \Psi_{\mathbf{q}} = \Theta_{\mathbf{q}} = \left\{ \begin{array}{c} 1/2L_1[(2L_1-1)(1-r)-(1-r^2)] \\ 1/2L_2[(2L_2-1)(1-r)-(1-r^2)] \\ 1/2L_3[(2L_3-1)(1-r)-(1-r^2)] \\ 1/2L_1[(2L_1-1)(1+r)-(1-r^2)] \\ 1/2L_2[(2L_2-1)(1+r)-(1-r^2)] \\ 1/2L_3[(2L_3-1)(1+r)-(1-r^2)] \\ 2L_1L_2(1-r) \\ 2L_2L_3(1-r) \\ 2L_3L_1(1-r) \\ L_1(1-r^2) \\ L_2(1-r^2) \\ L_3(1-r^2) \\ 2L_1L_2(1+r) \\ 2L_2L_3(1+r) \\ 2L_3L_1(1+r) \end{array} \right\} \tag{88}$$

The functions in (87) and (88) use area coordinates, L_i, for describing the triangular cross-section and a normalized coordinate, r, for the axial coordinate. Note that $L_1+L_2+L_3 = 1$; the L_i vary from 0 to 1 and r varies from -1 to $+1$. Only the isoparametric forms of this element are employed in TORO II.

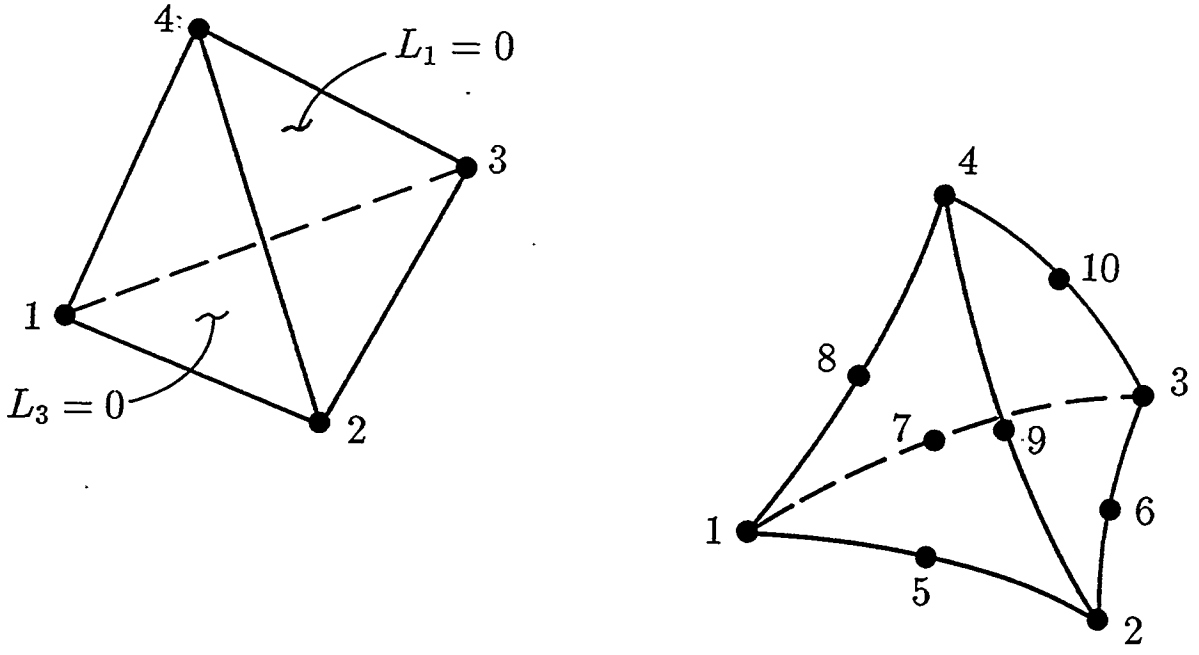

Figure 7: Three-dimensional tetrahedral elements.

4.5 Tetrahedral Element (3D)

The three-dimensional tetrahedron used in TORO II may be either a four-node or ten-node isoparametric element as shown in Figure 7. The linear element is defined by the functions

$$\mathbf{\Phi_l} = \mathbf{\Psi_l} = \mathbf{\Theta_l} = \left\{ \begin{array}{c} L_1 \\ L_2 \\ L_3 \\ L_4 \end{array} \right\} \tag{89}$$

while the quadratic element has shape functions of the form

$$\mathbf{\Phi_q} = \mathbf{\Psi_q} = \mathbf{\Theta_q} = \left\{ \begin{array}{c} L_1(2L_1 - 1) \\ L_2(2L_2 - 1) \\ L_3(2L_3 - 1) \\ L_4(2L_4 - 1) \\ 4L_1L_2 \\ 4L_2L_3 \\ 4L_3L_1 \\ 4L_1L_4 \\ 4L_2L_4 \\ 4L_3L_4 \end{array} \right\} \tag{90}$$

The functions in (89) and (90) are ordered as shown in the figure and are written in terms of the volume coordinates [13] for the element, where $L_1 + L_2 + L_3 + L_4 = 1$. Again, only the isoparametric forms of this element are considered.

4.6 Spatial Derivatives and Integrals

The construction of the various finite element coefficient matrices in (A24) requires the integration of combinations of the interpolation functions and their spatial derivatives over the volume (area) of the element. The integration process is most easily carried out in the normalized or natural coordinate system for each element since the limits of integration are simple and independent of the global coordinates. The shape functions presented in the previous sections were expressed in the natural coordinate system for each element. There remains the task of expressing spatial derivatives of the shape functions in the terms of same normalized coordinates. The following relations, based on the chain rule and the parametric mapping ideas, are needed

$$\left\{ \begin{array}{c} \frac{\partial \Lambda}{\partial s} \\ \frac{\partial \Lambda}{\partial t} \\ \frac{\partial \Lambda}{\partial r} \end{array} \right\} = \left[\begin{array}{ccc} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \end{array} \right] \left\{ \begin{array}{c} \frac{\partial \Lambda}{\partial x} \\ \frac{\partial \Lambda}{\partial y} \\ \frac{\partial \Lambda}{\partial z} \end{array} \right\} = \left[\begin{array}{ccc} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{array} \right] \left\{ \begin{array}{c} \frac{\partial \Lambda}{\partial x} \\ \frac{\partial \Lambda}{\partial y} \\ \frac{\partial \Lambda}{\partial z} \end{array} \right\} = \mathbf{J} \left\{ \begin{array}{c} \frac{\partial \Lambda}{\partial x} \\ \frac{\partial \Lambda}{\partial y} \\ \frac{\partial \Lambda}{\partial z} \end{array} \right\} \tag{91}$$

where $\mathbf{\Lambda}$ represents any of the element interpolation functions (*e.g.*, $\Phi_1, \Phi_q \ldots \Theta_1, \Theta_q$) and $\mathbf{J}$ is the Jacobian of the transformation from global coordinates x, y, z to the local element coordinates s, t, r. The parametric mapping scheme defined in the previous sections (*e.g.*, equations (80) or (86)) can be used to define the components of $\mathbf{J}$. That is,

$$\begin{aligned} J_{11} &= \frac{\partial \Upsilon^T}{\partial s} \mathbf{x} \quad ; \quad J_{12} = \frac{\partial \Upsilon^T}{\partial s} \mathbf{y} \quad ; \quad J_{13} = \frac{\partial \Upsilon^T}{\partial s} \mathbf{z} \\ J_{21} &= \frac{\partial \Upsilon^T}{\partial t} \mathbf{x} \quad ; \quad J_{22} = \frac{\partial \Upsilon^T}{\partial t} \mathbf{y} \quad ; \quad J_{23} = \frac{\partial \Upsilon^T}{\partial t} \mathbf{z} \\ J_{31} &= \frac{\partial \Upsilon^T}{\partial r} \mathbf{x} \quad ; \quad J_{32} = \frac{\partial \Upsilon^T}{\partial r} \mathbf{y} \quad ; \quad J_{33} = \frac{\partial \Upsilon^T}{\partial r} \mathbf{z} \end{aligned} \tag{92}$$

Inverting the transformation matrix in (91) provides the required definition of the spatial derivatives of the shape functions in terms of the local element coordinates

$$\left\{ \begin{array}{c} \frac{\partial \Lambda}{\partial x} \\ \frac{\partial \Lambda}{\partial y} \\ \frac{\partial \Lambda}{\partial z} \end{array} \right\} = \mathbf{J}^{-1} \left\{ \begin{array}{c} \frac{\partial \Lambda}{\partial s} \\ \frac{\partial \Lambda}{\partial t} \\ \frac{\partial \Lambda}{\partial r} \end{array} \right\} = \frac{1}{|\mathbf{J}|} \left[\begin{array}{ccc} \mathcal{J}_{11} & \mathcal{J}_{12} & \mathcal{J}_{13} \\ \mathcal{J}_{21} & \mathcal{J}_{22} & \mathcal{J}_{23} \\ \mathcal{J}_{31} & \mathcal{J}_{32} & \mathcal{J}_{33} \end{array} \right] \left\{ \begin{array}{c} \frac{\partial \Lambda}{\partial s} \\ \frac{\partial \Lambda}{\partial t} \\ \frac{\partial \Lambda}{\partial r} \end{array} \right\} \tag{93}$$

where $|\mathbf{J}|$ indicates the determinant of the Jacobian matrix $\mathbf{J}$. The components $\mathcal{J}_{ij}$ are complex functions of the components of $\mathbf{J}$ that can be obtained by analytically inverting the 3×3 Jacobian matrix. In practice, the Jacobian is usually inverted numerically. For the two-dimensional case the above equations are simplified substantially and permit analytic manipulation.

In performing integrations over the element volume it is also necessary to transform the integration variables and limits from the global coordinates to the local element coordinates. The differential elemental volume transforms according to

$$d\Omega = dx\,dy\,dz = |\mathbf{J}|\,ds\,dt\,dr \tag{94a}$$

while for two-dimensional geometries the planar area is transformed by

$$d\Omega = dx\,dy = |\mathbf{J}|\,ds\,dt \tag{94b}$$

and

$$d\Omega = r\,d\Theta\,dr\,dz = 2\pi\,r\,|\mathbf{J}|\,ds\,dt \tag{94c}$$

for axisymmetric geometries, where the circumferential dependence has been explicitly evaluated to produce the 2π factor. In the axisymmetric case the radius r would be interpolated by $r = \boldsymbol{\Upsilon}^T\mathbf{r}$. The integration limits for the integrals transform to the limits on the local coordinates s, t, r, *i.e.*, -1 to $+1$.

In the above equations the s, t, r variables for a brick element have been used for the purpose of explanation. Similar relations for a tetrahedral element can be derived by replacing s, t, r with L_1, L_2 and L_3. The L_4 variable does not enter the formulas due to the relation $L_1 + L_2 + L_3 + L_4 = 1$. Hybrid coordinates, such as those used in the prism element, are treated in an analogous manner. The two-dimensional case also follows the above procedures.

4.7 Matrix Evaluation

With the previous definitions it is now possible to derive a computational form for the matrix coefficients involved in the finite element equations of Section 3. For purposes of discussion, only a representative term from the matrix system will be considered in detail; the evaluation of the remaining terms follows in a similar manner.

