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A THEORETICAL AND NUMERICAL STUDY OF THE FLOW OF GRANUALAR MATERIALS

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SUMMARY

In this report the linearized stability equations for the flow of granular materials down an inclined plane are derived for a continuum model [cf. Goodman and Cowin (1971), Rajagopal and Massoudi (1990)]. The basic flow exhibits multiplicity of solutions, one in which the volume fraction decreases monotonically from the inclined plane to the free surface, and the other in which the volume fraction increases monotonically. The solutions to the basic equations were presented in the previous report. Next, we have to solve the stability equations, numerically.

INTRODUCTION

Granular materials are unlike solids in that they conform to the shape of the vessel containing them, thereby exhibiting fluid like characteristics. On the other hand, they cannot be considered a fluid, as it can be heaped. The characteristics of the particles that constitute the bulk solids are probably of major importance in influencing the characteristics of that bulk solids both at rest and during flow. Also it is very difficult to characterize bulk solids, which are composed of a variety of materials, *i.e.* mainly due to the fact that small variations in some of the properties of the particles such as the size, shape, hardness, density and surface roughness can result in very different behavior of the bulk. Furthermore, secondary factors such as the presence of moisture, the extent of prior compaction, the atmospheric temperature, etc., which are not directly properties of the particles but of the ambient, can have significant e ffect on the behavior of the bulk.

One approach used in the modeling of granular material is as a **continuum**, which assumes that the material properties of the ensemble may be represented by continuous functions so that the medium may be divided indefinitely without losing any of its defining properties. A continuum model for granular materials was proposed by Goodman & Cowin (1971, 1972) this theory was later refined by other investigators such as Cowin (1974a, b), Savage (1979), Ahmadi (1982a, 1982b), Mctigue (1982), Nunziato, et al. (1980), and Passman, et al. (1980). The other method used in the modeling of granular materials is the **kinetic theory approach**, which is generally used in the modeling of rapidily flowing granular materials [cf. Ackerman and Shen (1981), Ahmadi and Shahinpoor (1984), Hutter (1986a, b), Boyle and Massoudi (1989, 1990)].

GOVERNING EQUATIONS

The granular material is treated as a continuum and its stress tensor is modeled as proposed by Goodman and Cowin (1971) and Rajagopal and Massoudi (1990). The stress is given by

$$\mathbf{T} = \{ \beta_0(\nu) + \beta_1(\nu)\, grad\nu \,.\, grad\nu + \beta_2(\nu)\, tr\mathbf{D} \}\, \mathbf{1} + \beta_4(\nu)\, \nabla\nu \otimes \nabla\nu + \beta_3(\nu)\, \mathbf{D}. \tag{1}$$

In the above equation $\mathbf{T}$ denotes the Cauchy Stress, ν the volume fraction of the solid, $\mathbf{D}$ denotes the stretching tensor associated with the solid motion, $\beta_0(\nu)$ is similar to pressure in a compressible fluid and is given by the equation of state, $\beta_2(\nu)$ is akin to the second coefficient of viscosity in a compressible fluid, $\beta_1(\nu)$ and $\beta_4(\nu)$ are material parameters that reflect the distribution of the granular material and $\beta_3(\nu)$ is the viscosity of the granular material. The above model allows for normal-stress differences, a feature observed in granular materials. In general, the material properties β_0 through β_4 are functions of the density (*or* volume fraction ν), temperature, and the principal invariants of the stretching tensor $\mathbf{D}$, given by

$$\mathbf{D} = \frac{1}{2}\left[\, (grad\mathbf{u}) + (grad\mathbf{u})^T \,\right], \tag{2}$$

$\mathbf{u}$ being the velocity of the particles. In equation (1), $\mathbf{1}$ is the identity tensor, *grad* the gradient operator, ∇ denotes the Laplacian operator, $\otimes$ indicates the outer (dyadic) product of two vectors, and *tr* designates the trace of a tensor. Furthermore, ν is related to the bulk density of the material ρ, through

$$\rho = \gamma \nu, \tag{3}$$

where γ is the actual density of the grains at the place $\mathbf{x}$ and time $\mathbf{t}$ and the field ν is called the volume fraction (*or* the volume distribution) and is related to the porosity n *or* the void ratio e by

$$\nu = 1 - n = \frac{1}{1+e}; \qquad with\ 0 \le \nu < 1 \tag{4}$$

Consider the flow of granular material modeled by the above continuum model down an inclined plane (cf. Figure 1) due to the action of gravity [Savage (1979), Johnson and Jackson (1987), Johnson, Nott and Jackson (1990) and Richman and Marciniec (1991)]. In this problem we consider steady one dimensional flow of incompressible granular materials (*i.e.* γ = constant) down an inclined plane, where the angle of inclination is α. Also we assume that β_1, β_2, and β_4 are constants.

Following, Rajagopal and Massoudi (1990) we shall assume that β_0 has the structure

$$\beta_0(\nu) = k\nu \qquad (5)$$
$$\beta_3(\nu) = \beta_{30}(\nu + \nu^2), \qquad \textit{where, } \beta_{30} \textit{ is a constant}$$

The governing equations of motion are the conservation of mass, momentum, and energy. The conservation of mass is

$$\frac{\partial\rho}{\partial t} + div(\rho\,\mathbf{u}) = 0, \qquad (6)$$

where $\frac{\partial}{\partial t}$ is the partial derivative with respect to time. The balance of linear momentum is

$$div\mathbf{T} + \rho\,\mathbf{b} = \rho\,\frac{d\mathbf{v}}{dt}, \qquad (7)$$

where $\frac{d}{dt}$ is the material time derivative and **b** is the body force. Here, we shall consider the purely mechanical problem and shall hence not document the energy equation.

STABILITY ANALYSIS:

If the solution to the governing equations of motion is disturbed, then the solution is asymptotically stable if that disturbance eventually decays to zero and unstable if the disturbance grows in amplitude in such a way that the solution departs from initial state or reaches some constant value, yeilding a new solution. Once the stable or unstable sates are classified for the governing equations, then the locus which seperates the two classes of states is defined as marginal stablity or neutral stablity curve. Here, in the present analysis we determine the marginal stablity curves for the flow down an inclined plane. Linearized stablity doea not yeild sufficient conditions for stability. So if the solution is unstable to small disturbances then it will be unstable to finite disturbances, while on the otherhand, if the solution is stable to small disturbances it is not necessarily stable to finite disturbances.

