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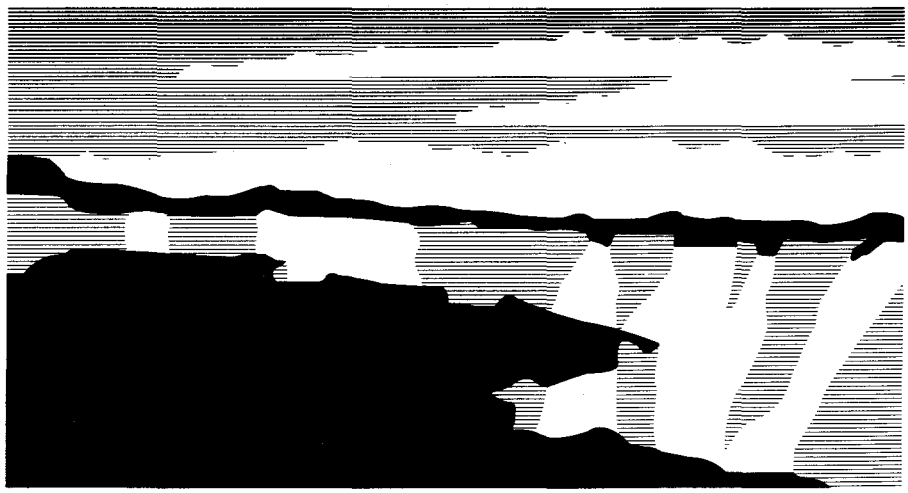

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PARTICLE HOPPING VS. FLUID-DYNAMICAL MODELS FOR TRAFFIC FLOW

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Although particle hopping models have been introduced into traffic science in the 1950s, their systematic use has only started recently. Two reasons for this are, that they are advantageous on modern computers, and that recent theoretical developments allow analytical understanding of their properties and therefore more confidence for their use. In principle, particle hopping models fit between microscopic models for driving and fluid-dynamical models for traffic flow. In this sense, they also help closing the conceptual gap between these two. This paper shows connections between particle hopping models and traffic flow theory. It shows that the hydrodynamical limits of certain particle hopping models correspond to the Lighthill-Whitham theory for traffic flow, and that only slightly more complex particle hopping models produce already the correct traffic jam dynamics, consistent with recent fluid-dynamical models for traffic flow. By doing so, this paper establishes that, on the macroscopic level, particle hopping models are at least as good as fluid-dynamical models. Yet, particle hopping models have at least two advantages over fluid-dynamical models: they straightforwardly allow *micro*scopic simulations, and they include stochasticity.

1 Introduction

Vehicular traffic has been a widely and thoroughly researched area in the 1950s and 60s. [1,2,3] Vehicular traffic theory can be broadly separated into two branches: Traffic *flow* theory, and *car-following* theory.

Traffic flow theory is concerned with finding relations between the three fundamental variables of traffic flow, which are velocity v, density ρ, and flow q. Only two of these variables are independent since they are related through $q = \rho\, v$.

Car-following theory regards traffic from a microscopic point of view: The behavior of each vehicle is modeled in relation to the vehicle ahead. As the definition indicates, this theory concentrates on single lane situations where a driver reacts to the movements of the vehicle ahead of him. Mathematical car-following theory uses differential delay equations.

A more recent addition to the development of vehicular traffic flow theory are *particle hopping models*. Imagine a one-dimensional chain of cells, each cell either empty, or occupied by exactly one particle. Movement of particles is achieved by particles jumping from one cell to another according to specific movement rules. In the context of vehicular traffic, one can imagine a road represented by cells which can fit exactly one car. A rough representation of car movements then is given by moving cars from one cell to another. Actually, the first proposition of such a model for vehicular traffic is from Gerlough [4] in 1956 and has been further extended by Cremer and coworkers. [5]

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This paper shows how particle hopping models fit into the context of traffic flow theory. It starts with short reviews of car following models (Section 2), and of fluid-dynamical traffic flow models (Section 3). Section 4 defines different particle hopping models which are of interest in the context of traffic flow. The paper continues by showing connections between the fluid-dynamical traffic flow models and particle hopping models. In some cases, these connections are exact and have long been established, but have never been viewed in the context of traffic theory. These cases are shown in Section 5. In other cases, the dynamic behavior of traffic jam clusters can be compared to instabilities in the partial differential equations. This description is only precise for the so-called cruise control limit of the particle hopping models, where jams cannot start spontaneously but have to be started by some external disturbance. This is described in Section 6. Section 7 explains in how far these results carry over to other models, in which jams initiate spontaneously due to fluctuations of driver's behavior. Summary and discussion conclude the paper.

This paper derives from[6], which discusses many of the issues of this paper on a more technical level.