Consider a typical term from the diagonal component of the double curl matrix given by equation (A24) as

$$\mathbf{K_{xx}} = \int_{\Omega_e} \nu_{zz} \frac{\partial \boldsymbol{\Phi}}{\partial y} \frac{\partial \boldsymbol{\Phi}^T}{\partial y}\, d\Omega \tag{95}$$

which will be evaluated for a three-dimensional, twenty-node, brick element. From the previous definitions in (93) and (94a), equation (95) can be written as

$$\mathbf{K_{xx}} = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} \nu_{zz} \underbrace{\frac{1}{|\mathbf{J}|}\left[\mathcal{J}_{21}\frac{\partial \boldsymbol{\Phi}_\mathbf{q}}{\partial s} + \mathcal{J}_{22}\frac{\partial \boldsymbol{\Phi}_\mathbf{q}}{\partial t} + \mathcal{J}_{23}\frac{\partial \boldsymbol{\Phi}_\mathbf{q}}{\partial r}\right]}_{\frac{\partial \boldsymbol{\Phi}_\mathbf{q}}{\partial y}}$$

$$\cdot \underbrace{\left[\mathcal{J}_{21}\frac{\partial \Phi_{\mathbf{q}}{}^T}{\partial s} + \mathcal{J}_{22}\frac{\partial \Phi_{\mathbf{q}}{}^T}{\partial t} + \mathcal{J}_{23}\frac{\partial \Phi_{\mathbf{q}}{}^T}{\partial r}\right]\frac{1}{|\mathbf{J}|}}_{\frac{\partial \Phi_{\mathbf{q}}{}^T}{\partial y}} |\mathbf{J}|\, dsdtdr \tag{96}$$

where the $\Phi_{\mathbf{q}}$ functions are given in (85). For an isoparametric (curve-sided) element the components of $\mathcal{J}_{ij}$ would also be evaluated using $\Upsilon = \Phi_{\mathbf{q}}$ from equation (85).

The above integral is of the general form

$$I = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} f(s,t,r)\, dsdtdr \tag{97}$$

where $f(s,t,r)$ is a rational function of the normalized coordinates. All of the element matrices are of this form and can be conveniently evaluated using a numerical quadrature procedure. That is, the integral in (97) can be evaluated by the formula

$$I = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} W_i W_j W_k f(s_i, t_j, r_k) \tag{98}$$

where W_i are weighting coefficients, s_i, t_j, r_k are quadrature points in the integration interval and n is the number of quadrature points in the formula. For linear and quadratic brick elements, TORO II generally employs a product Gauss quadrature rule as shown in (98) with $n = 2$ and $n = 3$, respectively; the two-dimensional quadrilaterals employ a similar scheme with a double sum used in (98) and $n = 2$ for a linear element and $n = 3$ for the quadratic elements. Other elements in the library are also evaluated using quadrature formulas, though the form of (97) and (98) are slightly different in these cases. For elements using volume or area coordinates, the limits on the definite integral in (97) run from 0 to 1. Also, these elements typically do not use a product rule but rather a single summation over the total number of quadrature points. In TORO II, the tetrahedral elements are evaluated with a four point or a five point formula and the triangular elements with a three point or a seven point rule. The prism uses a three point or a seven point integration rule in the triangular plane and a 2 or 3 point Gauss formula along the normalized axis. Nonproduct Gauss rules [15] are also available for use with the hexahedral elements and may offer some economic benefits over the product Gauss rules.

Application of the quadrature formula in (98) to the integral in equation (96) produces the element coefficient matrix $\mathbf{K}_{xx}$. Note that variable coefficients, such as the magnetic permeability or reluctivity, must be evaluated at the integration points if they vary over the element. Constant coefficients are of course removed from the integral and play no role in the quadrature procedure.

4.8 Element Boundary Conditions and Source Terms

In this section the construction of boundary conditions and volumetric source terms for the element matrix equations is considered. Though the required force vectors are numer-

ically evaluated in the same manner as the coefficient matrices, a number of additional assumptions and details are necessary that require further comment.

4.8.1 Volumetric Sources

The force vectors for the various matrix equations generally consist of two parts: a part due to volumetric sources and a part due to surface fluxes. Consider first the volumetric term associated with a specified component of the current density (*e.g.*, equation (A28))

$$\mathbf{F}_{\mathbf{J_X}} = \int_{\Omega_e} \Phi J_x \, d\Omega. \tag{99}$$

The source current is allowed to vary over the element in an arbitrary manner, which is indicated by $J_x\ (s,t,r)$. As given previously in (94) the elemental volume can also be written in terms of the normalized coordinates. Thus, in a computational form (99) becomes

$$\mathbf{F}_{\mathbf{J_X}} = \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} \Phi J_x\ (s,t,r) |\mathbf{J}| \, dsdtdr \tag{100}$$

for a three-dimensional brick element; similar forms are derivable for the other elements in two and three dimensions. The integral in (100) can be evaluated with a standard numerical quadrature rule to produce the force vector $\mathbf{F}_{\mathbf{J_X}}$. To accomplish the numerical integration, the current source must be evaluated at the quadrature points. If the current source depends on other variables, such as the field variables, temperature, spatial location, *etc.*, these quantities can be provided at the quadrature points through use of the element basis functions. Though a current source was used in the present example, the same procedure applies to all similar volumetric terms.

4.8.2 Surface Fluxes

The remaining force vectors in the matrix equations arise from surface fluxes distributed along element boundaries. These terms need only be considered for those element sides coinciding with the "exterior" boundaries of the problem domain; contributions from interior element boundaries are cancelled by adjoining elements. A typical surface flux vector for a magneotstatic problem is given by

$$\mathbf{F_V} = \int_{\Gamma_e} \Psi J_i n_i \, d\Gamma = \int_{\Gamma_e} \Psi \, f^J \, d\Gamma \tag{101}$$

where Γ_e is the surface of the element, $J_i n_i$ is the current flux normal to the surface and the remaining term is defined by the boundary condition in (20).

The computation of the indicated surface integral is most easily carried out in the normalized or natural coordinate system for the face (edge) of an element. This requires

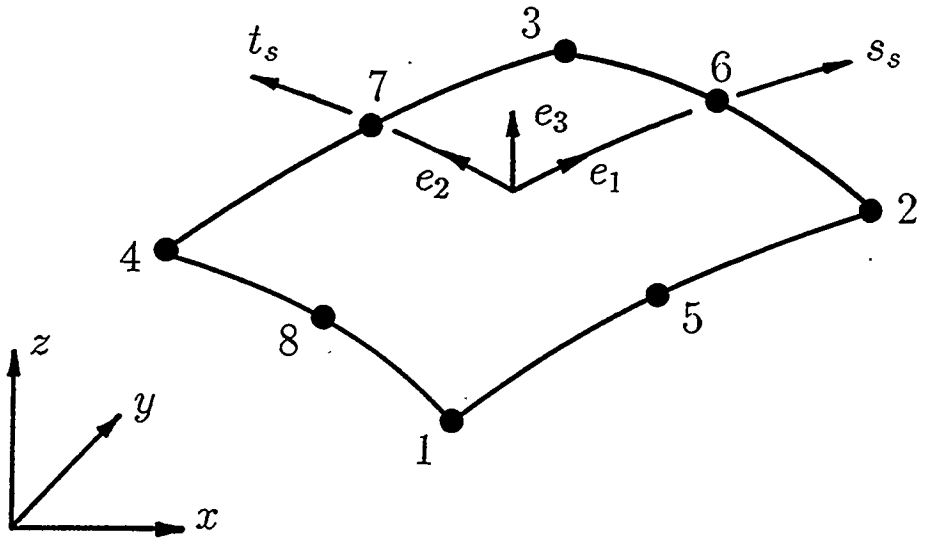

Figure 8: Nomenclature for element surface computations.

that the elemental surface area (edge length) $d\Gamma$, be related to the local surface coordinates. Consider the typical quadrilateral element face shown in Figure 8, where the vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ are defined as being tangent to the curvilinear coordinates, s_s and t_s. The e vectors are not necessarily unit vectors; the s_s and t_s coordinates are assumed to be natural coordinates for the element face. The elemental area $d\Gamma$ in terms of the global coordinates x, y, z is related to an elemental area in surface coordinates by

$$d\Gamma = |\mathbf{J_s}|\, ds_s dt_s \tag{102}$$

where $\mathbf{J_s}$ is the Jacobian of the coordinate transformation and $|\cdot|$ indicates the determinant. The determinant of the Jacobian can be written in terms of the e vectors as

$$|\mathbf{J_s}| = |\mathbf{e_1} \times \mathbf{e_2}| = \left[(\mathbf{e_1}\cdot\mathbf{e_1})(\mathbf{e_2}\cdot\mathbf{e_2}) - (\mathbf{e_1}\cdot\mathbf{e_2})^2\right]^{1/2} \tag{103}$$

The e vectors can be expressed in terms of the global coordinates by

$$\mathbf{e_1} = \begin{Bmatrix} \frac{\partial x}{\partial s_s} \\ \frac{\partial y}{\partial s_s} \\ \frac{\partial z}{\partial s_s} \end{Bmatrix} \quad ; \quad \mathbf{e_2} = \begin{Bmatrix} \frac{\partial x}{\partial t_s} \\ \frac{\partial y}{\partial t_s} \\ \frac{\partial z}{\partial t_s} \end{Bmatrix} \tag{104}$$

Using the parametric mapping concept allows

$$x = \hat{\boldsymbol{\Upsilon}}^T\mathbf{x} \quad ; \quad y = \hat{\boldsymbol{\Upsilon}}^T\mathbf{y} \quad ; \quad z = \hat{\boldsymbol{\Upsilon}}^T\mathbf{z} \tag{105}$$

where the $\hat{}$ notation indicates the restriction of the interpolation function to an element face (edge). The functions $\hat{\Upsilon}$ may be either linear or quadratic depending on the type of mapping used to describe the element geometry. Using (105) in (104) then

$$\mathbf{e_1} = \left\{ \begin{array}{c} \frac{\partial \hat{\Upsilon}^T}{\partial s_s}\mathbf{x} \\ \\ \frac{\partial \hat{\Upsilon}^T}{\partial s_s}\mathbf{y} \\ \\ \frac{\partial \hat{\Upsilon}^T}{\partial s_s}\mathbf{z} \end{array} \right\} \quad ; \quad \mathbf{e_2} = \left\{ \begin{array}{c} \frac{\partial \hat{\Upsilon}^T}{\partial t_s}\mathbf{x} \\ \\ \frac{\partial \hat{\Upsilon}^T}{\partial t_s}\mathbf{y} \\ \\ \frac{\partial \hat{\Upsilon}^T}{\partial t_s}\mathbf{z} \end{array} \right\} \tag{106}$$

Equations (103) and (106) provide a means for computing $|\mathbf{J_s}|$, thus allowing the transformation in (102) to be employed. Note that in two dimensions the above relations simplify and the elemental length of an element edge is given by

$$d\Gamma = \left[\left(\frac{\partial \hat{\Upsilon}^T}{\partial s}\mathbf{x} \right)^2 + \left(\frac{\partial \hat{\Upsilon}^T}{\partial s}\mathbf{y} \right)^2 \right]^{\frac{1}{2}} ds = \Delta \, ds \tag{107}$$

where s is the coordinate along the edge of an element.

To complete the specification of the integrand in (101) the variation of $J_i n_i$ with s_s and t_s is required. From the boundary condition definitions in equations (20) the normal current flux consists of

$$J_i n_i = f^J \tag{108}$$

For calculation of the boundary fluxes it is assumed that the applied flux, f^J, is a known functions of the surface (edge) coordinates, s_s, t_s. Then using the standard surface (edge) interpolation for the potential, the current flux vector can be written for the three-dimensional case as

$$\mathbf{F} = \int_{-1}^{+1} \int_{-1}^{+1} \hat{\Psi} f^J(s_s, t_s) |\mathbf{J_s}| \, ds_s dt_s \tag{109}$$

The integrals in (109) are evaluated using a numerical quadrature procedure over the element surface; in two dimensions only integrals along an element edge need to be considered. When f^J is variable over the surface it must be evaluated at the quadrature points and may depend on interpolated field variables or spatial location. The procedure used above is typical of all surface flux computations.