Consider solutions which consist of the basic flow plus an infinitesimal disturbance

$$\nu = \nu_0 + \varepsilon\, \nu_1 \tag{8}$$
$$u = u_0 + \varepsilon\, u_1 \tag{9}$$
$$v = \varepsilon\, u_2$$

where ν_0 and u_0 correspond to the basic solution of the governing equations and ν_1, u_1, u_2 represent the disturbance. It is assumed that for infinitesimal disturbances, the equations may be linearized *i.e.* the terms of order ε^2 and higher order can be neglected. The basic flow is assumed to have the form

$$\nu_0 = \nu_0(y)$$
$$u_0 = u_0(y)i \tag{10}$$

Now substituting equations (8) and (9) into conservation of mass and balance of linear momentum, the equations corresponding to the basic flow *i.e.* of order one are given by:

$$k \frac{d\nu_0}{dy} + 2\,(\beta_1 + \beta_4) \frac{d\nu_0}{dy} \frac{d^2\nu_0}{dy^2} = \gamma\, g\, \nu_0 \cos\alpha, \tag{11}$$

$$\beta_{30}\,(\nu_0 + \nu_0^2) \frac{d^2u_0}{dy^2} + \beta_{30}\,(1 + 2\,\nu_0) \frac{d\nu_0}{dy} \frac{du_0}{dy} = -2\,\gamma\, g\, \nu_0 \sin\alpha, \tag{12}$$

where g denotes the acceleration due to gravity and α is the angle of inclination of the plane.

We need to solve the basic solution equations (11) and (12) subject to the appropriate boundary conditions.

$$u_0 = f\left\{ k\, \nu_0 \sin\alpha + \beta_1 \sin\alpha \left\{\frac{d\nu_0}{dy}\right\}^2 + \frac{\beta_{30}}{2} (\nu_0 + \nu_0^2) \cos\alpha \frac{du_0}{dy} \right\}, \quad \text{at } y = 0 \text{ (on inclined plane)} \tag{13}$$

$$N = \int_0^h \nu_0\, dy\,. \tag{14}$$

and,

$$\frac{du_0}{dy} = 0$$

$$k\,\nu_0 + (\beta_1 + \beta_4)\left\{\frac{d\nu_0}{dy}\right\}^2 = 0 \qquad at\ y = h\ (at\ the\ free\ surface) \tag{15}$$

Notice that equations $(15)_{1,2}$ are the stress free conditions and equation (13) indicates the slip condition on the inclined plane in which f is the slip coefficient. Rajagopal and Massoudi (1990) and Rajagopal, Troy and Massoudi (1992) have showed that

$$k < 0. \tag{16}$$

The equations corresponding to order of ε are:

$$\frac{\partial \nu_1}{\partial t} + \nu_0\left\{\frac{\partial u_1}{\partial x} + \frac{du_2}{dy}\right\} + u_0\left\{\frac{\partial \nu_1}{\partial x}\right\} + \frac{d\nu_0}{dy}u_2 = 0 \tag{17}$$

$$\begin{aligned}
&k\frac{\partial \nu_1}{\partial x} + 2\beta_1\frac{d\nu_0}{dy}\frac{\partial^2 \nu_1}{\partial x \partial y} + \beta_2\frac{\partial}{\partial x}\left\{\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right\} + \beta_4\left\{\frac{d\nu_0}{dy}\frac{\partial^2 \nu_1}{\partial x\partial y} + \frac{d^2\nu_0}{dy^2}\frac{\partial \nu_1}{\partial x}\right\}\\
&\quad + \beta_{30}(\nu_0 + \nu_0^2)\frac{\partial^2 u_1}{\partial x^2} + \frac{\beta_{30}}{2}(1 + 2\nu_0)\frac{d\nu_0}{dy}\left\{\frac{\partial u_1}{\partial y} + \frac{du_2}{dx}\right\} + \frac{\beta_{30}}{2}(\nu_0 + \nu_0^2)\\
&\quad \left\{\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_2}{\partial x \partial y}\right\} + \frac{\beta_{30}}{2}(1 + 2\nu_0)\frac{d^2 u_0}{dy^2}\nu_1 + \frac{\beta_{30}}{2}(1 + 2\nu_0)\frac{du_0}{dy}\frac{\partial \nu_1}{\partial y}\\
&\quad + \beta_{30}\frac{du_0}{dy}\frac{d\nu_0}{dy}\nu_1 + \gamma g \sin\alpha\,\nu_1 = \gamma \nu_0\left\{\frac{\partial u_1}{\partial t} + u_0\frac{\partial u_1}{\partial x} + \frac{du_0}{dy}u_2\right\}
\end{aligned} \tag{18}$$

$$\begin{aligned}
&k\frac{\partial \nu_1}{\partial y} + 2\beta_1\frac{d\nu_0}{dy}\frac{\partial^2 \nu_1}{\partial y^2} + 2\beta_1\frac{d^2\nu_0}{dy^2}\frac{\partial \nu_1}{\partial y} + \beta_4\left\{\frac{d\nu_0}{dy}\frac{\partial^2\nu_1}{\partial x^2} + 2\frac{d\nu_0}{dy}\frac{\partial^2 \nu_1}{\partial y^2} + 2\frac{d^2\nu_0}{dy^2}\frac{\partial \nu_1}{\partial y}\right\}\\
&\quad + \beta_2\frac{\partial^2 u_1}{\partial x\partial y} + \beta_2\frac{\partial^2 u_2}{\partial y^2} + \frac{\beta_{30}}{2}\left\{(\nu_0 + \nu_0^2)\left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_1}{\partial x \partial y}\right) + (1 + 2\nu_0)\frac{du_0}{dy}\frac{\partial \nu_1}{\partial x}\right.\\
&\quad \left. + 2(1 + 2\nu_0)\frac{d\nu_0}{dy}\frac{\partial u_2}{\partial y} + 2(\nu_0 + \nu_0^2)\frac{\partial^2 u_2}{\partial y^2}\right\} - \gamma g \cos\alpha\,\nu_1\\
&\quad = \gamma \nu_0\left\{\frac{\partial u_2}{\partial t} + u_0\frac{\partial u_2}{\partial x}\right\}
\end{aligned} \tag{19}$$

subjected to the boundary conditions

$$u_1 = f\left\{ \left(k\,\nu_1 + 2\,\beta_1 \frac{d\nu_0}{dy}\frac{\partial \nu_1}{\partial y} + \beta_2 \left\{ \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \right\} + \beta_{30}\,(\nu_0 + \nu_0^2)\,\frac{\partial u_1}{\partial x} \right) sin\alpha \right.$$
$$\left. + \left(\beta_4 \frac{d\nu_0}{dy}\frac{\partial \nu_1}{\partial x} + \frac{\beta_{30}}{2}(\nu_0 + \nu_0^2) \left\{ \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right\} + \frac{\beta_{30}}{2}(1 + 2\,\nu_0)\, cos\alpha \frac{du_0}{dy}\nu_1 \right) cos\alpha \right\}$$

