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ANALYTIC SECOND- AND THIRD-ORDER ACHROMAT DESIGNS*

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I. INTRODUCTION

An achromat is a transport system that carries a beam without distorting its transverse phase space distribution. In this study, we apply the Lie algebraic technique [1-6] to a repetitive FODO array to make it either a second-order or a third-order achromat. (Achromats based on reflection symmetries [7,8] are not studied here.) We consider third-order achromats whose unit FODO cell layout is shown in Fig. 1. The second-order achromat layout is the same, except the octupoles are absent.

For the second-order achromats, correction terms (due to the finite bending of the dipoles) to the well-known formulae for the sextupole strengths are derived. For the third-order achromats, analytic expressions for the five octupole strengths are given. The quadrupole, sextupole and octupole magnets are assumed to be thin-lens elements. The dipoles are assumed to be sector magnets filling the drift spaces. More details of the analysis have been reported elsewhere.[9] We thank Y. Yan, H. Ye, J. Irwin and A. Dragt for their help.

II. ANALYSIS

We first calculate the Lie maps of each of the magnet elements. The map for a magnet element of length L is gven by $e^{-L:H:}$, where H is the Hamiltonian of the element. For a particle with $\delta = \Delta P/P_0$, we use (we ignore the path-length dynamics)

$$\text{thin quadrupole: } HL = \frac{1}{2F_k}\,(x^2-y^2)(1-\delta+\delta^2)$$

$$\text{thin sextupole: } HL = \frac{S_k}{3}\,(x^3-3xy^2)(1-\delta)$$

$$\text{thin octupole: } HL = \frac{O_k}{4}\,(x^4-6x^2y^2+y^4)$$

$$\text{sector dipole: } H = \frac{P_x^2+P_y^2}{2} + \frac{x^2}{2R^2} - \frac{x\delta}{R} + \frac{x(P_x^2+P_y^2)}{2R} - \frac{x^2\delta}{2R^2} + \frac{x\delta^2}{R} + \frac{(P_x^2+P_y^2)^2}{8} - \frac{x\delta^3}{R} + \frac{x^2\delta^2}{2R^2}\,, \qquad (1)$$

where R is the bending radius; F_k is the focal length of the k-th quadrupole; S_k and O_k are the k-th integrated sextupole and octupole strengths. Fringe fields are ignored.

Given the Hamiltonian H of an element, we factorize the element map as

$$e^{-L:H:} = e^{:H_2+H_3+H_4+\cdots:} = e^{:f_2:}e^{:f_3:}e^{:f_4:}e^{:O(X^5):}\,, \qquad (2)$$

where H_n and f_n are polynomials of order n in the variables $X = (x, P_x, y, P_y, \delta)$. We performed this factorization [3,5] and obtained

thin quadrupole:

$$f_3 = \frac{1}{2F_k}\,(x^2-y^2)\delta, \quad f_4 = -\frac{1}{2F_k}\,(x^2-y^2)\delta^2$$

thin sextupole:

$$f_3 = -\frac{S_k}{3}\,(x^3-3xy^2), \quad f_4 = \frac{S_k}{3}\,(x^3-3xy^2)\delta$$

thin octupole: $f_3 = 0$, $\quad f_4 = -\frac{O_k}{4}\,(x^4-6x^2y^2+y^4)$

sector dipole:

$$\begin{aligned}
f_3 = &-\frac{1}{6R^2}\sin^3\frac{L}{R}\,x^3 - \frac{1}{4R}\sin\frac{L}{R}\sin\frac{2L}{R}\,x^2P_x \\
&-\frac{1}{4}\cos\frac{L}{R}\sin\frac{2L}{R}\,xP_x^2 + \frac{R}{6}\,(1-\cos^3\frac{L}{R})\,P_x^3 \\
&-\frac{1}{2}\sin\frac{L}{R}\,xP_y^2 + \frac{x^2\delta}{2R}\sin\frac{L}{R}\,(\cos\frac{L}{R}+\sin^2\frac{L}{R}) \\
&+R\sin^2\frac{L}{2R}\,P_xP_y^2 - 2\sin^2\frac{L}{2R}\sin^2\frac{L}{R}\,xP_x\delta \\
&-\frac{R}{2}\sin^2\frac{L}{2R}\sin\frac{2L}{R}\,P_x^2\delta - \frac{1}{2}\,(L-R\sin\frac{L}{R})\,P_y^2\delta \\
&-\frac{x\delta^2}{2}\,(\sin^3\frac{L}{R}+\sin\frac{2L}{R}) + \frac{R}{2}\,(2-\cos\frac{L}{R})\,\sin^2\frac{L}{R}\,P_x\delta^2 \\
&+\frac{1}{12}\,(-6L+2R\sin^3\frac{L}{R}+3R\sin\frac{2L}{R})\,\delta^3\,,
\end{aligned}$$