4.8.3 Specified Potential Boundary Conditions

In addition to the ("natural") boundary conditions specified by the boundary integrals presented above, "essential" boundary conditions specifying particular values of the potentials must also be considered. Application of a specified potential boundary condition results in the field equation for that particular nodal point quantity being replaced by a constraint equation that enforces the proper boundary value. TORO II uses a penalty method [16] to implement this type of constraint condition.

5 Solution Procedures

The major computational effort in any finite element procedure occurs in the solution of the assembled matrix equations that describe the discretized problem. This is especially true in the case of highly nonlinear equations or problems with coupled physical phenomena, both of which can be found in the present case. In addition to computational efficiency these characteristics also introduce questions regarding the ability to achieve a solution, *i.e.*, convergence for a given set of data. The choice of a solution algorithm is therefore a critical element in the overall utility, robustness and efficiency of a computer code such as TORO II.

As described previously, the basic matrix problem of concern can be written as

$$\overline{\mathbf{M}}\dot{\mathbf{U}} + \overline{\mathbf{K}}\mathbf{U} = \overline{\mathbf{F}} \tag{110}$$

where $\overline{\mathbf{M}}$ represents a type of capacitance matrix, $\overline{\mathbf{K}}$ contains the diffusion terms, $\overline{\mathbf{F}}$ provides the boundary and volumetric forcing functions and the vector $\mathbf{U}$ represents all of the potentials in the problem. In the most general case, each term in (110) may depend explicitly on various field variables due to nonconstant properties and/or boundary conditions. In addition the matrices are large, sparse, and often symmetric in their structure; a proper ordering of the equations will produce a banded matrix system.

The general system shown in (110) has very specific forms as the problem type changes from static to time dependent to time-harmonic; the problem dimensionality also influences the matrix structure due to variations in the number of unknows that are active. TORO II has three basic solution options that correspond to static problems (electrostatics, current flow and magnetostatics), time-dependent problems (eddy currents) and time-harmonic problems (AC eddy currents). The following sections outline the available solution procedures for each type of problem.

5.1 Algorithms for Static Problems

The time-independent form of (110) is

$$\overline{\mathbf{K}}(\mathbf{U})\mathbf{U} = \overline{\mathbf{F}}(\mathbf{U}) \tag{111}$$

which is recognized as a system of nonlinear, algebraic equations. Consider first the case where $\overline{\mathbf{K}}$ and $\overline{\mathbf{F}}$ are not functions of $\mathbf{U}$. In this situation (111) reduces to a linear matrix equation which can be solved directly, without iteration. When (111) retains its nonlinear form, an iterative technique is required. TORO II currently employs a single type of iterative method, though it may be combined with other techniques to expand the possibilities for achieving a steady state solution.

5.1.1 Successive Substitution Method

A particularly simple iterative method with a large radius of convergence is the successive substitution (Picard, functional iteration) method described by

$$\overline{\mathbf{K}}(\mathbf{U}^n)\mathbf{U}^{n+1} = \overline{\mathbf{F}}(\mathbf{U}^n) \tag{112}$$

where the superscript indicates the iteration level. For the mildly nonlinear behavior typically found in potential problems, the rate of convergence of (112) is generally good, despite being a first-order method. An improvement in convergence rate can sometimes be realized by use of a relaxation formula where

$$\overline{\mathbf{K}}(\mathbf{U}^n)\mathbf{U}^* = \overline{\mathbf{F}}(\mathbf{U}^n) \tag{113}$$

and

$$\mathbf{U}^{n+1} = \alpha\mathbf{U}^n + (1-\alpha)\mathbf{U}^* \qquad 0 \leq \alpha < 1.$$

Strongly nonlinear problems may cause the above algorithms to exhibit a very slow rate of convergence or divergence. The standard approach to this difficulty is to switch to a second-order iterative procedure, such as Newton's method. However, in the present application Newton's method suffers an important shortcoming: the tangent matrix (Jacobian) used in the algorithm requires derivative data for material properties and boundary conditions. This would significantly complicate the data input and solution procedures. The present version of TORO II does not employ Newton's method though provisions have been made in the code for its future inclusion.

5.1.2 Continuation Method

Failure to achieve a converged solution using (112) or (113) can often be ascribed to the use of a poor initial guess ($\mathbf{U}^0$) for the iterative algorithm. There are two general approaches to the problem of generating good initial estimates for a solution vector and both involve some type of "tracking" of the solution. The first procedure is simply the method of false transients in which the solution is followed through use of a time parameter. The transient algorithms described in a later section are candidates for this approach with certain types of EM simulations.

A second method consists of incrementally approaching the final solution through a series of intermediate solutions. These intermediate solutions may be of physical interest or may simply be a means to obtain the required solution. The formal algorithms used to implement this procedure are termed continuation methods and can be used with either of the iterative methods in (112) and (113).

Assume that the solution in (111) depends continuously on some real parameter, λ. For potential problems, λ could be the magnitude of a volumetric source or the magnitude

of a boundary condition. Then (111) can be written in general as

$$\overline{\mathbf{K}}(\mathbf{U},\lambda)\mathbf{U} = \overline{\mathbf{F}}(\mathbf{U},\lambda) \tag{114}$$

which suggests the zeroth order continuation method

$$\overline{\mathbf{K}}(\mathbf{U}_\lambda^{\mathbf{n}},\lambda^{\mathbf{m}})\mathbf{U}_\lambda^{\mathbf{n+1}} = \overline{\mathbf{F}}(\mathbf{U}_\lambda^{\mathbf{n}},\lambda^{\mathbf{m}}) \tag{115}$$

where (115) is solved for a series of problems with increasing values of the continuation parameter $\lambda^m = \lambda^{m-1} + \Delta\lambda$. The converged solution, U_λ, at one value of λ is used as the starting solution at the next higher value of λ; the iterative method in (112) or (113) is used at each value of λ to achieve a converged solution. This technique is available in TORO II and can be used effectively with some very nonlinear problems.

5.1.3 Convergence Criteria

The use of an iterative solution method necessitates the definition of a convergence and stopping criteria to terminate the iteration process. The usual measure of convergence is a norm on the change in the solution vector between successive iterations. TORO II employs the discrete norm defined by

$$d_{n+1}^T = \frac{1}{U_{max}}\left[\sum_{i=1}^{N}\left(U_i^{n+1} - U_i^n\right)^2\right]^{\frac{1}{2}} \tag{116}$$

In the definition in (116) the subscript max indicates the maximum value of the variable found in the solution at iteration cycle $n+1$ and N is the total number of nodal points in the problem. The norm in (116) is employed separately for each variable in the problem.

The criteria for terminating the iteration process is based on a user supplied tolerance. The iterative algorithm is terminated when the following inequality is satisfied.

$$d_{n+1}^U \leq \epsilon^U \tag{117}$$

where ϵ^U is set by the user and has a default value of 0.001. The iterative process may also be terminated after a fixed number of iterations. This option acts as a backup criteria to prevent very slowly convergent or divergent problems from wasting computation time.

5.2 Algorithms for Time Dependent Problems

Equation (110) represents a discrete space, continuous time approximation to the original system of partial differential equations. A direct time integration procedure replaces the continuous time derivative with an approximation for the history of the dependent variables over a small portion of the problem time scale. The result is an incremental

procedure that advances the solution by discrete steps in time. In constructing such a procedure, questions of numerical stablity and accuracy must be considered.

A large body of literature is available on possible time integration schemes for equations of the diffusion type. Both implicit and explicit methods, as well as mode superposition, have been used successfully. Each of these approaches has its strong and weak points, most of which are problem dependent. In order to provide solution capabilities for as wide a range of problems as possible, three different integration schemes are used in TORO II. Two types of implicit methods are available, both of which make use of a predictor/corrector strategy to improve efficiency and accuracy. These procedures were originally developed by Gresho, *et al* [17] and are used in TORO II with only minor modifications. The third integration scheme is an explicit procedure that can be used effectively with the lower order elements available in the codes. All of the integration methods may be used with either a fixed time step or an adaptive time step selection algorithm.

5.2.1 Forward/Backward Euler Integration

The first-order implicit integration method used in TORO II employs a forward Euler scheme as a predictor with the backward Euler method functioning as the corrector step. Omitting the details of the derivation, the application of the explicit, forward Euler formula to equation (110) produces

$$\overline{\mathbf{M}}\mathbf{U}_p^{n+1} = \overline{\mathbf{M}}\mathbf{U}^n + \Delta t_n \left[\overline{\mathbf{F}}(\mathbf{U}^n) - \overline{\mathbf{K}}(\mathbf{U}^n)\mathbf{U}^n\right] \tag{118}$$

This can be written in a form that is more suitable for computation by replacing the bracketed term with a rearranged form of (110) to produce

$$\mathbf{U}_p^{n+1} = \mathbf{U}^n + \Delta t_n \dot{\mathbf{U}}^n \tag{119}$$

In equations (118) and (119) the superscript indicates the timeplane, the subscript p denotes a predicted value and $\Delta t_n = t_{n+1} - t_n$. By using the form shown in (119) a matrix inversion of $\overline{\mathbf{M}}$ is avoided; the "acceleration" vector $\dot{\mathbf{U}}^n$ is computed from a form of the corrector formula as shown below.

The corrector step of the first-order scheme is provided by the backward Euler (or fully implicit) method. When applied to equation (110) this implicit method yields

$$\overline{\mathbf{M}}\mathbf{U}^{n+1} = \overline{\mathbf{M}}\mathbf{U}^n + \Delta t_n \left[\overline{\mathbf{F}}(\mathbf{U}^{n+1}) - \overline{\mathbf{K}}(\mathbf{U}^{n+1})\mathbf{U}^{n+1}\right] \tag{120}$$

or in a form more suitable for computation

$$\left[\frac{1}{\Delta t_n}\overline{\mathbf{M}} + \overline{\mathbf{K}}(\mathbf{U}^{n+1})\right]\mathbf{U}^{n+1} = \frac{1}{\Delta t_n}\overline{\mathbf{M}}\mathbf{U}^n + \overline{\mathbf{F}}(\mathbf{U}^{n+1}) \tag{121}$$

The implicit nature of this method is evident from the form of (121), since it is in effect, a nonlinear, algebraic system for the variables in $\mathbf{U}$ at timeplane $n+1$.

The solution to (121) at timeplane $n+1$ can be achieved by an iteration procedure such as Picard's method. The rate of convergence of Picard's method is greatly increased if the initial solution estimate is "close" to the true solution. The solution predicted from (118) provides this initial guess for the iterative procedure in a cost-effective manner.

5.2.2 Adams-Bashforth/Trapezoid Rule Integration

An implicit integration method that is second-order accurate in time can be developed along the same lines as described above. A second-order equivalent to the forward Euler method is the variable step, Adams-Bashforth predictor given by

$$\mathbf{U}_p^{n+1} = \mathbf{U}^n + \frac{\Delta t_n}{2}\left[\left(2 + \frac{\Delta t_n}{\Delta t_{n-1}}\right)\dot{\mathbf{U}}^n - \left(\frac{\Delta t_n}{\Delta t_{n-1}}\right)\dot{\mathbf{U}}^{n-1}\right] \tag{122}$$

where $\Delta t_n = t_{n+1} - t_n$ and $\Delta t_{n-1} = t_n - t_{n-1}$. This formula can be used to predict the solution vector given two "acceleration" vectors from previous timeplanes; no matrix solution is required.

A compatible corrector formula for use with (122) is available in the form of the trapezoid rule. When applied to equation (110) the trapezoid rule produces

$$\left[\frac{2}{\Delta t_n}\overline{\mathbf{M}} + \overline{\mathbf{K}}(\mathbf{U}^{n+1})\right]\mathbf{U}^{n+1} = \frac{2}{\Delta t_n}\overline{\mathbf{M}}\mathbf{U}^n + \overline{\mathbf{M}}\dot{\mathbf{U}}^n + \overline{\mathbf{F}}(\mathbf{U}^{n+1}) \tag{123}$$

Equation (123) is observed to be a nonlinear, algebraic system for the vector $\mathbf{U}^{n+1}$ and can again be solved using an iterative procedure such as Picard's method.