$$u_2 = 0 \tag{20}$$

$$\int_0^h \nu_1 \, dy = 0 \tag{21}$$

$$\beta_4 \frac{d\nu_0}{dy}\frac{\partial \nu_1}{\partial x} + \frac{\beta_{30}}{2}(\nu_0 + \nu_0^2) \left\{ \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right\} + \frac{\beta_{30}}{2}(1 + 2\,\nu_0)\frac{du_0}{dy}\nu_1 = 0 \tag{22}$$

$$k\,\nu_1 + 2\,(\beta_1 + \beta_4)\frac{d\nu_0}{dy}\frac{\partial \nu_1}{\partial y} + \beta_2 \left\{ \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \right\} + \beta_{30}\,(\nu_0 + \nu_0^2)\frac{\partial u_2}{\partial y} = 0 \tag{23}$$

Equations $(20)_{1,2}$ are the boundary conditions at $y = 0$ on the inclined plane and equations (22) and (23) are the boundary conditions at $y = h$ at the free surface. The system of equations (11), (12), (17) (18) and (19) subject to the boundary conditions (13), (14), (15), (20) (21) (22) and (23) are non-dimensionalized by

$$\bar{y} = \frac{y}{h}; \quad \bar{x} = \frac{x}{h}; \quad \overline{u_0} = \frac{u_0}{U}; \quad \overline{u_1} = \frac{u_1}{U}; \quad \overline{u_2} = \frac{u_2}{U}; \quad \bar{t} = \frac{t\,U}{h}, \tag{24}$$

where h is a characteristic length and U is a reference velocity. Now, the above system of equations for the basic flow reduces to

$$R_1 \frac{d\nu_0}{d\bar{y}} + R_2 \frac{d\nu_0}{d\bar{y}}\frac{d^2\nu_0}{d\bar{y}^2} = \nu_0\, cos\alpha \tag{25}$$

$$R_3\,(\nu_0 + \nu_0^2)\frac{d^2\bar{u}_0}{d\bar{y}^2} + R_3\,(1 + 2\,\nu_0)\frac{d\nu_0}{d\bar{y}}\frac{d\bar{u}_0}{d\bar{y}} = -\nu_0\, sin\alpha \tag{26}$$

and the boundary conditions become

$$\bar{u}_0 = f\left\{ R_1\,\nu_0\, sin\alpha + R_5\, sin\alpha \left\{ \frac{d\nu_0}{dy} \right\}^2 \right.$$
$$\left. + R_3\,(\nu_0 + \nu_0^2)\, cos\alpha \frac{du_0}{dy} \right\}, \qquad at\ \bar{y} = 0\ (on\ inclined\ plane) \tag{27}$$

$$\bar{Q} = \int_0^1 \nu_0 \, d\bar{y} \tag{28}$$

and,

$$\frac{d\bar{u}_0}{d\bar{y}} = 0$$

$$R_1 \nu_0 + \frac{R_2}{2}\left\{\frac{d\nu_0}{d\bar{y}}\right\}^2 = 0 \qquad \text{at } \bar{y} = 1 \text{ (at the free surface)} \tag{29}$$

Then the non-dimensional equations for the order of ε are given by

$$\frac{\partial \nu_1}{\partial t} + \nu_0 \left\{ \frac{\partial \bar{u}_1}{\partial \bar{x}} + \frac{\partial \bar{u}_2}{\partial \bar{y}} \right\} + \bar{u}_0 \left\{ \frac{\partial \nu_1}{\partial \bar{x}} \right\} + \frac{d\nu_0}{d\bar{y}} \bar{u}_2 = 0 \tag{30}$$

$$R_1 \frac{\partial \nu_1}{\partial \bar{x}} + 2 R_5 \frac{d\nu_0}{d\bar{y}} \frac{\partial^2 \nu_1}{\partial \bar{x} \partial \bar{y}} + R_4 \frac{\partial}{\partial \bar{x}} \left\{ \frac{\partial \bar{u}_1}{\partial \bar{x}} + \frac{\partial \bar{u}_2}{\partial \bar{y}} \right\} + R_6 \left\{ \frac{d\nu_0}{d\bar{y}} \frac{\partial^2 \nu_1}{\partial \bar{x} \partial \bar{y}} + \frac{d^2\nu_0}{d\bar{y}^2} \frac{\partial \nu_1}{\partial \bar{x}} \right\}$$

$$+ 2 R_3 (\nu_0 + \nu_0^2) \frac{\partial^2 \bar{u}_1}{\partial \bar{x}^2} + R_3 (1 + 2\nu_0) \frac{d\nu_0}{d\bar{y}} \left\{ \frac{\partial \bar{u}_1}{\partial \bar{y}} + \frac{d\bar{u}_2}{d\bar{x}} \right\} + R_3 (\nu_0 + \nu_0^2)$$

$$\left\{ \frac{\partial^2 \bar{u}_1}{\partial \bar{y}^2} + \frac{\partial^2 \bar{u}_2}{\partial \bar{x} \partial \bar{y}} \right\} + R_3 (1 + 2\nu_0) \frac{d^2 \bar{u}_0}{d\bar{y}^2} \nu_1 + R_3 (1 + 2\nu) \frac{d\bar{u}_0}{d\bar{y}} \frac{\partial \nu_1}{\partial \bar{y}} + 2 R_3 \frac{d\bar{u}_0}{d\bar{y}} \frac{d\nu_0}{d\bar{y}} \nu_1$$

$$+ \sin\alpha \, \nu_1 = Fr \, \nu_0 \left\{ \frac{\partial \bar{u}_1}{\partial \bar{t}} + \bar{u}_0 \frac{\partial \bar{u}_1}{\partial \bar{x}} + \frac{d\bar{u}_0}{d\bar{y}} \bar{u}_2 \right\} \tag{31}$$