$$\begin{aligned}
f_4 = &\left[-\frac{x^2}{8R}\sin^3\frac{L}{R} - \frac{xP_x}{8}\sin\frac{L}{R}\sin\frac{2L}{R}\right. \\
&\left.-\frac{R}{8}\cos^2\frac{L}{R}\sin\frac{L}{R}\,P_x^2 - \frac{R}{8}\sin\frac{L}{R}\,P_y^2\right](P_x^2+P_y^2) \\
&+\frac{x^3\delta}{12R^2}\sin^3\frac{L}{R} + (\frac{1}{2}+\cos\frac{L}{R})\,\sin^2\frac{L}{2R}\sin\frac{L}{R}\,xP_x^2\delta \\
&+\left[\frac{R}{12}\,(3+4\cos\frac{L}{R}+5\cos\frac{2L}{R})\,P_x^3\delta\right. \\
&\left.+\frac{xP_y^2\delta}{4}\sin\frac{2L}{R} + \frac{R}{4}\,(3+\cos\frac{2L}{R})\,P_xP_y^2\delta\right]\sin^2\frac{L}{2R} \\
&-\frac{1}{4R}\,(\sin^3\frac{L}{R}+\sin\frac{2L}{R})\,x^2\delta^2 + \frac{1}{2}\sin^2\frac{L}{R}\,xP_x\delta^2 \\
&-\frac{R}{4}\,(1+3\cos\frac{L}{R})\,\sin^2\frac{L}{2R}\sin\frac{L}{R}\,P_x^2\delta^2 \\
&+\left[\frac{R}{2}\sin^4\frac{L}{2R}P_y^2 + (\cos\frac{L}{R}+\frac{1}{4}\sin^2\frac{L}{R})\,x\delta\right]\sin\frac{L}{R}\delta^2 \\
&-\frac{R}{2}\sin^2\frac{L}{R}\,P_x\delta^3 + \frac{\delta^4}{12}\,(6L-R\sin^3\frac{L}{R}-3R\sin\frac{2L}{R})\,.
\end{aligned} \qquad (3)$$

Having factorized the maps of all magnets, the total map $\mathcal{M}_{\text{cell}}$ of a cell is obtained by multiplying and concatenating the maps of the component elements [3,9]:

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$$\mathcal{M}_{\rm cell} = \prod_{i=1}^{N}(e^{:f_2^i:}e^{:f_3^i:}e^{:f_4^i:}) = e^{:h_2:}e^{:h_3:}e^{:h_4:}e^{:\mathcal{O}(X^5):} , \quad (4)$$

where

$$\mathcal{R} = e^{:h_2:} = \prod_{i=1}^{N} e^{:f_2^i:} , \qquad h_3 = \sum_{i=1}^{N} \tilde{f}_3^i$$
$$h_4 = \sum_{i=1}^{N} \tilde{f}_4^i + \frac{1}{2}\sum_{j>i=1}^{N} [\tilde{f}_3^i, \tilde{f}_3^j] . \quad (5)$$

In Eq.(5), $\tilde{f}^i$ means $\tilde{f}^i(X) = f^i(R_{N\to i}X)$ with $R_{N\to i}$ the linear map from the last element to the i-th element. The map of the N-cell achromat is $\mathcal{M} = \mathcal{M}_{\rm cell}^N$. The number of cells N is so that $\mu_{x,y}$ (the total phase advances in x are y) are both multiples of 2π, but avoid resonances.