5.2.3 Implicit Integration Procedures

The integration formulas outlined above form the basis for the implicit solution of time dependent eddy current problems in TORO II. The similarity of the first- and second-order methods makes it possible to include both procedures in a single, overall algorithm. The major steps in the time integration procedure are outlined here.

At the beginning of each time step it is assumed that all of the required solution and "acceleration" vectors are known and the time increment for the next step has been selected. To advance the solution from time t_n to time t_{n+1} then requires the following steps:

1) A tentative solution vector, $\mathbf{U}_p^{n+1}$, is computed using the predictor equations (119) or (122).

2) The corrector equations (121) or (123) are solved for the "true" solution, $\mathbf{U}^{n+1}$. This involves the iterative solution of (121) or (123) via Picard's method. The predicted values $\mathbf{U}_p^{n+1}$ are used to initialize the equation for the iteration procedure.

3) The "acceleration" vectors are updated using the new solution $\mathbf{U}^{n+1}$ and the "inverted" forms of the corrector formulas. For the first-order method the acceleration is computed from the backward Euler definition

$$\dot{\mathbf{U}}^{n+1} = \frac{1}{\Delta t_n}\left(\mathbf{U}^{n+1} - \mathbf{U}^n\right)$$

while the second-order accelerations are derived from the trapezoid rule

$$\dot{\mathbf{U}}^{n+1} = \frac{2}{\Delta t_n}\left(\mathbf{U}^{n+1} - \mathbf{U}^n\right) - \dot{\mathbf{U}}^n$$

4) A new integration time step is computed. The time step selection process is based on an analysis of the time truncation errors in the predictor and corrector formulas as described in Section 5.2.4. If a constant time step is being used, this step is omitted.

5) Return to step 1 for next time increment.

In actual implementation the Picard iteration process in step 2 is not carried to absolute convergence. Rather, a one-step correction is employed as advocated in [17]. This procedure is quite efficient and can be very accurate provided the time step is suitably controlled.

5.2.4 Time Step Control

Both of the implicit time integration procedures available in TORO II can be used with a fixed, user specified time step or a time step that changes only at certain points during the integration interval. However, the *a priori* selection and modification of a reasonable integration time step can be a difficult task, especially for a complex problem. One of the benefits of using the predictor/corrector algorithms described here is that it provides a rational basis for dynamically selecting the time step.

The detailed derivation of the time step selection formula is omitted here. The reader interested in further details is referred to [17]. The general ideas for the time step selection process come from the well-established procedures for solving ordinary differential equations. By comparing the time truncation errors for two time integration methods of comparable order, a formula can be developed to predict the next time step based on a user specified error tolerance. In the present case, the time truncation errors for the explicit predictor and implicit corrector steps are analyzed and provide the required formulas.

The time step estimation formula is given by [17] as

$$\Delta t_{n+1} = \Delta t_n \left(b \cdot \frac{\epsilon^t}{d_{n+1}} \right)^m \tag{124}$$

where $m = 1/2$, $b = 2$ for the first-order method and $m = 1/3$, $b = 3(1 + \Delta t_{n-1}/\Delta t_n)$ for the second-order scheme. The user specified error tolerance for the integration process is ϵ^t, which has a default value of 0.0001. The quantity d_{n+1} is an appropriate norm on the integration error, which is defined as the difference between the predicted solution and the corrected value. In TORO II the following norm is used

$$d^U_{n+1} = \frac{1}{U_{max}} \left[\sum_{i=1}^{N} \left(U_{i(n+1)} - U^p_{i(n+1)} \right)^2 \right]^{\frac{1}{2}} \tag{125}$$

where (125) is used for each variable in the problem. The largest norm is then used in (124) to compute the time step.

Unlike the procedure described in [17], TORO II always uses the newly computed time step derived from (124). If $\Delta t_{n+1} \leq 0.5 \Delta t_n$ a warning message is given to indicate a large reduction in the time step has occurred. However, the previous time step is not rejected nor recomputed.

5.2.5 Initialization

The predictor equations (119) and (122) require that one or more acceleration vectors be available at each timeplane in order to estimate a new solution vector. At the beginning of a transient solution these vectors are not generally available and thus a special starting procedure must be used. The approach taken in TORO II is to use the dissipative, backward Euler method for the first few steps and then switch to either of the standard predictor/corrector methods. This procedure has the advantage that any nonphysical features of the numerical model are quickly damped by the backward Euler scheme.

For the first time step, the implicit, backward Euler scheme is used alone; the second step uses a forward Euler predictor and backward Euler corrector. Both of these steps use a fixed, user supplied time step. At the third step, the usual predictor/corrector integration procedure begins and automatic time step selection is started, if this option has been requested. The initial time step supplied by the user to start the problem should be very conservative to prevent large time step reductions when the automatic selection procedure takes control.

5.2.6 Forward Euler Integration

The explicit integration method used in TORO II is based on the first-order, forward Euler method. This algorithm was previously defined in (118) and is written here as

$$\overline{\mathbf{M}}\mathbf{U}^{n+1} = \overline{\mathbf{M}}\mathbf{U}^{n} + \Delta t_n \left[\overline{\mathbf{F}}(\mathbf{U}^n) - \overline{\mathbf{K}}(\mathbf{U}^n)\mathbf{U}^n\right] \tag{126}$$

Inverting the capacitance matrix $\overline{\mathbf{M}}$ allows (126) to be written in a computationally effective form as

$$\mathbf{U}^{n+1} = \mathbf{U}^{n} + \Delta t_n \overline{\mathbf{M}}^{-1} \left[\overline{\mathbf{F}}(\mathbf{U}^n) - \overline{\mathbf{K}}(\mathbf{U}^n)\mathbf{U}^n\right] = \mathbf{U}^{n} + \Delta t_n \overline{\mathbf{M}}^{-1} \overline{\mathbf{F}}_{\mathbf{eff}} \tag{127}$$

As written in (127), this algorithm does not require the solution of a matrix system but simply the construction of an effective force vector from known data and a matrix-vector product.

The practical utility of (127) relies on two aspects of the algorithm. First, the inverse of the effective capacitance matrix must be easily obtainable (*i.e.*, computationally inexpensive). Also, the explicit nature of the method means that the stability of the algorithm must be considered when choosing an integration time step.

5.2.7 Matrix Diagonalization

A particular feature of the finite element method when used for time-dependent problems is the inherent coupling that occurs between nodal point time derivatives. By inspecting the form of the element level capacitance matrix, as shown in (A24), it is clear that M is not a simple diagonal matrix but has a banded structure. This structure carries over to the global matrix, $\overline{\mathbf{M}}$.

The inverse to $\overline{\mathbf{M}}$ could be directly computed for use in the explicit algorithm in (127). Unfortunately, the inverse of a banded matrix is a full matrix; the generally large size of $\overline{\mathbf{M}}$ for two and three-dimensional problems precludes the use of such an approach. For $\overline{\mathbf{M}}^{-1}$ to be computed efficiently, $\overline{\mathbf{M}}$ must have a diagonal form. Several strategies for diagonalizing $\overline{\mathbf{M}}$ have been proposed in the literature [16]. The approach taken in TORO II is to use the row-sum technique to approximate M at the element level. That is, the diagonal form of the element matrix, $\mathbf{M_D}$, is formed by the components given by

$$m_{ii}^{D} = \sum_{j=1}^{N} m_{ij} \quad ; \quad m_{ij}^{D} = 0 \quad i \neq j \tag{128}$$

where N is the number of degrees of freedom in the element and m_{ij} are components of the original capacitance matrix; the global matrix will also have the same diagonal form. In the algorithm given in (127) the term $\overline{\mathbf{M}}^{-1}$ is replaced with $\overline{\mathbf{M}}_{\mathbf{D}}^{-1}$, which is easily computed from $1/m_{ii}^{D}$.

Diagonalization (or "lumping") techniques, such as the row-sum method, have been widely used and investigated. It is known that the temporal response of the discretized equations is altered by such techniques, though good results can still be obtained with careful use. A major limitation to diagonalization (and row-sum in particular) is its restriction to low-order (linear) finite element approximations. Higher order basis functions generally produce poor results when used in a diagonalized form. This result implies that the explicit integration procedures in TORO II should only be used with the linear elements available in the code.

5.2.8 Stability and Time Step Control

Explicit integration methods are conditionally stable and thus require limits on the size of the integration time step. Conventional stability analyses of the forward Euler scheme for a diffusion equation [18] produce a time step restriction of the following form

$$\lambda \Delta t \leq 2 \tag{129}$$

where λ is the largest eigenvalue for the matrix system $\overline{\mathbf{M}}_{\mathbf{D}}^{-1}\overline{\mathbf{K}}$ from equation (127). The largest eigenvalue for the system can be bounded by the largest element eigenvalue, which in the present case is proportional to $1/h^2$ with h being a representative element dimension. From these results it is clear that the time step restriction for the explicit method is quite severe especially on highly refined meshes. Reference [1] has further details on the explicit time step control used in TORO II.

5.3 Algorithms for Time-Harmonic Problems

When time-harmonic problems with a phasor representation are considered, the basic matrix equation (110) becomes

$$i\omega\overline{\mathbf{M}}\mathbf{U_0} + \overline{\mathbf{K}}\mathbf{U_0} = \overline{\mathbf{F}} \tag{130}$$

which usually represents a linear set of complex equations. Equation (130) could be nonlinear if material properties or boundary conditions depend on the amplitude of the field variables; time harmonic variations in the properties are not permitted. The time-harmonic problem can also be expressed as

$$\tilde{\mathbf{K}}(\overline{\mathbf{U}}_\mathbf{0})\mathbf{U_0} = \overline{\mathbf{F}}(\overline{\mathbf{U}}_\mathbf{0}) \tag{131}$$

where it it is understood that $\tilde{\mathbf{K}}$ has complex coefficients, the solution vector $\mathbf{U_0}$ is also complex and the overbar indicates a magnitude. The form of equation (131) is the same as (111) for the static problems treated in Section 5.1. and the solution methods are exactly the same. The reorganization of (131) into real and imaginary components (see equation (72)) does not affect the solution algorithm though it does alter the allowable selections for the matrix solver. This is discussed in the next section.

5.4 Matrix Solution Procedures

When most of the algorithms of the previous sections are applied at a given iteration or time step, the result is a matrix equation of the form

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{132}$$

In the problems considered here the matrix $\mathbf{A}$ is large, sparse, banded and usually symmetric. A solution to (132) can be achieved by either an iterative or direct method. Historically, direct methods, such as the frontal method [19] or other forms of Gauss elimination, have been the solution methods of choice for most finite element applications. However, the computer memory and CPU inefficiency of direct methods with respect to large, three-dimensional problems, has produced renewed interest in iterative methods for (132).

The solution method used in TORO II is based on a preconditioned conjugate gradient (PCG) algorithm. The matrix solution technique is embedded in a PCG library package that was developed by Schunk and Shadid [20]. For application to the various symmetric EM problems, any of three different preconditions can be invoked: Jacobi, Polynomial and the Incomplete Choleski method. The unsymmetric system associated with time-harmonic applications requires an iterative method such as the generalized minimum residual method (GMRES). This unsymmetric method may also be used with a number of preconditioners. All of the preconditioners are derived from the assembled, global matrix, $\mathbf{A}$. To accommodate the storage requirements of $\mathbf{A}$, a standard sparse matrix format is used [21] that only records the nonzero entries of the $\mathbf{A}$ matrix. Complete details on the iterative solvers available in TORO II can be found in [20].