$$R_1 \frac{\partial \nu_1}{\partial \bar{y}} + R_2 \frac{d\nu_0}{d\bar{y}} \frac{\partial^2 \nu_1}{\partial \bar{y}^2} + R_2 \frac{d^2 \nu_0}{d\bar{y}^2} \frac{\partial \nu_1}{\partial \bar{y}} + R_6 \frac{d\nu_0}{d\bar{y}} \frac{\partial^2 \nu_1}{\partial \bar{x}^2} + R_4 \frac{\partial^2 \bar{u}_2}{\partial \bar{y}^2} + R_4 \frac{\partial^2 \bar{u}_1}{\partial \bar{x} \partial \bar{y}}$$

$$+ R_3 \left\{ (\nu_0 + \nu_0^2) \left(\frac{\partial^2 u_2}{\partial \bar{x}^2} + \frac{\partial u_1}{\partial \bar{x} \partial \bar{y}} \right) + (1 + 2\nu_0) \frac{du_0}{d\bar{y}} \frac{\partial \nu_1}{\partial \bar{x}} + 2 (1 + 2\nu_0) \frac{\partial \nu_0}{\partial \bar{y}} \frac{\partial \bar{u}_2}{\partial \bar{y}} + 2 (\nu_0 + \nu_0^2) \frac{\partial^2 \bar{u}_2}{\partial \bar{y}^2} \right\}$$

$$- \cos\alpha \, \nu_1 = Fr \, \nu_0 \frac{\partial \bar{u}_2}{\partial \bar{t}} + \bar{u}_0 \nu_0 \frac{\partial \bar{u}_2}{\partial \bar{x}} \tag{32}$$

and the boundary conditions become

$$u_1 = f \left\{ \left(R_1 \nu_1 + 2 R_5 \frac{d\nu_0}{d\bar{y}} \frac{\partial \nu_1}{\partial \bar{y}} + R_4 \left\{ \frac{\partial \bar{u}_1}{\partial \bar{x}} + \frac{\partial \bar{u}_2}{\partial \bar{y}} \right\} + 2 R_3 (\nu_0 + \nu_0^2) \frac{\partial \bar{u}_1}{\partial \bar{x}} \right) \sin\alpha \right.$$

$$\left. + \left(R_6 \frac{d\nu_0}{d\bar{y}} \frac{\partial \nu_1}{\partial \bar{x}} + R_3 (\nu_0 + \nu_0^2) \left\{ \frac{\partial \bar{u}_1}{\partial \bar{y}} + \frac{\partial \bar{u}_2}{\partial \bar{x}} \right\} + R_3 (1 + 2\nu_0) \cos\alpha \frac{d\bar{u}_0}{d\bar{y}} \nu_1 \right) \cos\alpha \right\}$$

$$\bar{u}_2 = 0 \tag{33}$$

$$\int_0^1 \nu_1 \, d\bar{y} = 0 \tag{34}$$

$$R_6 \frac{d\nu_0}{dy}\frac{\partial \nu_1}{\partial x} + R_3 (\nu_0 + \nu_0^2)\left\{\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right\} + R_3 (1 + 2\,\nu_0)\frac{du_0}{dy}\nu_1 = 0$$

$$R_1\,\nu_1 + R_2 \frac{d\nu_0}{dy}\frac{\partial \nu_1}{\partial y} + R_4\left\{\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right\} + 2\,R_3 (\nu_0 + \nu_0^2)\frac{\partial u_2}{\partial y} = 0 \tag{35}$$

Equations $(33)_{1,2}$ are the boundary conditions at $\bar{y} = 0$ on the inclined plane and equations $(35)_{1,2}$ are the boundary conditions at $\bar{y} = 1$ at the free surface. We shall assume the disturbances to be spatially periodic. That is the perturbed quantities have the form

$$\nu_1 = \nu_p(\bar{y})\, e^{s\bar{t}}\, e^{i\sigma\bar{x}} \tag{36}$$

$$\bar{u}_1 = U_{px}(\bar{y})\, e^{s\bar{t}}\, e^{i\sigma\bar{x}} \tag{37}$$

$$\bar{u}_2 = U_{py}(\bar{y})\, e^{s\bar{t}}\, e^{i\sigma\bar{x}} \tag{38}$$

Where ν_p is the amplitude of the volume distribution function,

U_{px} and U_{py} are the amplitudes of the perturbed velocity,

i is the imaginary number such that $i^2 = -1$,

σ is the wave number (real) and

$s = \zeta + i\,\omega$.

Then substituting (36), (37), and (38) into the equations (30), (31) and (32), and the corresponding bondary conditions (33), (34) and (35) we end up with

$$\nu_0 \frac{dU_{py}}{d\bar{y}} + \frac{d\nu_0}{d\bar{y}} U_{py} + i\,\sigma\,\nu_0\, U_{px} + (i\,\sigma\,\bar{u}_0 + s)\,\nu_p = 0 \tag{39}$$

$$R_3(\nu_0+\nu_0^2)\frac{d^2U_{px}}{d\bar{y}^2}+R_3(1+2\nu_0)\frac{d\nu_0}{d\bar{y}}\frac{dU_{px}}{d\bar{y}}+i\sigma\left\{R_4+R_3(\nu_0+\nu_0^2)\right\}\frac{dU_{py}}{d\bar{y}}$$

$$+\left\{2i\sigma R_5\frac{d\nu_0}{d\bar{y}}+i\sigma R_6\frac{d\nu_0}{d\bar{y}}+R_3(1+2\nu_0)\frac{d\bar{u}_0}{d\bar{y}}\right\}\frac{d\nu_p}{d\bar{y}}$$

$$+\sigma\left\{-\sigma R_4-2\sigma R_3(\nu_0+\nu_0^2)-\frac{Fr}{\sigma}\nu_0 s-iFr\nu_0\bar{u}_0\right\}U_{px}+\left\{i\sigma R_3(1+2\nu_0)\frac{d\nu_0}{d\bar{y}}-Fr\nu_0\frac{d\bar{u}_0}{d\bar{y}}\right\}U_{py}$$

$$+\left\{i\sigma R_1+iR_6\sigma\frac{d^2\nu_0}{d\bar{y}^2}+R_3(1+2\nu_0)\frac{d^2\bar{u}_0}{d\bar{y}^2}+2R_3\frac{d\nu_0}{d\bar{y}}\frac{d\bar{u}_0}{d\bar{y}}+\sin\alpha\right\}\nu_p=0 \tag{40}$$

$$R_2\frac{d\nu_0}{d\bar{y}}\frac{d^2\nu_p}{d\bar{y}^2}+\left\{R_1+R_2\frac{d^2\nu_0}{d\bar{y}^2}\right\}\frac{d\nu_p}{d\bar{y}}+\left\{R_4+2R_3(\nu_0+\nu_0^2)\right\}\frac{d^2U_{py}}{d\bar{y}^2}+2R_3(1+2\nu_0)\frac{d\nu_0}{d\bar{y}}\frac{dU_{py}}{d\bar{y}}$$