We now make a canonical coordinate transformation from (x, P_x, y, P_y) to $(\phi_x, A_x, \phi_y, A_y)$ by $x = \sqrt{2A_x\beta_x}\sin\phi_x + \eta\delta$, $P_x = \sqrt{\frac{2A_x}{\beta_x}}(\cos\phi_x - \alpha_x\sin\phi_x) + \eta'\delta$, and similarly for y and P_y without the η and η' terms, where $\beta_{x,y}, \alpha_{x,y}$ and η, η' are the Courant-Snyder and the dispersion functions [10]. The linear map generator h_2 becomes $h_2 = -\mu_x A_x - \mu_y A_y - \frac{1}{2}\bar{\alpha}_c\delta^2$ where and $\bar{\alpha}_c$ is the momentum compaction factor. We then decompose h_n in terms of the eigenmodes of $:h_2:$ as [5]

$$h_n = \sum_{a+b+c+d+e=n} C^n_{abcd,e}|abcd, e\rangle ,$$
$$|abcd, e\rangle \equiv \; .A_x^{(a+b)/2}A_y^{(c+d)/2}e^{i(a-b)\phi_x}e^{i(c-d)\phi_x}\delta^e \quad (6)$$

To reduce a nonlinear map to its normal form, it can be shown [11] that (in the absence of resonances) [2] all the non-secular terms can be transformed away via a symplectic similarity transformation leaving only terms with $a = b$ and $c = d$; i.e., terms depending on A_x, A_y and δ only. In particular, we have

$$h_3 = C^3_{1100,1}A_x\delta + C^3_{0011,1}A_y\delta + C^3_{0000,3}\delta^3 ,$$
$$h_4 = C^4_{2200,0}A_x^2 + C^4_{0022,0}A_y^2 + C^4_{1111,0}A_xA_y$$
$$+C^4_{1100,2}A_x\delta^2 + C^4_{0011,2}A_y\delta^2 + C^4_{0000,4}\delta^4 . \quad (7)$$

III. SECOND-ORDER ACHROMATS

For a second-order achromat, we follow Eqs. (6-7) and find the normal form of the unit cell is given by h_3 of Eq. (7) where

$$C^3_{1100,1} = \sum_{k=1,2}^{\rm quads}\left[\frac{1}{2F_k} - \lambda_k\eta(k)\right]\beta_x(k) + w_x ,$$
$$C^3_{0011,1} = -\sum_{k=1,2}^{\rm quads}\left[\frac{1}{2F_k} - \lambda_k\eta(k)\right]\beta_y(k) + w_y , \quad (8)$$

and

$$w_x = \sum_{k=1,2}^{\rm dipoles}\frac{1}{2}\sin^2(\frac{L}{R})\Big\{\frac{\beta_x(k)}{R}\Big[\sin\frac{L}{R} + \cot\frac{L}{R}$$
$$-\frac{\eta(k)}{R}\sin\frac{L}{R} - \eta'(k)\cos\frac{L}{R}\Big] + 2\alpha_x(k)\Big[1 - \cos\frac{L}{R}$$
$$+\frac{\eta(k)}{R}\cos\frac{L}{R} + \eta'(s)\cos\frac{L}{R}\cot\frac{L}{R}\Big]$$
$$+\gamma_x(k)R\Big[-\cos\frac{L}{R}\tan\frac{L}{2R} - \frac{\eta(k)}{R}\cos\frac{L}{R}\cot\frac{L}{R}$$
$$+(\cos\frac{L}{R} + \frac{1}{2}\sec^2\frac{L}{2R})\eta'(k)\Big]\Big\} ,$$
$$w_y = \sum_{k=1,2}^{\rm dipoles}\frac{1}{2}\gamma_y(k)R\Big[\sin\frac{L}{R} - \frac{L}{R} - \eta(k)\sin\frac{L}{R}$$
$$+\eta'(k)(1 - \cos\frac{L}{R})\Big] . \quad (9)$$

The lattice functions are evaluated at the two quadrupoles in Eq. (8) and at the ends of the two dipoles in Eq. (9). In the limit of weak bending with $\epsilon_1 = \frac{L}{R} \ll 1$, we have

$$w_x \simeq \epsilon_1\sum_s^D \alpha_x(s)\eta'(s) + \frac{1}{4}\gamma_x(s)(3L\eta'(s) - 2\eta(s)) ,$$
$$w_y \simeq \epsilon_1\sum_s^D \frac{1}{4}\gamma_y(s)(L\eta'(s) - 2\eta(s)) . \quad (10)$$