2 Car-following theory

Many of us have learned heuristic driving rules such as "Abstand halber Tacho" or "Leave two seconds time headway". Since time headway is equal to v/gap, where v is velocity and gap is the front-bumper-to-end-bumper distance, one obtains as a rough driving rule $v \propto V(gap, \ldots)$, where V is a preferred velocity function, which is roughly linear in gap and may also depend on other variables. $\propto$ means "proportional to". This proportionality is also roughly confirmed by measurement.[3]

Making this time-dependent, one may expect that people attempt to reach this velocity after some time τ, which is a delay time, summarizing the effects of reaction, vehicle inertia, etc. This leads to

$$v(t+\tau) = V(gap(t), \ldots) \ , \tag{1}$$

which has, with $V \propto gap$, for example been used by Newell[7] and Whitham.[8]

Two other car-following relations are of particular importance in the literature:

- Herman and coworkers (see[1,2,3]) have used the form

$$a(t+\tau) = const \cdot \frac{v(t)^m}{gap(t)^l} \cdot \Delta v(t) \ , \tag{2}$$

  where $a := \frac{dv}{dt}$ is the acceleration, Δv is the velocity difference to the vehicle ahead, and $const$, m, and l are empirical constants.

  This equation is usually seen as a stimulus-response-relation, where Δv is the stimulus, $a(t+\tau)$ is the response, and $const \cdot v(t)^m/gap(t)^l$ is the sensitivity. One can, though, also get some intuition for this equation on purely formal grounds:[2] Time-deriving $v(t+\tau) \propto gap(t)$ leads to $a(t+\tau) \propto \Delta v(t)$. Adding a factor $const \cdot v(t)^m/gap(t)^m$ leads to Eq. 2.

- More recently, the following equation has been used:[9]

$$a(t) \approx \frac{1}{\tau}\left[V(gap(t)) - v(t)\right] \ . \tag{3}$$

Heuristically, one can obtain this equation by Taylor-expanding Eq. 1,[8] which leads to

$$v(t) + \tau \cdot \dot{v}(t) + \ldots = V(gap(t)) \ , \tag{4}$$

and after rearranging to Eq. 3. Note that this derivation is only heuristic, and the mathematical solutions of Eq. 4 and of Eq. 1 for $\tau \neq 0$ may be different, which should be investigated in more detail.

3 Fluid-dynamical models for traffic flow

3.1 Models without inertia

In traffic flow theory, models can be roughly distinguished into two different classes: the ones which assume ***instantaneous*** adaption of velocity to density, and the ones where this adaption needs some time because it has to overcome inertia. The first class is the simpler one; the basic equation here is

$$\partial_t \rho + \partial_x q = 0 \ , \tag{5}$$

where ρ is the density, q is the flow or current or throughput, and ∂_t and ∂_x are partial derivatives with respect to time and space, respectively. The equation is just the equation of continuity, and it simply expresses mass conservation.

In order to make this work, one has to give the flow as a function of density, $q = f(\rho)$, for example $q \propto \rho(1-\rho)$ (which would be the Greenshield relation, see[2]).

Physically, one finds behind these equations that velocity adapts instantaneously to the surrounding density, i.e. $v \equiv q/\rho = f(\rho)/\rho = F(\rho \text{ only})$. This is exactly the well-known theory of Lighthill and Whitham,[10] and a much is known how to handle these equations.

Note that, following a theoretical physics tradition, some variables are made free of units before they are used. For example, density here is renormalized so that it is between zero and one; in traffic one would achieve this by dividing it by the jam density:

$$\rho \text{ [no unit]} := \frac{\rho_{real} \text{ [number of cars per km]}}{\rho_{jam} \text{ [number of cars per km]}} \ , \tag{6}$$

where possible units are indicated in the brackets.

3.2 Models including inertia

Models including momentum consist of a second equation describing the fact that velocity does in reality ***not*** adapt instantaneously to the density. An often used form of such a non-instantaneous velocity adaption term is[11,12]

$$a \equiv \frac{\mathrm{d}v}{\mathrm{d}t} \equiv \partial_t v + v\,\partial_x v = \frac{1}{\tau}\left[V(\rho) - v\right] + \frac{c_o^2}{\rho}\,\partial_x \rho + \nu\,\partial_x^2 v \ . \tag{7}$$

The equation says that individual acceleration (left hand side) is proportional to the following three effects (right hand side):

- Difference between desired speed $V(\rho)$ and actual speed v. Note that this term is essentially the same as $\frac{1}{\tau}[V(gap(t)) - v]$ from the car-following models.

- Gradient of the density: If traffic gets denser in the driving direction, one slows down. A possible formal derivation of this term is that $V(gap)$ is actually not symmetric around the car and thus has to be expanded into $V(gap) = V(\rho + \Delta x/2) = V(\rho) + \frac{1}{2}\Delta x \cdot V' \cdot \partial_x \rho + \ldots$, thus leading to a gradient term of the density.[13]

- A spatial smoothing (diffusion) term. This effect can better be derived as a space-averaging effect;[6] it is thus not necessary to conceptually connect this term to individual accelerations.

Note that the limit $\tau \to 0$ makes the velocity adaption infinitely fast, i.e. returning to the instantaneous adaption case.

A more comprehensive review of the fluid-dynamical equations needed here can be found, e.g., in[6].