$$+i\sigma\left\{R_4+R_3(\nu_0+\nu_0^2)\right\}\frac{dU_{px}}{d\bar{y}}+\left\{-\sigma^2R_6\frac{d\nu_0}{d\bar{y}}+i\sigma R_3(1+2\nu_0)\frac{d\bar{u}_0}{d\bar{y}}-\cos\alpha\right\}\nu_p$$

$$+\left\{-\sigma^2R_3(\nu_0+\nu_0^2-Fr\,\nu_0 s-i\sigma\nu_0\bar{u}_0\right\}U_{py}=0 \tag{41}$$

subjected to the boundary conditions

$$U_{px}=f\left\{A_1\frac{dU_{py}}{d\bar{y}}+A_2\frac{dU_{px}}{d\bar{y}}+A_3\frac{d\nu_p}{d\bar{y}}+iA_4U_{py}+iA_5U_{px}+(A_6+iA_7)\nu_p\right\}$$

$$U_{py}=0 \tag{42}$$

$$\int_0^1\nu_p\,d\bar{y}=0 \tag{43}$$

$$R_3(\nu_0+\nu_0^2)\left\{\frac{dU_{px}}{d\bar{y}}+i\sigma U_{py}\right\}+\left\{i\sigma R_6\frac{d\nu_0}{d\bar{y}}+R_3(1+2\nu_0)\frac{d\bar{u}_0}{d\bar{y}}\right\}\nu_p=0$$

$$\left\{R_1\nu_p+2R_3(\nu_0+\nu_0^2)+R_4\right\}\frac{dU_{py}}{d\bar{y}}+R_2\frac{d\nu_0}{d\bar{y}}\frac{d\nu_p}{d\bar{y}}+i\sigma R_4U_{px}=0 \tag{44}$$

Equations $(42)_{1,2}$ are the boundary conditions at $\bar{y}=0$ on the inclined plane and equations $(44)_{1,2}$ are the boundary conditions at $\bar{y}=1$ at the free surface. Here equation (39) is used in equations (40) and (41) so that the ν_p is eliminated from the two equations. In the above equations we set $s=0$ as we are interested in the marginal stability curve. Then we have to solve two eqautions in which the order of one of the equations is increased by one. Also, in the present problem the free surface is fixed, *i.e.* same as the basic solution domain ($\bar{y}=1$). But, in the real problem the perturbed domain will not be the same as the basic solution domain. if the perturbed domain is not fixed, there is a difficulty associated with that as we do not have the base solution (numerical) in the perturbed domain. The final equations are

$$S_1 \frac{d^2 U_{px}}{d\bar{y}^2} + (S_2 + i\, S_3) \frac{d^2 U_{py}}{d\bar{y}^2} + (S_4 + i\, S_5) \frac{d U_{px}}{d\bar{y}} + (S_6 + i\, S_7) \frac{d U_{py}}{d\bar{y}}$$

$$+ (S_8 + i\, S_9)\, U_{px} + (S_{10} + i\, S_{11})\, U_{py} = 0 \tag{45}$$

$$i\, S_{12} \frac{d^3 U_{py}}{d\bar{y}^3} + (S_{13} + i\, S_{14}) \frac{d^2 U_{py}}{d\bar{y}^2} + (S_{15} + i\, S_{16}) \frac{d U_{py}}{d\bar{y}} + S_{17} \frac{d^2 U_{px}}{d\bar{y}^2}$$

$$+ (S_{18} + i\, S_{19}) \frac{d U_{px}}{d\bar{y}} + (S_{20} + i\, S_{21})\, U_{py} + (S_{22} + i\, S_{23})\, U_{px} = 0 \tag{46}$$

subjected to the boundary conditions

$$U_{px} = f\left(i\, C_1 \frac{d^2 U_{py}}{d\bar{y}^2} + (C_2 + i\, C_3) \frac{d U_{py}}{d\bar{y}} + C_4 \frac{d U_{px}}{d\bar{y}} + (C_5 + i\, C_6)\, U_{py} + (C_7 + i\, C_8)\, U_{px} \right)$$

$$U_{py} = 0 \tag{47}$$

$$\int_0^1 \frac{1}{\bar{u}_0} \left\{ \frac{d(\nu_0\, U_{py})}{d\bar{y}} + i\, \sigma\, \nu_0\, U_{px} \right\} d\bar{y} = 0 \tag{48}$$

$$(B_1 + i\, B_2) \frac{d U_{py}}{d\bar{y}} + B_3 \frac{d U_{px}}{d\bar{y}} + (B_4 + i\, B_5)\, U_{py} + (B_6 + i\, B_7)\, U_{px} = 0$$

$$i\, B_8 \frac{d^2 U_{py}}{d\bar{y}^2} + (B_9 + i\, B_{10}) \frac{d U_{py}}{d\bar{y}} + B_{11} \frac{d U_{px}}{d\bar{y}} + i\, B_{12}\, U_{py} + (B_{13} + i\, B_{14})\, U_{px} = 0 \tag{49}$$

Equations $(47)_{1,2}$ are the boundary conditions at $\bar{y} = 0$ on the inclined plane and equations $(49)_{1,2}$ are the boundary conditions at $\bar{y} = 0$ at the free surface.

where,

$$S_1 = R_3\, (\nu_0 + \nu_0^2)\, \sigma\, \bar{u}_0^2$$

$$S_2 = -2\,R_5 \frac{d\nu_0}{d\bar{y}} \nu_0 \sigma\, \bar{u}_0 - R_6\, \sigma\, \bar{u}_0 \frac{d\nu_0}{d\bar{y}} \nu_0$$

$$S_3 = R_3\,(1 + 2\,\nu_0) \frac{d\bar{u}_0}{d\bar{y}} \nu_0\, \bar{u}_0$$

$$S_4 = R_3\, \sigma\, \bar{u}_0\,(1 + 2\,\nu_0) \left\{ \frac{d\nu_0}{d\bar{y}} \bar{u}_0 - \nu_0 \frac{d\bar{u}_0}{d\bar{y}} \right\}$$

$$S_5 = -\,\sigma^2\, \bar{u}_0\, \nu_0 \left\{ 2\,R_5 + R_6 \right\} \frac{d\nu_0}{d\bar{y}}$$