To form a second-order achromat, we set the two C-coefficients to zero, and obtain

$$S_1 = \frac{1}{2\eta(1)F_1} + \frac{\beta_y(2)w_x + \beta_x(2)w_y}{\eta(1)[\beta_x(1)\beta_y(2) - \beta_x(2)\beta_y(1)]} ,$$
$$S_2 = \frac{1}{2\eta(2)F_2} - \frac{\beta_y(1)w_x + \beta_x(1)w_y}{\eta(2)[\beta_x(1)\beta_y(2) - \beta_x(2)\beta_y(1)]} . \quad (11)$$

The first terms usually dominate and give the well known results. The correction terms with w_x and w_y are normally, but not always, small.

IV. THIRD-ORDER ACHROMATS

We also studied the case of a third-order achromat. An algebraic program using Mathematica was developed to do the analysis. Here, we only report our results. The normal form of the third-order generator for a unit cell is given by Eq.(9) with

$$C^4_{2200,0} = -\frac{3}{8}\sum_{k=1}^{5}\beta_x(k)^2O_k + w_{xx} ,$$
$$C^4_{1111,0} = \frac{3}{2}\sum_{k=1}^{5}\beta_x(k)\beta_y(k)O_k + w_{xy} ,$$
$$C^4_{0022,0} = -\frac{3}{8}\sum_{k=1}^{5}\beta_y(k)^2O_k + w_{yy} ,$$
$$C^4_{1100,2} = -\frac{3}{2}\sum_{k=1}^{5}\beta_x(k)\eta(k)O_k + w_{xd} ,$$
$$C^4_{0011,2} = \frac{3}{2}\sum_{k=1}^{5}\beta_y(k)\eta(k)O_k + w_{yd} , \quad (12)$$

and (when $\epsilon_1 = \frac{L}{R} \ll 1$)

$$w_{xx} \simeq \csc\frac{3\mu_x}{2}(2 + 3\cos\mu_x)\prod_s^S\frac{S_s}{4}\beta_x(s)^{\frac{3}{2}} - \frac{3}{16}\frac{L}{}\sum_s^D\gamma_x(s)^2$$

$$+\frac{1}{8}\csc\frac{3\mu_x}{2}(3\cos\frac{\mu_x}{2}+2\cos\frac{3\mu_x}{2})\sum_s^S S_s^2\beta_x(s)^3 \ ,$$

$$w_{xy} \simeq -\frac{L}{4}\sum_s^D \gamma_x(s)\gamma_y(s) \ - \ \frac{1}{2}\cot\frac{\mu_x}{2}\sum_s^S S_s^2\beta_x(s)^2\beta_y(s)$$

$$-\csc(\frac{\mu_x}{2}+\mu_y)\csc(\frac{\mu_x}{2}-\mu_y)\sin 2\mu_y\sum_s^S \frac{S_s^2}{4}\beta_x(s)\beta_y(s)^2$$

$$+\left[\csc(\frac{\mu_x}{2}+\mu_y)-\csc(\frac{\mu_x}{2}-\mu_y)-\csc\frac{\mu_x}{2}\sum_s^S\frac{\beta_x(s)}{\beta_y(s)}\right]$$

$$\times\frac{1}{2}\prod_s^S S_s\sqrt{\beta_x(s)}\beta_y(s) \ ,$$

$$w_{yy} \simeq -\frac{3\,L}{16}\sum_s^D \gamma_y^2(s) \ + \ \frac{1}{16}\sum_s^S S_s^2\beta_x(s)\beta_y(s)^2$$

$$\times\left[4\cot\frac{\mu_x}{2}+\sin\mu_x\csc(\frac{\mu_x}{2}+\mu_y)\csc(\frac{\mu_x}{2}-\mu_y)\right]$$

$$+\frac{1}{8}\left[4\csc\frac{\mu_x}{2}+\csc(\frac{\mu_x}{2}+\mu_y)+\csc(\frac{\mu_x}{2}-\mu_y)\right]$$

$$\times\prod_s^S S_s\sqrt{\beta_x(s)}\beta_y(s) \ ,$$

$$w_{xd} \simeq -\frac{3\,L}{4}\sum_s^D \gamma_x(s)\eta'(s)^2 \ - \ \sum_s^S \beta_x(s)(\frac{1}{2F_s}-S_s\beta_x(s))$$