Prigogine, Herman, and coworkers developed a kinetic theory for traffic flow,[14] where the Lighthill-Whitham situation can be obtained as a limiting case of the kinetic theory.

4 Definitions of particle hopping models

This section defines several particle hopping models which are candidate models for traffic flow. They all have in common that they are defined on a lattice of, say, length L, where L is the number of sites, and that each site can be either empty, or occupied by exactly one particle. Also, in all models particles can only move in one direction. The number of particles, N, is conserved except at the boundaries.

These models are sometimes also called cellular automata (CA).[15] Particle hopping models and CA are not exactly the same, although the definitions are overlapping. All CA models in this paper are particle hopping models; the inverse is true except for the ASEP (see below).

This section starts out with the Stochastic Traffic Cellular Automaton (STCA), which has been proposed for traffic flow by Nagel and Schreckenberg,[16] and which is currently implemented as a microsimulation option for large scale traffic simulation projects both in the United States[17] and in Germany.[18] The STCA includes strong randomness in the rules. Setting this randomness to zero reduces the STCA to be a much simpler, deterministic model, which, when restricting oneself to maximum velocity $v_{max} = 1$, turns out to a well known cellular automaton model. In the third model of this section, randomness is re-introduced, but in this case by changing the update algorithm: Whereas in the first two models all particles are updated synchronously based in "old" information, in this third model, particles are selected in random sequence for individual updates.

4.1 *The Stochastic Traffic Cellular Automaton (STCA)*

The Stochastic Traffic Cellular Automaton (STCA) is defined as follows. Each particle (= car) can have an integer velocity between 0 and v_{max}, which is often taken to be 5.[16] The complete configuration at time-step t is stored, and the configuration at time-step $t+1$ is computed from that (parallel or synchronous update). For each particle, the following steps are done in parallel:

- Find number of empty sites ahead (= *gap*) at time t.
- If $v > gap$ (**too fast**), then slow down to $v := gap$. [rule 1]
- Else if $v < gap$ (**enough headway**) and $v < v_{max}$, then accelerate by one: $v := v + 1$. [rule 2]
- **Randomization:** If after the above steps the velocity is larger than zero ($v > 0$), then, with probability p, reduce v by one. [rule 3]
- **Particle propagation:** Each particle moves v sites ahead. [rule 4]

Note that, because of integer arithmetic, conditions like $v > gap$ and $v \geq gap + 1$ are equivalent.

The randomization (rule 3) condenses three different properties of human driving into one computational operation: Fluctuations at maximum speed, overreactions at braking, and retarded (noisy) acceleration.

Despite its simplicity, this model is astonishingly successful in reproducing realistic behavior such as start-stop-waves (Fig. 1) and realistic fundamental diagrams (Fig. 2, compare with[28]).[16]

Due to the given discretization of space and time, proper units are often omitted in the context of particle hopping or cellular automata models. Proper units here would be: $[gap]$ = number of cells, $[v]$ = number of cells per time step, $[t]$ = number of time steps, etc. For that reason, it is possible to write something like $v < gap$, which properly would have to be $v < gap/(\text{time step})$.

4.2 *The deterministic limit of the STCA (CA-184)*

One can take the deterministic limit of the STCA by setting the randomization probability p equal to zero, which just amounts to skipping the randomization step. See Figs. 3 and 4. It turns out that, when using maximum velocity $v_{max} = 1$, this is equivalent[19] to the cellular automaton rule 184 in Wolfram's notation,[15] which is why I use the notation CA-184.

4.3 *The Asymmetric Stochastic Exclusion Process (ASEP)*

The probably most-investigated particle hopping model is the Asymmetric Stochastic Exclusion Process (ASEP) (e.g.[19,20,21,22]). It is defined as follows:

- Pick one particle randomly. [rule 1]
- If the site to the right is free, move the particle to that site. [rule 2]