$$S_6 = -\,2\,\sigma\, \bar{u}_0 \left(2\,R_5 + R_6 \right) \left\{ \frac{d\nu_0}{d\bar{y}} \right\}^2 + \sigma\,(2\,R_5 + R_6)\, \nu_0 \frac{d\nu_0}{d\bar{y}} \frac{d\bar{u}_0}{d\bar{y}}$$

$$-\,R_1\, \sigma\, \bar{u}_0\, \nu_0 - R_6\, \sigma\, \nu_0\, \bar{u}_0 \frac{d^2\nu_0}{d\bar{y}^2}$$

$$S_7 = \sigma^2\, \bar{u}_0^2 \left(R_4 + R_3\,(\nu_0 + \nu_0^2) \right) + 2\,R_3\,(1 + 3\,\nu_0)\, \bar{u}_0 \frac{d\nu_0}{d\bar{y}} \frac{d\bar{u}_0}{d\bar{y}}$$

$$-\,R_3\,(1 + 2\,\nu_0)\nu_0 \left\{ \frac{d\bar{u}_0}{d\bar{y}} \right\}^2 + R_3\,(1 + 2\,\nu_0)\nu_0\, \bar{u}_0 \frac{d^2\bar{u}_0}{d\bar{y}^2} + \bar{u}_0\, \nu_0\, sin\alpha$$

$$S_8 = -\,\sigma^3\, \bar{u}_0^2 \left(R_4 + 2\,R_3\,(\nu_0 + \nu_0^2) \right) + R_3\, \sigma\,(1 + 2\,\nu_0) \frac{d\bar{u}_0}{d\bar{y}} \left(-\,\bar{u}_0 \frac{d\nu_0}{d\bar{y}} + \nu_0 \frac{d\bar{u}_0}{d\bar{y}} \right)$$

$$-\,R_3\, \sigma\, \bar{u}_0\, \nu_0\,(1 + 2\,\nu_0) \frac{d^2\bar{u}_0}{d\bar{y}^2} - 2\,R_3\, \sigma\, \bar{u}_0\, \nu_0 \frac{d\bar{u}_0}{d\bar{y}} \frac{d\nu_0}{d\bar{y}} - \nu_0\, \sigma\, \bar{u}_0 sin\alpha$$

$$S_9 = -\,\sigma^2\, Fr\, \nu_0\, \bar{u}_0^3 - \sigma^2\, \bar{u}_0\,(2\,R_5 + R_6) \left\{ \frac{d\nu_0}{d\bar{y}} \right\}^2 + \sigma^2\, \nu_0\,(2\,R_5 + R_6) \frac{d\bar{u}_0}{d\bar{y}} \frac{d\nu_0}{d\bar{y}}$$

$$-\,R_1\, \sigma^2\, \nu_0\, \bar{u}_0 - \sigma^2\, R_6\, \nu_0\, \bar{u}_0 \frac{d^2\nu_0}{d\bar{y}^2}$$

$$S_{10} = -Fr\,\sigma\,\nu_0\,\bar{u}_0^2\,\frac{d\bar{u}_0}{d\bar{y}} - \sigma\,\bar{u}_0\,(2R_5 + R_6)\,\frac{d\nu_0}{d\bar{y}}\,\frac{d^2\nu_0}{d\bar{y}^2}$$

$$+\,\sigma\,(2R_5 + R_6)\,\frac{d\bar{u}_0}{d\bar{y}}\left\{\frac{d\nu_0}{d\bar{y}}\right\}^2 - R_1\,\sigma\,\bar{u}_0\,\frac{d\nu_0}{d\bar{y}} - R_6\,\sigma\,\bar{u}_0\,\frac{d\nu_0}{d\bar{y}}\,\frac{d^2\nu_0}{d\bar{y}^2}$$

$$S_{11} = R_3\,(1 + 2\nu_0)\,\bar{u}_0\left\{\sigma^2\,\bar{u}_0\,\frac{d\nu_0}{d\bar{y}} + \frac{d\bar{u}_0}{d\bar{y}}\,\frac{d^2\nu_0}{d\bar{y}^2} + \frac{d\nu_0}{d\bar{y}}\,\frac{d^2\bar{u}_0}{d\bar{y}^2}\right\}$$

$$+\,R_3\,\frac{d\bar{u}_0}{d\bar{y}}\,\frac{d\nu_0}{d\bar{y}}\left(2\,\bar{u}_0\,\frac{d\nu_0}{d\bar{y}} - (1 + 2\nu_0)\,\frac{d\bar{u}_0}{d\bar{y}}\right) + \bar{u}_0\,sin\alpha\,\frac{d\nu_0}{d\bar{y}}$$

$$S_{12} = R_2\,\nu_0\,\bar{u}^2\,\frac{d\nu_0}{d\bar{y}}$$

$$S_{13} = \sigma\,\bar{u}_0^3\left\{R_4 + 2R_3\,(\nu_0 + \nu_0^2)\right\}$$

$$S_{14} = R_2\,\frac{d\nu_0}{d\bar{y}}\left\{3\,\bar{u}_0^2\,\frac{d\nu_0}{d\bar{y}} - 2\,\nu_0\,\bar{u}_0\,\frac{d\bar{u}_0}{d\bar{y}}\right\} + \nu_0\,\bar{u}_0^2\left\{R_1 + R_2\,\frac{d^2\nu_0}{d\bar{y}^2}\right\}$$

$$S_{15} = -R_3\,(1 + 2\nu_0)\,\sigma\,\bar{u}_0^2\left\{\nu_0\,\frac{d\bar{u}_0}{d\bar{y}} - 2\,\bar{u}_0\,\frac{d\nu_0}{d\bar{y}}\right\}$$

$$S_{16} = R_2\,\frac{d\nu_0}{d\bar{y}}\left\{3\,\bar{u}_0^2\,\frac{d^2\nu_0}{d\bar{y}^2} - 2\,\bar{u}_0\,\frac{d\nu_0}{d\bar{y}}\,\frac{d\bar{u}_0}{d\bar{y}} - \bar{u}_0\,\nu_0\,\frac{d^2\bar{u}_0}{d\bar{y}^2} + 2\,\nu_0\left\{\frac{d\bar{u}_0}{d\bar{y}}\right\}^2 - 2\,\bar{u}_0\,\frac{d\bar{u}_0}{d\bar{y}}\,\frac{d\nu_0}{d\bar{y}}\right\}$$