6 Pre- and Post-Processing

The TORO II program was designed to be a self-contained analysis package with the necessary options to set up a problem, solve for the required dependent variables and analyze the resultant solution in terms of derived quantitites. The present section documents some of the numerical procedures used in the pre-solution and data analysis sections of the program.

6.1 Mesh Generation

TORO II contains no mesh generation capability and relies completely on external mesh generation software for a geometric description of the problem. The code reads mesh generation data from a standard format file called EXODUS II [22]. A complete description of the mesh generation interface to TORO II is available in the user's manual [4].

6.2 Auxiliary Field Computations

When the electric scalar potential, V, and magnetic vector potential, $\mathbf{A}$, have been computed for an electromagnetic problem, a number of auxiliary fields may be evaluated over the computational domain. These fields are all derived from the basic definitions given in equations (28) and (30), the constitutive relations (16) and (17) and the force and volume heating definitions in (9) and (10).

Electric Field Variables

The electric field in a region is given in terms of the potentials by equation (30)

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{133}$$

This may be computed on an element-by-element basis within TORO II depending on the availability of the two potentials within a given material. For static problems, the time derivative is absent and only the potential gradient may be available. For regions with specified electric currents the gradient part of the $\mathbf{E}$ field is computed from an inverted form of Ohm's law, $\mathbf{E} = \sigma^{-1}\mathbf{J}$. For time-harmonic problems the real and imaginary parts of the $\mathbf{E}$ field are derived from

$$\begin{aligned} \mathbf{E}^{\mathbf{R}} &= -\nabla V^R + \omega \mathbf{A}^{\mathbf{I}} \\ \mathbf{E}^{\mathbf{I}} &= -\nabla V^I - \omega \mathbf{A}^{\mathbf{R}} \end{aligned} \tag{134}$$

and the modulus and phase angles for the field are defined by

$$|\mathbf{E}| = \sqrt{(\mathbf{E^R})^2 + (\mathbf{E^I})^2} \tag{135}$$

$$\phi = tan^{-1}(\frac{\mathbf{E^I}}{\mathbf{E^R}}) \tag{136}$$

The components of $\mathbf{E}$ in (133) and (134) are computed by using the standard finite element approximations for V and $\mathbf{A}$,

$$V(x_i, t) = \Psi^T(x_i)\mathbf{V}(t)$$

$$A_i(x_i, t) = \Phi^T(x_i)\mathbf{A_i}(t)$$

and the relations for the local spatial derivatives as derived in Section 4.6. For example, the gradient of V is obtained from

$$\begin{Bmatrix} \frac{\partial \Psi}{\partial x} \\ \frac{\partial \Psi}{\partial y} \\ \frac{\partial \Psi}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial \Psi}{\partial s} \\ \frac{\partial \Psi}{\partial t} \\ \frac{\partial \Psi}{\partial r} \end{Bmatrix} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} & \mathcal{J}_{13} \\ \mathcal{J}_{21} & \mathcal{J}_{22} & \mathcal{J}_{23} \\ \mathcal{J}_{31} & \mathcal{J}_{32} & \mathcal{J}_{33} \end{bmatrix} \begin{Bmatrix} \frac{\partial \Psi}{\partial s} \\ \frac{\partial \Psi}{\partial t} \\ \frac{\partial \Psi}{\partial r} \end{Bmatrix}$$

Using these definitions the electric field components in (133) become

$$\begin{aligned} E_x &= -\frac{1}{|\mathbf{J}|}\left(\mathcal{J}_{11}\frac{\partial \Psi^T}{\partial s}\mathbf{V} + \mathcal{J}_{12}\frac{\partial \Psi^T}{\partial t}\mathbf{V} + \mathcal{J}_{13}\frac{\partial \Psi^T}{\partial r}\mathbf{V}\right) - \Phi^T\dot{\mathbf{A}}_\mathbf{x} \\ E_y &= -\frac{1}{|\mathbf{J}|}\left(\mathcal{J}_{21}\frac{\partial \Psi^T}{\partial s}\mathbf{V} + \mathcal{J}_{22}\frac{\partial \Psi^T}{\partial t}\mathbf{V} + \mathcal{J}_{23}\frac{\partial \Psi^T}{\partial r}\mathbf{V}\right) - \Phi^T\dot{\mathbf{A}}_\mathbf{y} \\ E_z &= -\frac{1}{|\mathbf{J}|}\left(\mathcal{J}_{31}\frac{\partial \Psi^T}{\partial s}\mathbf{V} + \mathcal{J}_{32}\frac{\partial \Psi^T}{\partial t}\mathbf{V} + \mathcal{J}_{33}\frac{\partial \Psi^T}{\partial r}\mathbf{V}\right) - \Phi^T\dot{\mathbf{A}}_\mathbf{z} \end{aligned} \tag{137}$$

Similar expressions can be derived for the field components in cylindrical coordinates and for the case of a time-harmonic problem where both real and imaginary components must be computed.

The current density, if not specified, can be obtained from the electric field and Ohm's law $\mathbf{J} = \sigma \cdot \mathbf{E}$ where for the general case the conductivity σ is a tensor. The components of $\mathbf{J}$ then are

$$\begin{aligned} J_x &= \sigma_{xx}E_x + \sigma_{xy}E_y + \sigma_{xz}E_z \\ J_y &= \sigma_{yx}E_x + \sigma_{yy}E_y + \sigma_{yz}E_z \\ J_z &= \sigma_{zx}E_x + \sigma_{zy}E_y + \sigma_{zz}E_z \end{aligned} \tag{138}$$

with a similar form for the cylindrical geometry case. When time-harmonic problems are considered the equations in (138) are employed for both real and imaginary components of $\mathbf{J}$ and the modulus and phase are computed from definitions equivalent to (135)-(136).

The Ohmic or Joule heating for a region can be defined once the electric field and/or current has been evaluated. From equation (10) then

$$Q = \mathbf{J} \cdot \mathbf{E} = \sigma \mathbf{E} \cdot \mathbf{E} = \sigma^{-1} \mathbf{J} \cdot \mathbf{J}$$

where the components of $\mathbf{E}$ and $\mathbf{J}$ are defined in (137) and (138). For the time-harmonic case an average heating over a cycle is required and this is defined by

$$< Q >=< \mathrm{Re}\{\mathbf{J}\} \cdot \mathrm{Re}\{\mathbf{E}\} >= \sigma < \mathrm{Re}\{\mathbf{E}\} \cdot \mathrm{Re}\{\mathbf{E}\} >= \sigma^{-1} < \mathrm{Re}\{\mathbf{J}\} \cdot \mathrm{Re}\{\mathbf{J}\} >$$

which can be rewritten to avoid the explicit averaging as

$$< Q >= \frac{1}{2}\mathrm{Re}\{\mathbf{J} \cdot \mathbf{E}^*\} = \frac{\sigma}{2}\mathrm{Re}\{\mathbf{E} \cdot \mathbf{E}^*\} = \frac{\sigma^{-1}}{2}\mathrm{Re}\{\mathbf{J} \cdot \mathbf{J}^*\} \tag{139}$$

where Re indicates the real part of the field and the $*$ denotes the complex conjugate of the field. Using the current density definition as an example

$$\begin{aligned} \mathbf{J} &= (\mathbf{J}^{\mathbf{R}} + i\,\mathbf{J}^{\mathbf{I}})\,(\cos\omega t + i\,\sin\omega t) \\ \mathbf{J}^* &= (\mathbf{J}^{\mathbf{R}} - i\,\mathbf{J}^{\mathbf{I}})\,(\cos\omega t - i\,\sin\omega t) \end{aligned} \tag{140}$$

Evaluating the product $\mathbf{J} \cdot \mathbf{J}^*$ from (140) and taking the real part leads to

$$< Q >= \frac{\sigma^{-1}}{2}[(\mathbf{J}^{\mathbf{R}})^2 + (\mathbf{J}^{\mathbf{I}})^2] \tag{141}$$

which is the average volume heating.

Magnetic Field Variables

The magnetic flux for a region is defined in terms of the magnetic potential by

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{142}$$

which can be written in component form as

$$\begin{aligned} B_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ B_y &= \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ B_z &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{aligned} \tag{143}$$

with a similar definition for other coordinate systems. Using the shape functions for the components of $\mathbf{A}$ and the definitions of the spatial derivatives given above and in Section 4.7, then a typical component of the magnetic flux becomes

$$B_x = \frac{1}{|\mathbf{J}|}\left(\mathcal{J}_{21}\frac{\partial \mathbf{\Phi}^T}{\partial s}\mathbf{A_z} + \mathcal{J}_{22}\frac{\partial \mathbf{\Phi}^T}{\partial t}\mathbf{A_z} + \mathcal{J}_{23}\frac{\partial \mathbf{\Phi}^T}{\partial r}\mathbf{A_z}\right)$$

$$-\frac{1}{|\mathbf{J}|}\left(\mathcal{J}_{31}\frac{\partial\Phi^T}{\partial s}\mathbf{A_y}+\mathcal{J}_{32}\frac{\partial\Phi^T}{\partial t}\mathbf{A_y}+\mathcal{J}_{33}\frac{\partial\Phi^T}{\partial r}\mathbf{A_y}\right) \tag{144}$$

The remaining components follow in a similar manner. The time-harmonic case utilizes (142) to produce real and imaginary components of $\mathbf{B}$; the modulus and phase angle are computed from the definitions given previously.

The magnetic field $\mathbf{H}$ is computed from the magnetic flux based on the constitutive relation $\mathbf{H}=\nu\mathbf{B}$ where ν is a tensor. The components of $\mathbf{H}$ for the general case are

$$\begin{aligned} H_x &= \nu_{xx}B_x+\nu_{xy}B_y+\nu_{xz}B_z \\ H_y &= \nu_{yx}B_x+\nu_{yy}B_y+\nu_{yz}B_z \\ H_z &= \nu_{zx}B_x+\nu_{zy}B_y+\nu_{zz}B_z \end{aligned} \tag{145}$$

with a similar form for the cylindrical geometry case. When time-harmonic problems are considered the equations in (145) are employed for both real and imaginary components of $\mathbf{H}$ and the modulus and phase are computed from definitions equivalent to (135)-(136).

The magnetic part of the Lorentz force can be evaluated once the currents and magnetic flux are known. From equation (9) the force vector is

$$\mathbf{F}=\mathbf{J}\times\mathbf{B}$$

where the components of $\mathbf{F}$ are

$$\begin{aligned} F_x &= J_yB_z-J_zB_y \\ F_y &= J_zB_x-J_xB_z \\ F_z &= J_xB_y-J_yB_x \end{aligned} \tag{146}$$

The components of $\mathbf{J}$ and $\mathbf{B}$ are defined in (138) and (143). An equation similar to (146) may be written for the cylindrical geometry case. For time-harmonic problems the Lorentz force must be averaged over a cycle and

$$<\mathbf{F}>=<\mathrm{Re}\{\mathbf{J}\}\times\mathrm{Re}\{\mathbf{B}\}>=\frac{1}{2}\mathrm{Re}\{\mathbf{J}\times\mathbf{B}^*\} \tag{147}$$

As before, the $\mathbf{J}$ and $\mathbf{B}^*$ fields may be written as

$$\begin{aligned} \mathbf{J} &= (\mathbf{J^R}+i\,\mathbf{J^I})\,(\cos\omega t+i\,\sin\omega t) \\ \mathbf{B}^* &= (\mathbf{B^R}-i\,\mathbf{B^I})\,(\cos\omega t-i\,\sin\omega t) \end{aligned} \tag{148}$$

Taking the real part of the cross product then produces the Lorentz force as

$$<\mathbf{F}>=\frac{1}{2}(\mathbf{J^R}\times\mathbf{B^R}+\mathbf{J^I}\times\mathbf{B^I}) \tag{149}$$

with components

$$< F_x >= \frac{1}{2}(J_y^R B_z^R - J_z^R B_y^R + J_y^I B_z^I - J_z^I B_y^I)$$
$$< F_y >= \frac{1}{2}(J_z^R B_x^R - J_x^R B_z^R + J_z^I B_x^I - J_x^I B_z^I) \quad (150)$$
$$< F_z >= \frac{1}{2}(J_x^R B_y^R - J_y^R B_x^R + J_x^I B_y^I - J_y^I B_x^I)$$

A final quantity that is sometimes of interest for two-dimensional problems with out-of-plane currents, is the magnetic stream function. By definition

$$B_x = \frac{\partial \psi}{\partial y} \qquad ; B_y = \frac{\partial \psi}{\partial y} \quad (151)$$

for planar geometries and

$$B_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \qquad ; B_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \quad (152)$$

for axisymmetric regions. The stream function could be computed from (151)-(152) by integrating the components of **B** along closed contours (*e.g.*, element boundaries) as is typical in fluid mechanics codes. However, the stream function is also related to the magnetic potential and this provides a more direct route for computation. For the axisymmetric case $\psi = rA_\theta$ and in the planar geometries $\psi = A_z$. The magnetic stream function is thus most useful for visualization of axisymmetric problems.