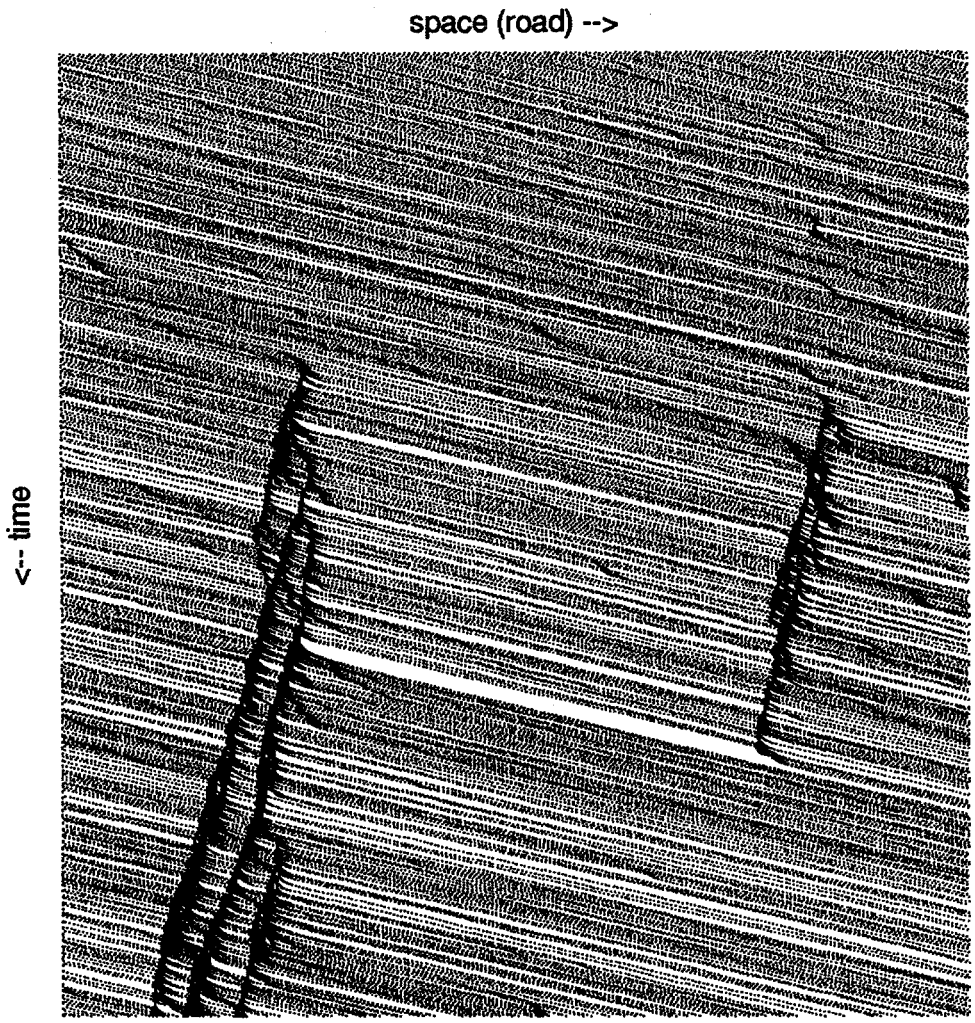

Figure 1: Space-time plot for the STCA, $v_{max} = 5$, $\rho = 0.09$ (i.e. slightly above $\rho(q_{max})$), starting from ordered initial conditions. The ordered state is meta-stable, i.e. "survives" for about 300 iterations until is spontaneously separates into jammed regions and into regions with $\rho = \rho(q_{max})$.

For a space-time plot of the ASEP, see Fig. 5.

The ASEP is closely related to CA-184/1 and STCA/1, where "/1" means "with $v_{max} = 1$". The difference actually only is in the manner in which sites are updated. CA-184 and STCA update all sites synchronously, whereas ASEP uses a random serial sequence.

Going from the ASEP to CA-184/1, i.e. changing the update from randomly asynchronous to deterministic synchronous, produces very different dynamics [19] (compare Fig. 5 to 3 and 4). Re-introducing the randomness via the randomization (rule 3) in the STCA again leads to different results (Fig. 1).

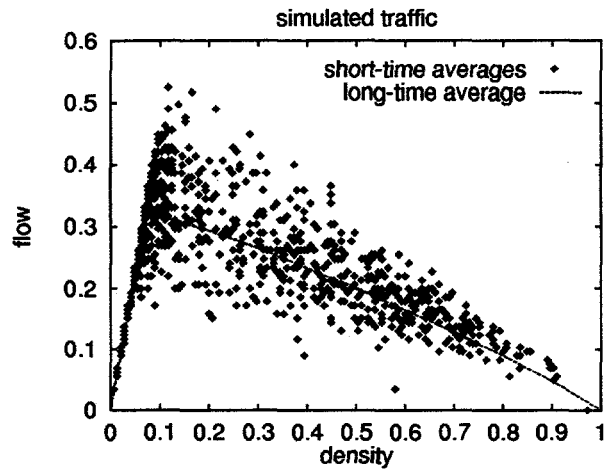

Figure 2: Flow-density fundamental diagram for the STCA. Short-time averages are taken over 300 simulation steps and thus mimic the 5-minute averages often taken in reality.