$$+\,2\left\{R_1 + R_2\,\frac{d^2\nu_0}{d\bar{y}^2}\right\}\left\{\bar{u}_0^2\,\frac{d\nu_0}{d\bar{y}} - \bar{u}_0\,\nu_0\,\frac{d\bar{u}_0}{d\bar{y}}\right\} - R_6\,\sigma^2\,\bar{u}_0^2\,\nu_0\,\frac{d\nu_0}{d\bar{y}} - \bar{u}_0^2\,\nu_0\,cos\alpha$$

$$S_{17} = -R_2\,\sigma\,\nu_0\,\bar{u}_0^2\,\frac{d\nu_0}{d\bar{y}}$$

$$S_{18} = R_2\,\sigma\,\bar{u}_0\,\frac{d\nu_0}{d\bar{y}}\left\{-2\,\bar{u}_0\,\frac{d\nu_0}{d\bar{y}} + \nu_0\,\frac{d\bar{u}_0}{d\bar{y}}\right\}$$

$$S_{19} = \sigma^2\,\bar{u}_0^3\,(R_4 + R_3\,(\nu_0 + \nu_0^2))$$

$$S_{20} = -R_3\,\sigma\,\bar{u}_0^{2}\left\{ (1+2\nu_0)\frac{d\nu_0}{d\bar{y}}\frac{d\bar{u}_0}{d\bar{y}} + \bar{u}_0(\nu_0+\nu_0^2) \right\}$$

$$S_{21} = R_2\frac{d\nu_0}{d\bar{y}}\left\{ \bar{u}_0^{2}\frac{d^3\nu_0}{d\bar{y}^3} - \bar{u}_0\frac{d^2\nu_0}{d\bar{y}^2}\frac{d\bar{u}_0}{d\bar{y}} - \bar{u}_0\frac{d\nu_0}{d\bar{y}}\frac{d^2\bar{u}_0}{d\bar{y}^2} + 2\frac{d\nu_0}{d\bar{y}}\left\{\frac{d\bar{u}_0}{d\bar{y}}\right\}^2 - \bar{u}_0\frac{d\bar{u}_0}{d\bar{y}}\frac{d^2\nu_0}{d\bar{y}^2} \right\}$$

$$+\left\{ R_1 + R_2\frac{d^2\nu_0}{d\bar{y}^2} \right\}\left\{ \bar{u}_0^{2}\frac{d^2\nu_0}{d\bar{y}^2} - \bar{u}_0\frac{d\nu_0}{d\bar{y}}\frac{d\bar{u}_0}{d\bar{y}} \right\} - R_6\,\sigma^2\,\bar{u}_0^{2}\left\{\frac{d\nu_0}{d\bar{y}}\right\}^2 - \sigma^2\nu_0\,\bar{u}_0^{4} - \bar{u}_0^{2}\cos\alpha\frac{d\nu_0}{d\bar{y}}$$

$$S_{22} = R_2\frac{d\nu_0}{d\bar{y}}\,\sigma\left\{ \bar{u}_0^{2}\frac{d^2\nu_0}{d\bar{y}^2} + \bar{u}_0\frac{d\nu_0}{d\bar{y}}\frac{d\bar{u}_0}{d\bar{y}} + \bar{u}_0\,\nu_0\frac{d^2\bar{u}_0}{d\bar{y}^2} - 2\nu_0\left\{\frac{d\bar{u}_0}{d\bar{y}}\right\}^2 + \bar{u}_0\frac{d\bar{u}_0}{d\bar{y}}\frac{d\nu_0}{d\bar{y}} \right\}$$

$$-\left\{ R_1 + R_2\frac{d^2\nu_0}{d\bar{y}^2} \right\}\left\{ -\bar{u}_0^{2}\,\sigma\frac{d^2\nu_0}{d\bar{y}^2} + \bar{u}_0\,\nu_0\,\sigma\frac{d\bar{u}_0}{d\bar{y}} \right\} + R_6\,\sigma^3\,\bar{u}_0^{2}\,\nu_0\frac{d\nu_0}{d\bar{y}} + \sigma\,\nu_0\,\bar{u}_0^{2}\cos\alpha$$

$$S_{23} = -\sigma^2 R_3\,\nu_0\,\bar{u}_0^{2}(1+2\nu_0)\frac{d\bar{u}_0}{d\bar{y}} \tag{50}$$

$$A_1 = R_4\sin\alpha$$

$$A_2 = R_3(\nu_0+\nu_0^2)\cos\alpha$$

$$A_3 = 2R_5\sin\alpha\frac{d\nu_0}{d\bar{y}}$$

$$A_4 = \sigma\sin\alpha\left\{ R_4 + 2R_3(\nu_0+\nu_0^2) \right\}$$

$$A_5 = \sigma R_3\cos\alpha(\nu_0+\nu_0^2)$$

$$A_6 = R_1\sin\alpha + R_3(1+2\nu_0)\cos\alpha\frac{d\bar{u}_0}{d\bar{y}}$$

$$A_7 = \sigma R_6\cos\alpha\frac{d\nu_0}{d\bar{y}} \tag{51}$$

$$B_1 = -R_6\, \nu_0\, \sigma \frac{d\nu_0}{d\bar{y}}$$

$$B_2 = R_3\, \nu_0\, (1 + 2\,\nu_0) \frac{d\bar{u}_0}{d\bar{y}}$$

$$B_3 = R_3\, \sigma\, \bar{u}_0\, (\nu_0 + \nu^2)$$

$$B_4 = -R_6\, \sigma \left\{ \frac{d\nu_0}{d\bar{y}} \right\}^2$$

$$B_5 = R_3\, (1 + 2\,\nu_0) \frac{d\nu_0}{d\bar{y}} \frac{d\bar{u}_0}{d\bar{y}} + R_3\, \sigma^2\, \bar{u}_0\, (\nu_0 + \nu_0^2$$

$$B_6 = -R_3\, \sigma\, \nu_0\, (1 + 2\,\nu_0) \frac{d\bar{u}_0}{d\bar{y}}$$

$$B_7 = -R_6\, \sigma^2\, \nu_0 \frac{d\nu_0}{d\bar{y}}$$

$$B_8 = R_2\, \nu_0\, \bar{u}_0 \frac{d\nu_0}{d\bar{y}}$$

$$B_9 = R_4\, \sigma\, \bar{u}_0^2 + 2\, R_3\, \sigma\, \bar{u}_0^2\, (\nu_0 + \nu_0^2)$$

$$B_{10} = 2\, R_2\, \bar{u}_0 \left\{ \frac{d\nu_0}{d\bar{y}} \right\}^2 - R_2\, \nu_0 \frac{d\nu_0}{d\bar{y}} \frac{d\bar{u}_0}{d\bar{y}} + R_1\, \nu_0\, \bar{u}_0$$