The definitions given above are sufficient to define the various field components at any point s_0, t_0, r_0 within an element. In TORO II the field components are evaluated in the interior of each element at selected integration points. For quadrilateral and hexahedral elements the selected interior points are typically the $2 \times 2 \times 2$ Gauss points as recommended in [23]. Other element types also have recommended interior points for acurate derivative evaluation. Note that auxiliary fields computed from derivatives of a field variable are discontinuous between elements. To produce a continuous field distribution, the integration point values are linearly extrapolated to the nodes of each element and averaged between all connected elements. Fields that may not be continuous across material interfaces, such as the current density, are not averaged across material boundaries. The Joule heating and Lorentz force fields are defined only as an element quantity and are assumed uniform within an element.

6.3 Graphical Output

TORO II contains no graphics capability and relies completely on external visualization software. The code outputs solution data in a standard format file called EXODUS II [22] that can be accessed by any of several graphics packages. Details are available in [4].

7 References

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Appendix A - Finite Element Equations for Quasi-Static Electromagnetics

The detailed manipulations required to transform the basic field equations, written in terms of the potentials, into a discrete system are presented here for the case of a conducting material region. The development of the equations for various dielectric regions is also summarized and the finite element form of the gauge condition is derived. Finally, the discrete forms of the equations describing the static field problems are developed from the results of the more general eddy current problem. Though the outline of the numerical approximation procedure in Section 3 began with the definition of the finite element approximations, the process to be described here begins with the weighted residual statement of the boundary value problem and introduces the element approximations at a later stage.

Conduction Region, Ω_C

The starting point for the development is the set of vector and scalar equations describing the magnetic and electric potentials as given in (31) and (32)

$$\nabla \times (\nu \cdot \nabla \times \mathbf{A}) = -\sigma \cdot \frac{\partial \mathbf{A}}{\partial t} - \sigma \cdot \nabla V \tag{A1}$$

$$\nabla \cdot \left(-\sigma \cdot \nabla V - \sigma \cdot \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \tag{A2}$$

The weak or weighted integral forms corresponding to (A1) and (A2) are obtained by defining a vector weighting function $\mathbf{W}$ and a scalar weighting function W, multiplying (A1) and (A2) by the appropriate function and integrating each equation over the conducting region. That is,

$$\int_{\Omega_C} \mathbf{W} \cdot \nabla \times (\nu \cdot \nabla \times \mathbf{A})\, d\Omega + \int_{\Omega_C} \mathbf{W} \cdot \sigma \cdot \frac{\partial \mathbf{A}}{\partial t}\, d\Omega + \int_{\Omega_C} \mathbf{W} \cdot \sigma \cdot \nabla V\, d\Omega = 0 \tag{A3}$$

and

$$\int_{\Omega_C} W \nabla \cdot (-\sigma \cdot \nabla V)\, d\Omega + \int_{\Omega_C} W \nabla \cdot (-\sigma \cdot \frac{\partial \mathbf{A}}{\partial t})\, d\Omega = 0 \tag{A4}$$

The weak forms in (A3) and (A4) may be further transformed by utilizing Gauss' theorem to reduce the highest order derivative terms. Proceeding first with the magnetic potential equation, let $\mathbf{H} = \nu \cdot \nabla \times \mathbf{A}$ and rewrite (A3) as

$$\int_{\Omega_C} \mathbf{W} \cdot \nabla \times \mathbf{H}\, d\Omega + \int_{\Omega_C} \mathbf{W} \cdot \sigma \cdot \frac{\partial \mathbf{A}}{\partial t}\, d\Omega + \int_{\Omega_C} \mathbf{W} \cdot \sigma \cdot \nabla V\, d\Omega = 0 \tag{A5}$$

By a vector identity

$$\nabla \cdot (\mathbf{W} \times \mathbf{H}) = \mathbf{H} \cdot \nabla \times \mathbf{W} - \mathbf{W} \cdot \nabla \times \mathbf{H}$$

which allows (A5) to be written as

$$\int_{\Omega_C} \mathbf{H}\cdot\nabla\times\mathbf{W}\,d\Omega - \int_{\Omega_C} \nabla\cdot(\mathbf{W}\times\mathbf{H})\,d\Omega + \int_{\Omega_C} \mathbf{W}\cdot\sigma\cdot\frac{\partial\mathbf{A}}{\partial t}\,d\Omega + \int_{\Omega_C} \mathbf{W}\cdot\sigma\cdot\nabla V\,d\Omega = 0 \quad (A6)$$

Introducing the divergence theorem for the second integral and the vector identity $(\mathbf{W}\times\mathbf{H})\cdot\mathbf{n} = \mathbf{W}\cdot(\mathbf{H}\times\mathbf{n})$ then (A6) becomes

$$\int_{\Omega_C} \mathbf{H}\cdot\nabla\times\mathbf{W}\,d\Omega - \int_{\Gamma_C} \mathbf{W}\cdot(\mathbf{H}\times\mathbf{n})\,d\Gamma + \int_{\Omega_C} \mathbf{W}\cdot\sigma\cdot\frac{\partial\mathbf{A}}{\partial t}\,d\Omega + \int_{\Omega_C} \mathbf{W}\cdot\sigma\cdot\nabla V\,d\Omega = 0 \quad (A7)$$

where $\mathbf{n}$ is the outward normal to the boundary Γ_C. Finally, reintroducing the definition of $\mathbf{H}$ and rearranging produces a weighted integral form of the magnetic potential equation

$$\int_{\Omega_C} \mathbf{W}\cdot\sigma\cdot\frac{\partial\mathbf{A}}{\partial t}\,d\Omega + \int_{\Omega_C} \nabla\times\mathbf{W}\cdot(\nu\cdot\nabla\times\mathbf{A})\,d\Omega = -\int_{\Omega_C} \mathbf{W}\cdot\sigma\cdot\nabla V\,d\Omega + \int_{\Gamma_C} \mathbf{W}\cdot(\nu\cdot\nabla\times\mathbf{A})\times\mathbf{n}\,d\Gamma \quad (A8)$$

As a final step, the natural boundary condition for the magnetic potential may be introduced. Note that the boundary integral in (A8) contains the boundary condition specified in equation (35). Making the appropriate substitution leads to the required form of the integral statement

$$\int_{\Omega_C} \mathbf{W}\cdot\sigma\cdot\frac{\partial\mathbf{A}}{\partial t}\,d\Omega + \int_{\Omega_C} \nabla\times\mathbf{W}\cdot(\nu\cdot\nabla\times\mathbf{A})\,d\Omega = -\int_{\Omega_C} \mathbf{W}\cdot\sigma\cdot\nabla V\,d\Omega + \int_{\Gamma_C} \mathbf{W}\cdot\mathbf{f}^{\mathbf{H}}\,d\Gamma \quad (A9)$$

The electric potential equation (A4) is transformed in a similar manner. Using the divergence theorem on the first two integrals in (A4) leads to

$$-\int_{\Omega_C} \nabla W\cdot(-\sigma\cdot\nabla V)\,d\Omega - \int_{\Omega_C} \nabla W\cdot\left(-\sigma\cdot\frac{\partial\mathbf{A}}{\partial t}\right)d\Omega = -\int_{\Gamma_C} W\left(-\sigma\cdot\frac{\partial\mathbf{A}}{\partial t} - \sigma\cdot\nabla V\right)\cdot\mathbf{n}\,d\Gamma \quad (A10)$$

This may be rearranged into a standard form as

$$\int_{\Omega_C} \nabla W\cdot\sigma\cdot\nabla V\,d\Omega + \int_{\Omega_C} \nabla W\cdot\sigma\cdot\frac{\partial\mathbf{A}}{\partial t}\,d\Omega = \int_{\Gamma_C} W\left(\sigma\cdot\frac{\partial\mathbf{A}}{\partial t} + \sigma\cdot\nabla V\right)\cdot\mathbf{n}\,d\Gamma \quad (A11)$$

which is a weighted integral form of the electric potential equation. Again, the natural boundary condition for the potential, equation (36), may be substituted into (A11) to obtain the final form of the weighted integral

$$\int_{\Omega_C} \nabla W\cdot\sigma\cdot\nabla V\,d\Omega + \int_{\Omega_C} \nabla W\cdot\sigma\cdot\frac{\partial\mathbf{A}}{\partial t}\,d\Omega = \int_{\Gamma_C} W f^J\,d\Gamma \quad (A12)$$

The weak or weighted integral forms of the field equations in (A9) and (A12) form the basis for the computational method. The region Ω_C is discretized into an assemblage of finite elements and the weighted integral statements are applied to each element. Within

each element the vector magnetic potential and scalar electric potential are approximated by expansions of the form

$$\mathbf{A}(\mathbf{x},t) = \Phi^T(\mathbf{x})\mathbf{A}_\mathbf{x}(t)\mathbf{e}_\mathbf{x} + \Phi^T(\mathbf{x})\mathbf{A}_\mathbf{y}(t)\mathbf{e}_\mathbf{y} + \Phi^T(\mathbf{x})\mathbf{A}_\mathbf{z}(t)\mathbf{e}_\mathbf{z} + \qquad (A13)$$

$$V(\mathbf{x},t) = \Psi^T(\mathbf{x})\mathbf{V}(t) \qquad (A14)$$

which are written here for the three-dimensional cartesian case with the $\mathbf{e}_\mathbf{i}$ being unit vectors for the coordinate system. Similar expressions can be constructed for the axisymmetric and two-dimensional cases. In (A13) and (A14) Φ and Ψ represent vectors of interpolation functions, $\mathbf{A}_\mathbf{i}$ and $\mathbf{V}$ are vectors of nodal point unknowns and superscript T indicates a vector transpose. Note that the assumed approximations for the dependent variables are semi-discrete with the spatial dependence being discretized through interpolation and the temporal dependence remaining continuous. For a Galerkin method the weight functions $\mathbf{W}$ and W are selected to be the same functions as used to represent the variables. That is

$$\mathbf{W}(\mathbf{x}) = \Phi(\mathbf{x})\mathbf{e}_\mathbf{x} + \Phi(\mathbf{x})\mathbf{e}_\mathbf{y} + \Phi(\mathbf{x})\mathbf{e}_\mathbf{z} \qquad (A15)$$

$$W(\mathbf{x}) = \Psi(\mathbf{x}) \qquad (A16)$$

Substituting the definitions in (A13)-(A16) into the weighted residual equations in (A9) and (A12) produces the following set of discrete equations for each element

$$\mathbf{M}\dot{\mathbf{A}} + \mathbf{K}\mathbf{A} + \mathbf{N}\mathbf{V} = \mathbf{F}_\mathbf{A} \qquad (A17)$$

and

$$\mathbf{N}^T\dot{\mathbf{A}} + \mathbf{L}\mathbf{V} = \mathbf{F}_\mathbf{V} \qquad (A18)$$

where the superposed dot indicates a time derivative and equation (A17) represents the three component equations for the magnetic potential.