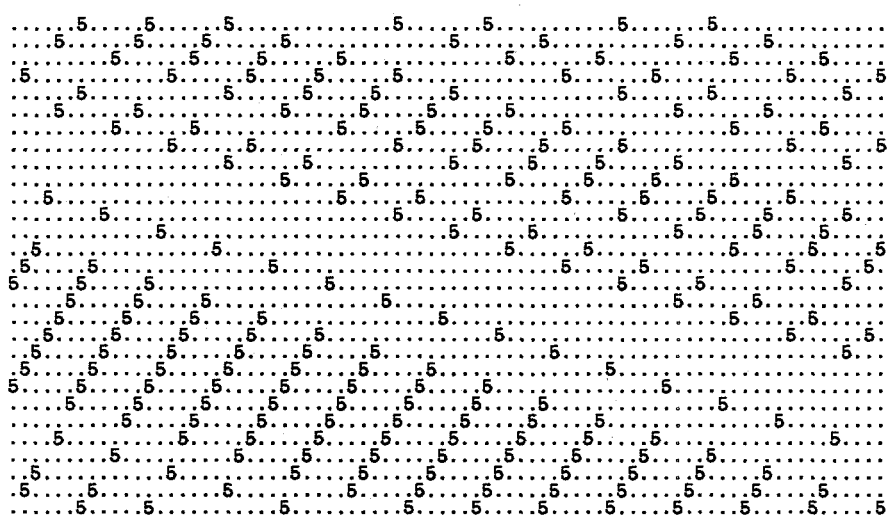

Figure 3: Space-time plot for CA-184/5 and subcritical density. Here, dots represent empty spaces, and integer numbers denote the velocity of the particle at that position. Note that one can follow individual particles from left to right. The wave velocity in this regime is $c = v_{max}$, i.e. the holes between platoons, which move with v_{max}, are already the waves.

5 Particle hopping models and fluid dynamics

Writing about both particle hopping and fluid-dynamical models for traffic flow does not make much sense as long as one cannot compare them. Fortunately, such a comparison is possible and will turn out to be quite instructive. Actually, for some

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..3...1.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2.
.3...1.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..
3...1.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3
...1.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3.
..1.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3..
.1.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...
1.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4
.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4.
2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4..
..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4...
.3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....
3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5
...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.
..4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5..
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.5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5..
5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5...
.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5....
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..5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5.....5.
.5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5.....5..
5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5.....5...
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Figure 4: Space-time plot for CA-184/5 and supercritical density. Not that one can still follow individual particles from the left to right. Wave velocity here is $c = q' = -1$, as can be easily seen.

of the mentioned particle hopping models, fluid-dynamical limits are known exactly. By fluid-dynamical limit one technically means the limit where the grid size Δx goes to zero, both the number of grid points, n, and the number of particles, N, go to infinity, while one keeps the system size, $L = n \cdot \Delta x$ and the density $\rho = N/n$ both constant. More intuitively, a fluid-dynamical description of a particle hopping model is a description where one averages over enough particles so that the granularity of the original system is no longer visible.

5.1 ASEP/1

The classic stochastic asymmetric exclusion process corresponds to the noisy Burgers equation. More precisely, the hydrodynamic limit of the ASEP particle process is a diffusion equation

$$\partial_t \rho + \partial_x q = D\, \partial_x^2 \rho + \eta \tag{8}$$

with a current [19,20] of $q = \rho\,(1-\rho)$. Interestingly, this is exactly the Lighthill-Whitham case, specialized to the Greenshields flow relation, with terms added for noise and diffusion. *In other words, the ASEP/1 particle hopping process and the Lighthill-Whitham-theory (plus noise plus diffusion), specialized to the case of the Greenshields flow-density relation, describe the same behavior.* In consequence, many phenomena of this particle hopping process can be understood using the Lighthill-Whitham theory.

Inserting the flow relation into Eq. 8 yields $\partial_t \rho + \partial_x \rho - \partial_x \rho^2 = D\, \partial_x^2 \rho + \eta$, which is a Burgers-type equation.[23] In the steady state, this model shows kinematic waves (= small jams), which are produced by the noise and damped by diffusion (Fig. 5). These non-dispersive waves move forwards (wave velocity $c = \mathrm{d}q/\mathrm{d}\rho = 1 - 2\rho > 0$) for $\rho < 1/2$ and backwards ($c < 0$) for $\rho > 1/2$. At $\rho = 1/2$, the wave velocity is exactly zero ($c = 0$), and this is the point of maximum throughput.[21]