$$B_{11} = -R_2\, \sigma\, \bar{u}_0\, \nu_0 \frac{d\nu_0}{d\bar{y}}$$

$$B_{12} = R_2\, \bar{u}_0 \frac{d\nu_0}{d\bar{y}} \frac{d^2\nu_0}{d\bar{y}^2} - R_2 \frac{d\bar{u}_0}{d\bar{y}} \left\{ \frac{d\nu_0}{d\bar{y}} \right\}^2 + R_1\, \bar{u}_0 \frac{d\nu_0}{d\bar{y}}$$

$$B_{13} = -R_2\, \sigma\, \bar{u}_0 \left\{ \frac{d\nu_0}{d\bar{y}} \right\}^2 + R_2\, \sigma\, \nu_0 \frac{d\nu_0}{d\bar{y}} \frac{d\bar{u}_0}{d\bar{y}} - \sigma\, R_1\, \nu_0\, \bar{u}_0$$

$$B_{14} = \sigma^2\, R_4\, \bar{u}_0^2 \qquad (52)$$

$$C_1 = 2\,R_5\,\bar{u}_0\,v_0\,sin\alpha\,\frac{dv_0}{d\bar{y}}$$

$$C_2 = \sigma\,\bar{u}_0\left\{ R_4\,\bar{u}_0\,sin\alpha - R_6\,v_0\,cos\alpha\,\frac{dv_0}{d\bar{y}}\right\}$$

$$C_3 = 2\,R_5\left\{\frac{dv_0}{d\bar{y}}\right\}^2 sin\alpha\left\{2\,\bar{u}_0 - \frac{d\bar{u}_0}{d\bar{y}}\right\} + R_1\,\bar{u}_0\,v_0\,sin\alpha + R_3\,(1+2\,v_0)\,\bar{u}_0\,v_0\,cos\alpha\,\frac{d\bar{u}_0}{d\bar{y}}$$

$$C_4 = \sigma\,\bar{u}_0\left\{ R_3\,\bar{u}_0\,cos\alpha\,(v_0 + v_0^2) - 2\,R_5\,v_0\,sin\alpha\,\frac{dv_0}{d\bar{y}}\right\}$$

$$C_5 = -\,\sigma\,R_6\,\bar{u}_0\,cos\alpha\left\{\frac{d\bar{u}_0}{d\bar{y}}\right\}^2$$

$$C_6 = 2\,R_5\,sin\alpha\,\frac{dv_0}{d\bar{y}}\left\{\bar{u}_0\,\frac{d^2v_0}{d\bar{y}^2} - \frac{dv_0}{d\bar{y}}\,\frac{d\bar{u}_0}{d\bar{y}}\right\} + R_1\,\bar{u}_0\,sin\alpha\,\frac{dv_0}{d\bar{y}}$$
$$+ R_3\,\bar{u}_0\,cos\alpha\left\{\sigma^2\,(v_0 + v_0^2)\,\bar{u}_0 + (1+2\,v_0)\,\frac{d\bar{u}_0}{d\bar{y}}\,\frac{dv_0}{d\bar{y}}\right\}$$

$$C_7 = 2\,R_5\,sin\alpha\,\sigma\,\bar{u}_0\,\frac{dv_0}{d\bar{y}}\left\{-\frac{dv_0}{d\bar{y}} + \frac{d\bar{u}_0}{d\bar{y}}\right\} - R_1\,\sigma\,v_0\,\bar{u}_0\,sin\alpha - R_3\,(1+2\,v_0)\,cos\alpha\,\frac{d\bar{u}_0}{d\bar{y}}$$

$$C_8 = \sigma^2\,\bar{u}_0^2\,sin\alpha\left\{R_4 + 2\,R_3\,(v_0 + v_0^2)\right\} - R_6\,\sigma^2\,v_0\,\bar{u}_0\,cos\alpha\,\frac{dv_0}{d\bar{y}} \qquad (53)$$

Now, the non-dimensional parameters R_1, R_2, R_3, R_4, R_5, R_6 and Fr are given by

$$R_1 = \frac{k}{h\,\gamma\,g}; \qquad R_2 = \frac{2\,(\beta_1 + \beta_4)}{h^3\,\gamma\,g}$$

$$R_3 = \frac{\beta_{30}\,U}{2\,h^2\,\gamma\,g}; \qquad R_4 = \frac{\beta_2\,U}{h^2\,\gamma\,g}$$

$$R_5 = \frac{\beta_1}{h^3\,\gamma\,g}; \qquad R_6 = \frac{\beta_4}{h^3\,\gamma\,g}; \qquad Fr = \frac{U^2}{h\,g}, \qquad (54)$$

These dimensionless parameters do have physical interpretations. R_1 is the ratio of the pressure force to the gravity force. R_2, R_5 and R_6 are the ratio of volume distribution force to the gravity force. R_3 and R_4 are the ratio of the viscous force to the gravity force.

FUTURE WORK

The system of equations (45) and (46) with the boundary conditions (27), (48), and (49) and subject to the restriction (16) will be solved numerically using a colocation method and IMSL routines to obtain the marginal stability curves. The equations (25) and (26) subjected to the boundary conditions (27), (28) and (29) govern the basic flow and solutions to the same were presented in the previous report. We shall use these solutions to study the linearized stability problem.

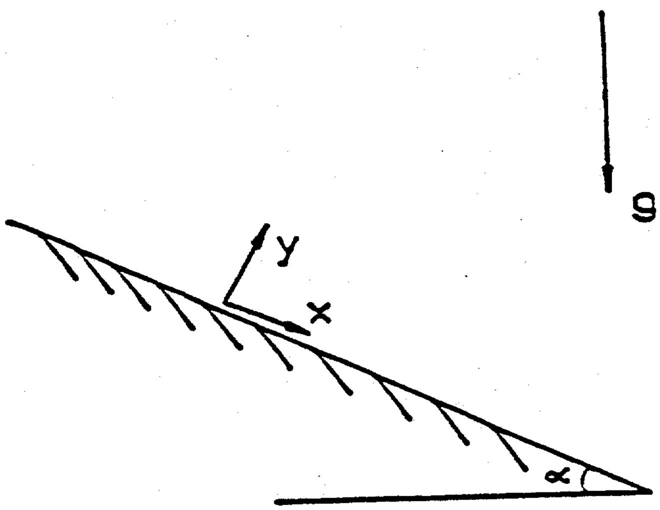

Figure 1. Flow Down An Inclined Plane

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