The matrix system shown above is unsymmetric and is of an undesirable form from the standpoint of time integration. To restore symmetry, the following definition proposed by Chari, *et al.* [24] is employed

$$\mathbf{V} \equiv \frac{\partial \mathbf{v}}{\partial t} = \dot{\mathbf{v}} \qquad (A19)$$

Using this definition equations (A17) and (A18) can be rewritten as

$$\mathbf{M}\dot{\mathbf{A}} + \mathbf{K}\mathbf{A} + \mathbf{N}\dot{\mathbf{v}} = \mathbf{F}_\mathbf{A} \qquad (A20)$$

and

$$\mathbf{N}^T\dot{\mathbf{A}} + \mathbf{L}\dot{\mathbf{v}} = \mathbf{F}_\mathbf{V} \qquad (A21)$$

and in a completely assembled form

$$\begin{bmatrix} \mathbf{M} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{L} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{A}} \\ \dot{\mathbf{v}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{A} \\ \mathbf{v} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_\mathbf{A} \\ \mathbf{F}_\mathbf{V} \end{Bmatrix} \qquad (A22)$$

A general symbolic form for the system in (A22) can be written as

$$\overline{\mathbf{M}}\dot{\mathbf{U}} + \overline{\mathbf{K}}\mathbf{U} = \overline{\mathbf{F}} \tag{A23}$$

where

$$\mathbf{U}^T = \{\mathbf{A_1^T}, \mathbf{A_2^T}, \mathbf{A_3^T}, \mathbf{v^T}\}.$$

The component matrices defined in (A17)-(A22) are defined by the following integrals that arise from the weighted residual statements. The matrices are written here in terms of vector notation; the explicit forms of the matrices needed for computation are described in Appendix B.

$$\begin{aligned}
\mathbf{M} &= \int_{\Omega_e} \mathbf{\Phi} \cdot \sigma \cdot \mathbf{\Phi}^T \, d\Omega \\
\mathbf{K} &= \int_{\Omega_e} \nabla \times \mathbf{\Phi} \cdot \nu \cdot \nabla \times \mathbf{\Phi}^T \, d\Omega \\
\mathbf{N} &= \int_{\Omega_e} \mathbf{\Phi} \cdot \sigma \cdot \nabla \mathbf{\Psi}^T \, d\Omega \\
\mathbf{N}^T &= \int_{\Omega_e} \nabla \mathbf{\Psi} \cdot \sigma \cdot \mathbf{\Phi}^T \, d\Omega \\
\mathbf{L} &= \int_{\Omega_e} \nabla \mathbf{\Psi} \cdot \sigma \cdot \nabla \mathbf{\Psi}^T \, d\Omega \\
\mathbf{F_A} &= \int_{\Gamma_e} \mathbf{\Phi} \cdot \mathbf{f^H} \, d\Gamma \\
\mathbf{F_V} &= \int_{\Gamma_e} \mathbf{\Psi} f^J \, d\Gamma
\end{aligned} \tag{A24}$$

Dielectric and Conduction Region with Source Currents, Ω_D, Ω_J

The development of the finite element equations for dielectric regions or regions with specified currents follows the same procedure as outlined above but with a reduced set of partial differential equations. Since dielectric or free space regions and specified current regions differ only by the presence of a volumetric source, they may be treated as a single domain. The starting point for the development is equation (31) that describes the magnetic potential

$$\nabla \times (\nu \cdot \nabla \times \mathbf{A}) = \mathbf{J_0} \tag{A25}$$

Following the previously described procedure, the weighted residual statement corresponding to (A25) is

$$\int_{\Omega_C} \nabla \times \mathbf{W} \cdot (\nu \cdot \nabla \times \mathbf{A}) \, d\Omega = \int_{\Omega_C} \mathbf{W} \cdot \mathbf{J_0} \, d\Omega + \int_{\Gamma_C} \mathbf{W} \cdot \mathbf{f^H} \, d\Gamma \tag{A26}$$

which is analogous to (A9) for conduction regions. The finite element approximation for (A26) again makes use of the definitions in (A13) and (A15) and leads to the following matrix system

$$\mathbf{KA} = \mathbf{F_A} + \mathbf{F_J} \tag{A27}$$

Equation (A27) is the free space and source current equivalent of (A22). Note that for free space regions $\mathbf{F_J} = 0$ by definition while regions with specified currents will have $\mathbf{F_J}$ defined by

$$\mathbf{F_J} = \int_{\Omega_e} \Phi \cdot \mathbf{J_0}\, d\Omega \tag{A28}$$

Gauge Condition

When the magnetic potential must be made unique, a projection method is used to define a scalar correction variable. The gauge condition is given by equation (42)

$$\nabla \cdot \nabla S = -\nabla \cdot \mathbf{A}^* \tag{A29}$$

where $\mathbf{A}^*$ is the unconstrained magnetic potential. A weak form for (A29) is developed in the standard way by defining a scalar weight function W, and writing the following weighted residual statement

$$\int_\Omega W \nabla \cdot \nabla S\, d\Omega = -\int_\Omega W \nabla \cdot \mathbf{A}^*\, d\Omega \tag{A30}$$

Use of the divergence theorem in (A30) leads to

$$\int_\Omega \nabla W \cdot \nabla S\, d\Omega = -\int_\Omega W \nabla \cdot \mathbf{A}^*\, d\Omega + \int_\Gamma W f^S\, d\Gamma \tag{A31}$$

which is the required integral equation for the gauge condition. Note that the natural boundary condition for the variable S has been used in (A31); the boundary condition f^S specifies the gradient of S normal to the boundary.

A finite element approximation to (A31) follows again the previously defined procedure. The variable S is approximated by an expansion

$$S(\mathbf{x}, t) = \Theta^T(\mathbf{x})\mathbf{S}(t) \tag{A32}$$

Using the definition in (A13) for $\mathbf{A}^*$, the weighted residual statement in (A31) along with the Galerkin criteria, $W = \Theta$, produces the matrix equation

$$\mathbf{NS} = \mathbf{F_S} + \mathbf{F_{A^*}} \tag{A33}$$

where the components of (A33) are defined by

$$\begin{aligned} \mathbf{N} &= \int_{\Omega_e} \nabla\Theta \cdot \nabla\Theta^T\, d\Omega \\ \mathbf{F_S} &= \int_{\Gamma_e} \Theta f^S\, d\Gamma \\ \mathbf{F_{A^*}} &= \int_{\Gamma_e} \Theta \cdot \nabla\Phi^T\, d\Omega\, \mathbf{A}^* \end{aligned} \tag{A34}$$

Static Field Equations

The finite element equations for the simplified static field problems are developed in the same manner as outlined in the previous sections. For some situations, the equations are merely subsets of the more complex problems treated above.

Electrostatics

The basic equation for electrostatics is (43)

$$\nabla \cdot (\epsilon \cdot \nabla V) = -\rho \tag{A35}$$

This has a weak form that is similar to the gauge equation

$$\int_\Omega \nabla W \cdot \epsilon \cdot \nabla V \, d\Omega = -\int_\Omega W \rho \, d\Omega + \int_\Gamma W f^J \, d\Gamma \tag{A36}$$

where the natural boundary condition is a specification of the current (gradient of the potential) normal to the boundary. Let the electric potential be approximated by the steady form of the finite element representation given in (A14)

$$V(\mathrm{x}) = \Psi^T(\mathrm{x})\mathrm{V} \tag{A14}$$

Using the Galerkin definition in (A16), equation (A36) leads to the matrix equation

$$\mathbf{L}_\epsilon = \mathbf{F}_\rho + \mathbf{F}_\mathbf{V} \tag{A37}$$

where

$$\mathbf{L}_\epsilon = \int_{\Omega_e} \nabla \Psi \cdot \epsilon \cdot \nabla \Psi^T \, d\Omega$$

$$\mathbf{F}_\rho = \int_{\Gamma_e} \Psi \rho \, d\Omega \tag{A38}$$

$$\mathbf{F}_\mathbf{V} = \int_{\Gamma_e} \Psi f^J \, d\Gamma$$

Steady Current Flow

The steady current flow problem is a subset of the general eddy current formulation and can bedefined immediately as

$$\mathbf{LV} = \mathbf{F}_\mathbf{V} \tag{A39}$$

where the matrix and vector are defined in (A24).

Magnetostatics

The magnetostatics problem is also a subset of the general eddy current problem and corresponds to the equations for a free space region with specified currents. The relevant finite element equations are given by (A27)

$$\mathbf{KA} = \mathbf{F_A} + \mathbf{F_J} \qquad (A40)$$

Appendix B - Matrix Equations for Quasi-Static Electromagnetics

In this section the matrix components of the eddy current equations are presented for the case of a Cartesian coordinate system in three dimensions. A few matrices are also given for the reduced dimensionality case of an axisymmetric geometry, since these have a form that is somewhat different than the planar case. In all situations the general anisotropic material models are employed; the isotropic case follows by a straightforward elimination of off-diagonal tensor components. Also, the matrices associated with static problems and the time-harmonic approximation will not be presented separately, since they can be obtained easily from the following basic forms. The component matrices defined here are derived from equation (A24) in the previous appendix. Note that derivatives are expressed in terms of the global x, y, z coordinates, which for purposes of computation would be converted to local element coordinates via the isoparametric transformation given in equuation (93).

Magnetic Mass Matrix

The magnetic mass matrix is defined by

$$\mathbf{M} = \int_{\Omega_e} \Phi \cdot \sigma \cdot \Phi^T \, d\Omega \tag{B1}$$

which has components

$$\begin{aligned}
\mathbf{M_{xx}} &= \int_{\Omega_e} \Phi \sigma_{xx} \Phi^T \, d\Omega \qquad \mathbf{M_{xy}} = \int_{\Omega_e} \Phi \sigma_{xy} \Phi^T \, d\Omega \qquad \mathbf{M_{xz}} = \int_{\Omega_e} \Phi \sigma_{xz} \Phi^T \, d\Omega \\
\mathbf{M_{yx}} &= \int_{\Omega_e} \Phi \sigma_{yx} \Phi^T \, d\Omega \qquad \mathbf{M_{yy}} = \int_{\Omega_e} \Phi \sigma_{yy} \Phi^T \, d\Omega \qquad \mathbf{M_{yz}} = \int_{\Omega_e} \Phi \sigma_{yz} \Phi^T \, d\Omega \\
\mathbf{M_{zx}} &= \int_{\Omega_e} \Phi \sigma_{zx} \Phi^T \, d\Omega \qquad \mathbf{M_{zy}} = \int_{\Omega_e} \Phi \sigma_{zy} \Phi^T \, d\Omega \qquad \mathbf{M_{zz}} = \int_{\Omega_e} \Phi \sigma_{zz} \Phi^T \, d\Omega
\end{aligned} \tag{B2}$$