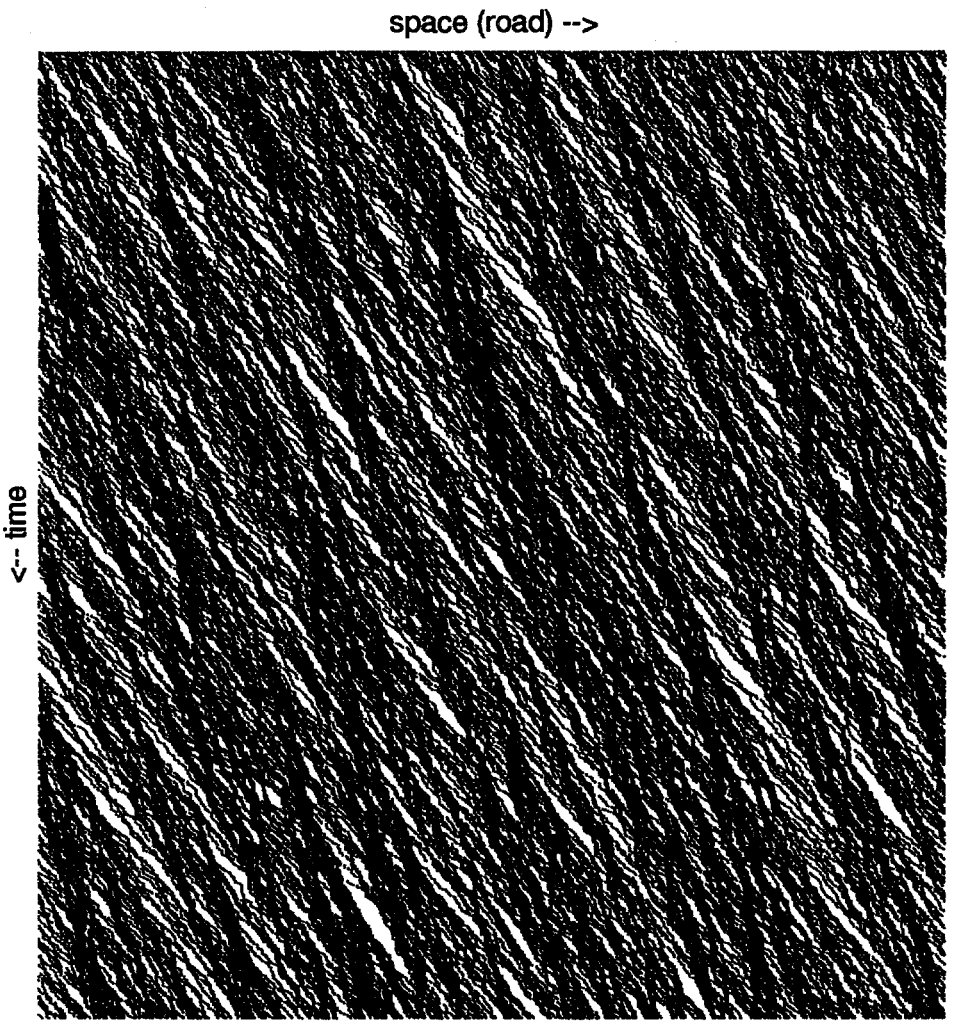

Figure 5: Space-time plot for random sequential update, $v_{max} = 1$ (ASEP/1), and $\rho = 0.3$. Black dots represent particles; consecutive horizontal lines represent configurations at consecutive time steps. Clearly, the kinematic waves are moving forwards. For $\rho > 1/2$, the kinematic waves would be moving backwards.

The drawback of this model with respect to traffic flow is that it does neither have a regime of laminar flow nor "real", big jams (see also Fig. 5). Because of the random sequential update, vehicles with average speed $\overline{v}$ fluctuate severely around their average position given by $\overline{v}\,t$. As a result, they always "collide" with their neighbors, even at very low densities, leading to "mini-jams" everywhere. This is clearly unrealistic for light traffic.

5.2 CA-184

Using a maximum velocity higher than one does not change the general behavior of CA-184. It therefore makes sense to directly discuss the general case.

The CA-184/1 is the deterministic counterpart of the ASEP/1. But taking away the noise from the particle update completely changes the dynamics.[19] The model now corresponds to the non-diffusive, non-noisy equation of continuity

$$\partial_t \rho + \partial_x q = \partial_t \rho + q' \, \partial_x \rho = 0 \tag{9}$$

with

$$q' = \frac{\mathrm{d}q}{\mathrm{d}\rho} = \begin{cases} v_{max} & \text{for } \rho < \rho_{qmax} \\ -1 & \text{for } \rho > \rho_{qmax} \end{cases}, \tag{10}$$

which is, except at $\rho = \rho_{qmax}$, a linear flow-density relation. The intersection point of the fundamental diagram divides two phenomenological regimes: light traffic ($\rho < \rho_{qmax}$; Fig. 3; wave velocity $c = q' = v_{max}$) and dense traffic ($\rho > \rho_{qmax}$; Fig. 4; wave velocity $c = -1$).

Note that this is again the Lighthill-Whitham theory, this time without the addition of noise or diffusion, and with a different flow-density relation than before.

6 Cruise control limit of the STCA (STCA-CC)

No fluid-dynamical limits for the other particle hopping models are known, although some analytic approximations for average behavior exist.[24] Another, more intuitive, way to gain further insight is to look into the traffic jam dynamics of the different models. In order to separate out the traffic jam dynamics from other effects, as a first step one would like to modify the models in such a way that always only one jam at a time exists. This is achieved by introducing the "cruise control limit" (see also[25] for more details). Here, fluctuations at free driving (i.e. when $v = v_{max}$ and $gap \geq v_{max}$) are set to zero. The result is that traffic in these models, once all vehicles are in the free driving regime, remains deterministic and laminar for all times. A single jam can then be initiated by perturbing one single car by, say, stopping it and letting it re-accelerate. In general, many different choices for the local perturbation give rise to the same large scale behavior. The perturbed car eventually re-accelerates back to maximum velocity. In the meantime, though, a following car may have come too close to the disturbed car and has to slow down. This initiates a chain reaction – an emergent traffic jam (see Fig. 6).

Since the STCA-CC has no fluctuations at free driving, the maximum flow one can reach is with all cars at maximum speed and $gap = v_{max}$. Therefore, one *can* manually achieve flows which follow, for $\rho \leq \rho_{c2}$, the same q-ρ-relationship as the CA-184, where ρ_{c2} now denotes the density of maximum flow of the deterministic model CA-184, i.e. $\rho_{c2} = 1/(v_{max} + 1)$.