$$+\frac{1}{2}\cot\mu_x\sum_s^S\left[\beta_x(s)(\frac{1}{2F_s}-S_s\beta_x(s))\right]^2$$

$$+\csc\mu_x\prod_s^S\beta_x(s)(\frac{1}{2F_s}-S_s\beta_x(s))$$

$$+\csc\frac{\mu_x}{2}\sum_{1,2}^S\frac{S_2}{2}\beta_x(2)\eta(1)(S_1\eta(1)-\frac{1}{F_1})\sqrt{\beta_x(1)\beta_x(2)}$$

$$+\frac{1}{2}\cot\frac{\mu_x}{2}\sum_s^S S_s\eta(s)(S_s\eta(s)-\frac{1}{F_s})\beta_x(s)^2 \ ,$$

$$w_{yd} \simeq -\frac{L}{4}\sum_s^D \gamma_y(s)\eta'(s)^2 \ + \ \sum_s^S \beta_y(s)(\frac{1}{2F_s}-S_s\beta_x(s))$$

$$+\frac{1}{2}\cot\mu_y\sum_s^S\left[\beta_y(s)(\frac{1}{2F_s}-S_s\beta_x(s))\right]^2$$

$$+\csc\mu_y\prod_s^S\beta_y(s)(\frac{1}{2F_s}-S_s\beta_x(s))$$

$$-\csc\frac{\mu_x}{2}\sum_{1,2}^S\frac{S_2}{2}\beta_y(2)\eta(1)(S_1\eta(1)-\frac{1}{F_1})\sqrt{\beta_x(1)\beta(2)}$$

$$-\frac{1}{2}\cot\frac{\mu_x}{2}\sum_s^S S_s\eta(s)(S_s\eta(s)-\frac{1}{F_s})\beta_x(s)\beta_y(s) \ . \tag{13}$$

Exact expressions of the w-coefficients are too lengthy to be included here.

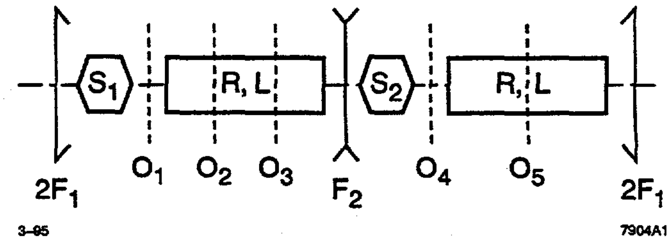

Figure. 1. Unit cell of an achromat layout.

The required octupole strengths are such that the five C-coefficients in Eq. (12) are equal to zero. For the case when two of the octupoles are located next to the two sextupoles and the other three are at the $\frac{1}{3}$, $\frac{2}{3}$, and the $\frac{1}{2}$ locations of the two bending magnets, we find

$$
\begin{aligned}
&O_1 \simeq \frac{a+b}{6f^3D}, \quad O_2 \simeq \frac{81(c+d)}{2fD} \ , \quad O_3 \simeq \frac{81(c-d)}{2fD} \\
&O_4 \simeq \frac{a-b}{6f^3D} \ , \quad O_5 \simeq \frac{128e}{3(2f^2-1)D} \ , \\
&a = 2f(1360-22846f^2-74476f^4+695809f^6 \\
&\quad -1438146f^8+1200096f^{10}-326592f^{12}) \ , \\
&b = -352-3360f^2+233290f^4-1070910f^6 \\
&\quad +1917603f^8-1364850f^{10}+361584f^{12} \ , \\
&c = 6f(-42+1076f^2-7409f^4+16306f^6-14368f^8 \\
&\quad +4032f^{10}) \ , \\
&d = 8-394f^2+5322f^4-16907f^6+14866f^8-4464f^{10} \ , \\
&e = -368+10536f^2-92342f^4+307222f^6-470547f^8 \\
&\quad +330642f^{10}-81648f^{12} \ , \\
&D = (4f^2-1)^2(3f^2-4)(10-173f^2-261f^4+324f^6)L^3\epsilon_1^2 \ .
\end{aligned} \tag{14}
$$

We have defined the dimensionless parameter $f=\frac{2F_1}{L}$ and have assumed that $\epsilon_1=\frac{L}{R}\ll 1$ and $|\frac{F_1+F_2}{L}|\ll 1$.

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