Magnetic Diffusion Matrix

The magnetic diffusion or double curl matrix is defined by

$$\mathbf{K} = \int_{\Omega_e} \nabla \times \Phi \cdot \nu \cdot \nabla \times \Phi^T \, d\Omega \tag{B3}$$

which has components

$$\mathbf{K_{xx}} = \int_{\Omega_e} \frac{\partial \Phi}{\partial y} \nu_{zz} \frac{\partial \Phi^T}{\partial y} + \frac{\partial \Phi}{\partial z} \nu_{yy} \frac{\partial \Phi^T}{\partial z} - \frac{\partial \Phi}{\partial z} \nu_{yz} \frac{\partial \Phi^T}{\partial y} - \frac{\partial \Phi}{\partial y} \nu_{zy} \frac{\partial \Phi^T}{\partial z} \, d\Omega$$

$$
\begin{aligned}
\mathbf{K_{xy}} &= \int_{\Omega_e} -\frac{\partial\Phi}{\partial y}\nu_{zz}\frac{\partial\Phi^T}{\partial x} + \frac{\partial\Phi}{\partial z}\nu_{yz}\frac{\partial\Phi^T}{\partial y} - \frac{\partial\Phi}{\partial z}\nu_{yx}\frac{\partial\Phi^T}{\partial z} + \frac{\partial\Phi}{\partial y}\nu_{zx}\frac{\partial\Phi^T}{\partial z}\,d\Omega \\
\mathbf{K_{xz}} &= \int_{\Omega_e} -\frac{\partial\Phi}{\partial z}\nu_{yy}\frac{\partial\Phi^T}{\partial x} + \frac{\partial\Phi}{\partial z}\nu_{yx}\frac{\partial\Phi^T}{\partial y} - \frac{\partial\Phi}{\partial y}\nu_{zx}\frac{\partial\Phi^T}{\partial y} + \frac{\partial\Phi}{\partial y}\nu_{zy}\frac{\partial\Phi^T}{\partial x}\,d\Omega \\
\mathbf{K_{yx}} &= \int_{\Omega_e} -\frac{\partial\Phi}{\partial x}\nu_{zz}\frac{\partial\Phi^T}{\partial y} + \frac{\partial\Phi}{\partial x}\nu_{zy}\frac{\partial\Phi^T}{\partial z} - \frac{\partial\Phi}{\partial z}\nu_{xy}\frac{\partial\Phi^T}{\partial z} + \frac{\partial\Phi}{\partial z}\nu_{xz}\frac{\partial\Phi^T}{\partial y}\,d\Omega \\
\mathbf{K_{yy}} &= \int_{\Omega_e} \frac{\partial\Phi}{\partial z}\nu_{xx}\frac{\partial\Phi^T}{\partial z} + \frac{\partial\Phi}{\partial x}\nu_{zz}\frac{\partial\Phi^T}{\partial x} - \frac{\partial\Phi}{\partial z}\nu_{xz}\frac{\partial\Phi^T}{\partial x} - \frac{\partial\Phi}{\partial x}\nu_{zx}\frac{\partial\Phi^T}{\partial z}\,d\Omega \qquad (B4) \\
\mathbf{K_{yz}} &= \int_{\Omega_e} -\frac{\partial\Phi}{\partial z}\nu_{xx}\frac{\partial\Phi^T}{\partial y} + \frac{\partial\Phi}{\partial x}\nu_{zx}\frac{\partial\Phi^T}{\partial y} - \frac{\partial\Phi}{\partial x}\nu_{zy}\frac{\partial\Phi^T}{\partial x} + \frac{\partial\Phi}{\partial z}\nu_{xy}\frac{\partial\Phi^T}{\partial x}\,d\Omega \\
\mathbf{K_{zx}} &= \int_{\Omega_e} -\frac{\partial\Phi}{\partial x}\nu_{yy}\frac{\partial\Phi^T}{\partial z} + \frac{\partial\Phi}{\partial y}\nu_{xy}\frac{\partial\Phi^T}{\partial z} - \frac{\partial\Phi}{\partial y}\nu_{xz}\frac{\partial\Phi^T}{\partial y} + \frac{\partial\Phi}{\partial x}\nu_{yz}\frac{\partial\Phi^T}{\partial y}\,d\Omega \\
\mathbf{K_{zy}} &= \int_{\Omega_e} -\frac{\partial\Phi}{\partial y}\nu_{xx}\frac{\partial\Phi^T}{\partial z} + \frac{\partial\Phi}{\partial y}\nu_{xz}\frac{\partial\Phi^T}{\partial x} - \frac{\partial\Phi}{\partial x}\nu_{yz}\frac{\partial\Phi^T}{\partial x} + \frac{\partial\Phi}{\partial x}\nu_{yx}\frac{\partial\Phi^T}{\partial z}\,d\Omega \\
\mathbf{K_{zz}} &= \int_{\Omega_e} \frac{\partial\Phi}{\partial y}\nu_{xx}\frac{\partial\Phi^T}{\partial y} + \frac{\partial\Phi}{\partial x}\nu_{yy}\frac{\partial\Phi^T}{\partial x} - \frac{\partial\Phi}{\partial y}\nu_{xy}\frac{\partial\Phi^T}{\partial x} - \frac{\partial\Phi}{\partial x}\nu_{yx}\frac{\partial\Phi^T}{\partial y}\,d\Omega
\end{aligned}
$$

The components of the diffusion matrix can be written in cylindrical coordinates using the standard definitions of the curl operator. The result is a series of terms that are analogous to the components of equation (B4). The utility of the cylindrical form is primarily for use in problems with reduced dimensionality, *i.e.*, axisymmetric geometries. The type of axisymmetric problem considered depends on the orientation of the current and which components of the magnetic vector potential are active. When the problem is independent of the azimuthal coordinate, θ, two components of the magnetic potential are required to describe the field, $A_r(r,z)$ and $A_z(r,z)$. The components of the diffusion matrix for this case are

$$
\begin{aligned}
\mathbf{K_{rr}} &= \int_{\Omega_e} \frac{\partial\Phi}{\partial z}\nu_{\theta\theta}\frac{\partial\Phi^T}{\partial z}\,d\Omega \qquad \mathbf{K_{rz}} = \int_{\Omega_e} -\frac{\partial\Phi}{\partial z}\nu_{\theta\theta}\frac{\partial\Phi^T}{\partial r}\,d\Omega \\
\mathbf{K_{zr}} &= \int_{\Omega_e} -\frac{\partial\Phi}{\partial r}\nu_{\theta\theta}\frac{\partial\Phi^T}{\partial z}\,d\Omega \qquad \mathbf{K_{zz}} = \int_{\Omega_e} \frac{\partial\Phi}{\partial r}\nu_{\theta\theta}\frac{\partial\Phi^T}{\partial r}\,d\Omega \qquad (B5)
\end{aligned}
$$

Note that only one component of the reluctivity tensor is utilized in these terms and the problem is therefore isotropic. When the problem geometry is such that only the azimuthal component of the vector potential is active (*e.g.*, currents are in the azimuthal direction and are invariant with this coordinate) then the matrix components of the diffusion operator are limited to

$$
\begin{aligned}
\mathbf{K_{\theta\theta}} &= \int_{\Omega_e} \frac{\partial\Phi}{\partial r}\nu_{zz}\frac{\partial\Phi^T}{\partial r} + \frac{\partial\Phi}{\partial z}\nu_{rr}\frac{\partial\Phi^T}{\partial z}\,d\Omega \\
&+ \int_{\Omega_e} \nu_{zz}\frac{\Phi\Phi}{r^2} + \nu_{zz}\frac{\Phi}{r}\frac{\partial\Phi^T}{\partial r} + \nu_{zz}\frac{\partial\Phi}{\partial r}\frac{\Phi^T}{r}\,d\Omega \qquad (B6)
\end{aligned}
$$

$$-\int_{\Omega_e} \frac{\partial \Phi}{\partial z}\nu_{rz}\frac{\partial \Phi^T}{\partial r} + \frac{\partial \Phi}{\partial r}\nu_{zr}\frac{\partial \Phi^T}{\partial z} + \frac{\partial \Phi}{\partial z}\nu_{rz}\frac{\Phi^T}{r} + \frac{\Phi}{r}\nu_{zr}\frac{\partial \Phi^T}{\partial z}\, d\Omega$$

Here four components of the reluctivity tensor are present and the problem can be anisotropic in the $r-z$ plane.

Electric-Magnetic Coupling Matrices

The coupling matrices between the electric and magnetic field variables are defined by

$$\mathbf{N} = \int_{\Omega_e} \Phi \cdot \sigma \cdot \nabla \Psi^T \, d\Omega \qquad (B7)$$

and

$$\mathbf{N}^T = \int_{\Omega_e} \nabla \Psi \cdot \sigma \cdot \Phi^T \, d\Omega \qquad (B8)$$

which has components

$$\begin{aligned}
\mathbf{N_{xx}} &= \int_{\Omega_e} \Phi\sigma_{xx}\frac{\partial \Psi^T}{\partial x}\, d\Omega \qquad \mathbf{N_{xy}} = \int_{\Omega_e} \Phi\sigma_{xy}\frac{\partial \Psi^T}{\partial y}\, d\Omega \qquad \mathbf{N_{xz}} = \int_{\Omega_e} \Phi\sigma_{xz}\frac{\partial \Psi^T}{\partial z}\, d\Omega \\
\mathbf{N_{yx}} &= \int_{\Omega_e} \Phi\sigma_{yx}\frac{\partial \Psi^T}{\partial x}\, d\Omega \qquad \mathbf{N_{yy}} = \int_{\Omega_e} \Phi\sigma_{yy}\frac{\partial \Psi^T}{\partial y}\, d\Omega \qquad \mathbf{N_{yz}} = \int_{\Omega_e} \Phi\sigma_{yz}\frac{\partial \Psi^T}{\partial z}\, d\Omega \qquad (B9) \\
\mathbf{N_{zx}} &= \int_{\Omega_e} \Phi\sigma_{zx}\frac{\partial \Psi^T}{\partial x}\, d\Omega \qquad \mathbf{N_{zy}} = \int_{\Omega_e} \Phi\sigma_{zy}\frac{\partial \Psi^T}{\partial y}\, d\Omega \qquad \mathbf{N_{zz}} = \int_{\Omega_e} \Phi\sigma_{zz}\frac{\partial \Psi^T}{\partial z}\, d\Omega
\end{aligned}$$

The component of $\mathbf{N}^T$ are defined by the transpose of the above components.

Electric Mass Matrix

The electric mass matrix is defined by

$$\mathbf{L} = \int_{\Omega_e} \nabla \Psi \cdot \sigma \cdot \nabla \Psi^T \, d\Omega \qquad (B10)$$

which has components

$$\begin{aligned}
\mathbf{L_{xx}} &= \int_{\Omega_e} \frac{\partial \Psi}{\partial x}\sigma_{xx}\frac{\partial \Psi^T}{\partial x}\, d\Omega \qquad \mathbf{L_{xy}} = \int_{\Omega_e} \frac{\partial \Psi}{\partial x}\sigma_{xy}\frac{\partial \Psi^T}{\partial y}\, d\Omega \qquad \mathbf{L_{xz}} = \int_{\Omega_e} \frac{\partial \Psi}{\partial x}\sigma_{xz}\frac{\partial \Psi^T}{\partial z}\, d\Omega \\
\mathbf{L_{yx}} &= \int_{\Omega_e} \frac{\partial \Psi}{\partial y}\sigma_{yx}\frac{\partial \Psi^T}{\partial x}\, d\Omega \qquad \mathbf{L_{yy}} = \int_{\Omega_e} \frac{\partial \Psi}{\partial y}\sigma_{yy}\frac{\partial \Psi^T}{\partial y}\, d\Omega \qquad \mathbf{L_{yz}} = \int_{\Omega_e} \frac{\partial \Psi}{\partial y}\sigma_{yz}\frac{\partial \Psi^T}{\partial z}\, d\Omega \\
\mathbf{L_{zx}} &= \int_{\Omega_e} \frac{\partial \Psi}{\partial z}\sigma_{zx}\frac{\partial \Psi^T}{\partial x}\, d\Omega \qquad \mathbf{L_{zy}} = \int_{\Omega_e} \frac{\partial \Psi}{\partial z}\sigma_{zy}\frac{\partial \Psi^T}{\partial y}\, d\Omega \qquad \mathbf{L_{zz}} = \int_{\Omega_e} \frac{\partial \Psi}{\partial z}\sigma_{zz}\frac{\partial \Psi^T}{\partial z}\, d\Omega
\end{aligned} \qquad (B11)$$

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