Above a certain ρ_{c1}, these flows are unstable to small local perturbations. This density will turn out to be a "critical" density; for that reason I will use $\rho_c \equiv \rho_{c1}$.

Now assume that a single jam has been initiated in an infinite system of density ρ_0. It is straightforward to see[25] that $n(t)$, the number of cars in the jam, follows a usually biased, absorbing random walk, where $n(t) = 0$ (jam dissolved) is the

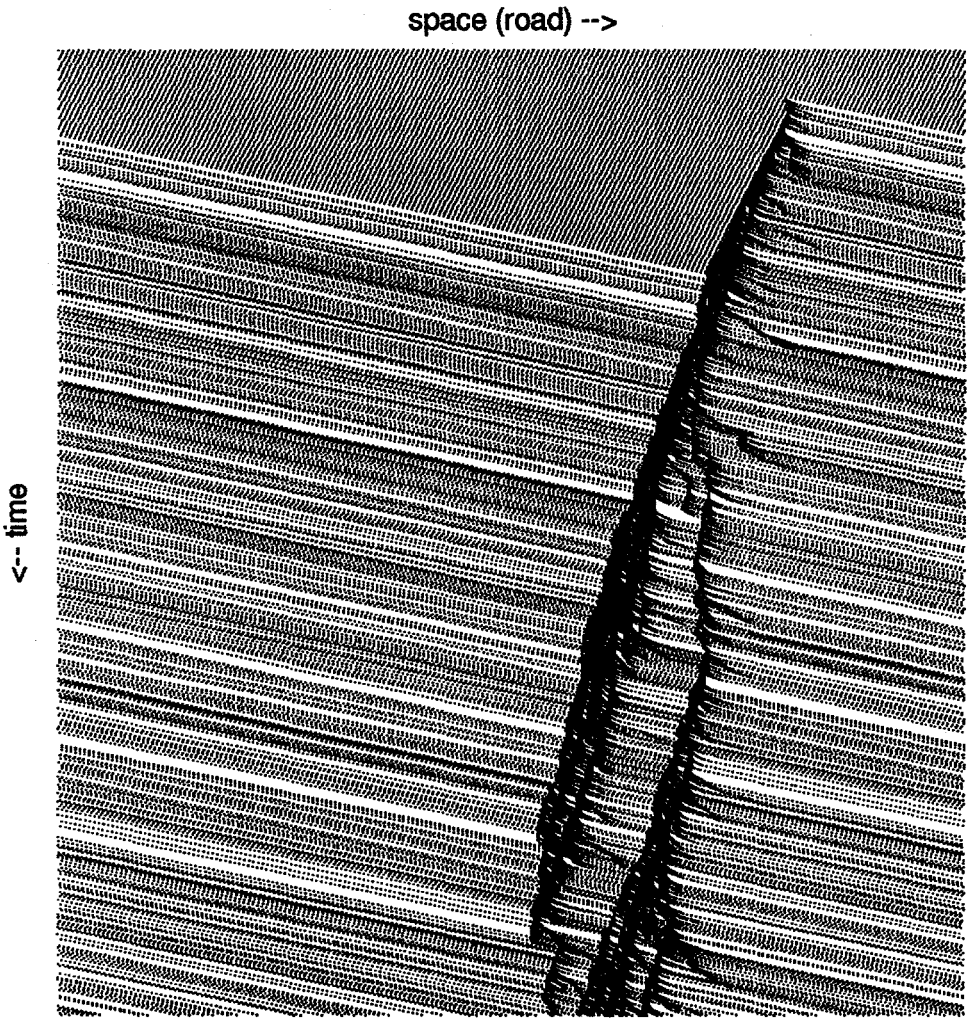

Figure 6: Emergent traffic jam in the cruise control limit in a closed system, $v_{max} = 5$, $\rho = 0.09$, i.e. slightly above critical. The flow is started in a deterministic, supercritical configuration. A single disturbance starts a jam.

absorbing state: Every time a new car arrives at the end of the jam, $n(t)$ increases by one, which is governed by the inflow rate, q_{in}. Every time a car leaves the jam at the outflow side, $n(t)$ decreases by one, and this is governed by the outflow rate, q_{out}. When $q_{in} = q_{out}$, $n(t)$ follows an unbiased absorbing random walk. $q_{in} \neq q_{out}$ introduces a bias or drift term $\propto (q_{in} - q_{out}) \cdot t$. The statistics of such absorbing random walks can be calculated exactly. See[25] for more details.

For $q_{in} < q_{out}$, jams shrink and eventually dissolve. For $q_{in} > q_{out}$, jams grow forever. For $q_{in} = q_{out}$, the jams "cannot decide" and show large, "critical", fluctuations, which can be described by critical exponents.[25] For that reason, $q_c :=$

q_{out} is the critical flow, and $\rho_c := \rho(q_{out})$ is the critical density. Note that these quantities are entirely given by the outflow behavior.

Especially for theoretical considerations, one often uses a closed system, i.e. with periodic boundary conditions ("traffic in a closed loop"). Such a situation is actually shown in Fig. 6. One clearly sees that once the system "feels" its finiteness (i.e. the outflow from the jam also becomes its inflow), then the situation changes: The initially supercritical jam stops to grow, and the system is now best described as having undergone a phase separation: from a one-phase system with supercritical laminar traffic into a two-phase system, where one phase consists of exactly critical outflow traffic, and the other phase is the jam. — It is clear from these remarks that, in a large closed system which is only slightly perturbed, maximum throughput occurs at $\rho = \rho_c$: For $\rho < \rho_c$, jams quickly dissolve, therefore flow is $q = v_{max} \cdot \rho < q_{max} = v_{max} \cdot \rho_c$. For $\rho > \rho_c$, the system is a mixture of laminar flow regions operating at q_{max}, and jams, operating at lower flow. Only at $\rho = \rho_c$, the *whole* system operates at q_{max}. All this is made more precise in [25].

This picture is consistent with recent results both in fluid-dynamical models and mathematical car-following models for traffic flow:

- Traffic simulations using a fluid-dynamical model starting from nearly homogeneous conditions eventually form stable waves. [26,27] One has, though, to distinguish between small amplitude and large amplitude instabilities. [27] It seems that the large amplitude instability (i.e. the one which is *not* found by linear stability analysis) is the one which corresponds more closely to the $q_{in} > q_{out}$ description.

- The separation of traffic into laminar and jammed phases can also be found in deterministic continuous mathematical car-following models. [9]

7 Returning to the Stochastic Traffic CA (STCA)

Returning to the full STCA, the important difference is that jams now start spontaneously and independently of other jams because vehicles fluctuate even at maximum speed. One consequence is that the supercritical laminar flow (i.e. $\rho_c < \rho < \rho_{c2}$) now is meta-stable at best, and eventually *spontaneously* (i.e. without external perturbation) decomposes into laminar regions with $\rho = \rho_c$, and jams (see Fig. 1).

A more precise explanation of this is as follows. Laminar traffic will always and at all densities, due to small fluctuations, have small disturbances which can develop into jams. The inflow to the jam decides if such a jam can become long-lived or not: Since the average outflow q_{out} is fixed by the driving dynamics, $q_{in} > q_{out}$ makes the jam (in the average) long-lived, $q_{in} < q_{out}$ not.

With this dynamical explanation in mind, it is fairly straightforward to explain the high variations in the short time measurements. Measuringin, in a situation like in Fig. 1, at a fixed position (i.e. along an arbitrary vertical line), one can measure arbitrary combinations of supercritical laminar traffic, critical laminar traffic, jams, or traffic during acceleration or slowing down, thereby obtaining the characteristic

data cloud of real-world fundamental diagrams.[28,29] This dynamical picture also makes precise Treiterer's hysteresis argument.[30,29] See[6] for more details.

8 Summary and discussion

At a first glance, particle hopping models seem a somewhat crude approximation of real world traffic. Yet, they produce surprisingly realistic dynamics, for example with respect to start-stop wave formation and with respect to fundamental diagrams. The reason for this is that even when the microscopic dynamics is only crudely represented, the macroscopic behavior can still be very realistic. This has already been known for some time and in some cases even been proven for example for the lattice gas methods for Navier-Stokes-equations.[31] More interesting in the context of traffic flow theory are results for one-dimensional systems, and they turn out to be even more instructive than expected.

After a short review of car-following and fluid-dynamical models for vehicular traffic, this paper defines a certain particle hopping model, called STCA for Stochastic Traffic Cellular Automaton. After that, sub-cases or variations of this model are discussed, which make it arguably less realistic, but have the advantage that these cases are well understood. It turns out that two of these sub-cases are described by certain cases of the Lighthill-Whitham theory, which has been used in the traffic context for about 40 years now. And in addition, the way in which the STCA goes beyond these models (by including momentum) is exactly the same way in which recent fluid-dynamical work goes beyond Lighthill-Whitham theory. Moreover, one can show that certain important aspects of the traffic jam dynamics of the STCA are phenomenologically the same as in the modern fluid-dynamical models. Yet, the STCA goes even beyond that at least with respect to fluctuations, which the STCA includes but the fluid-dynamical theories do not.

Thus, one learns that particle hopping models, crude as they are on the microscopic level, are "good" enough to lead to reasonable behavior on the fluid-dynamical level.

It is clear that there are limits to how well particle hopping models will be able to represent microscopic properties of traffic. Sometimes, it will be possible to expand the particle hopping model, for example by choosing a higher resolution,[32] but often enough, it will be necessary to resort to a higher fidelity model. Nevertheless, the body of theory which is already available or currently being developed for particle hopping models puts them into a special position here: Understanding what a model does is the best way of knowing what it cannot do.

In situations where one is interested in mostly macroscopic quantities which could be obtained in principle by a fluid-dynamical model, but needs microscopic ingredients such as individual trip plans, or fast and slow vehicles, or – most importantly – some information about fluctuations, then particle hopping models are a good way to go, especially if one wants to save computational resources.

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