

DOE/PC/90180-T15

Final Report: A Theoretical and Numerical Study of the Flow of Granular Materials Down an Inclined Plane

P. I.:
Professor K. R. Rajagopal
Department of Mechanical Engineering
University of Pittsburgh
Pittsburgh, PA 15261

Student/Degree: R. Gudhe / Ph. D.

Institution: University of Pittsburgh, Pittsburgh, PA 15261

Grant No.: DE-AC22-91PC90180

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1. PROJECT PERSONNEL

K.R. Rajagopal

A. Role in the Project

Rajagopal is the principal investigator for this project. He has supervised a doctoral student during the course of this project.

B. Principal areas of research

Rajagopal's principal areas of research are non-linear fluid and solid mechanics. He has published over 150 papers in the above areas. Recently, he has worked in the area of mechanics of granular materials and fluidization.

C. Time commitment to the project

Rajagopal devotes approximately 2 months during the year, including summer.

D. Education

Ph.D. in Mechanics, University of Minnesota, 1978.

M.S. in Aerospace and Mechanical Engineering, Illinois Institute of Technology, 1974.

B.Tech. in Mechanical Engineering, Indian Institute of Technology, India, 1973.

E. Experience

Rajagopal received his B.Tech from Indian Institute of Technology, Madras in 1973, his M.S. from the Illinois Institute of Technology, Chicago in Mechanical Engineering in 1974 and his Ph.D. from the University of Minnesota in Mechanics in 1978. He is currently James T. MacLeod Professor of Mechanical Engineering (1990), Professor of Mathematics & Statistics (1986) and Professor of Surgery (1994). He was a postdoctoral lecturer at the University of Michigan (1978-80), Assistant Professor at the Catholic University of America (1980-82), Assistant Professor at the University of Pittsburgh (1982-84), Associate Professor (1984-85) and Professor (1985). Rajagopal was also a postdoctoral fellow at the Macromolecular Research Center at the University of Michigan (Jun - Sep 1980), Visiting Scientist at the National Bureau of Standards (1980-82).

F. Professional activities

Societies: Member ASME, Society of Rheology, American Academy of Mechanics, Society of Engineering Science, Society for Natural Philosophy, Society for Industrial and Applied Mathematics.

Editorial Board: Advances in mathematics for Applied Sciences, Archive for Rational Mechanics and Analysis, International Journal of Engineering Science, International Journal of Mathematics and Mathematical Sciences, International Journal of Non-Linear Mechanics, International Journal for Engineering Analysis and Design, Journal of Mathematical and Physical Sciences, Meccanica, Mathematical Models and Methods in the Applied Sciences, Stability and Applied Analysis of Continuous Media, Continuum Mechanics and Thermodynamics.

G. Some recent publications not emanating from the present project (1991-present)

- A Numerical Study of a K-BKZ Fluids Between Rotating Plates, Journal of Non-Newtonian Fluid Mechanics, Vol. 38, pp. 289-312 (1991) (with R.X. Dai and A.Z. Szeri).
- Couette Flow with Entrained Solid Particles, In Recent Advances in Mechanics of Structured Continua, AMD-Vol. 117, ASME, New York, pp. 97-105 (1991) (with G. Johnson and M. Massoudi).

- Flow of a Non-Newtonian Fluid Past a Wedge, *Acta Mechanica*, Vol. 88, pp. 113-123 (1991) (with V.K. Garg).
- Flow of a Shear Thinning Fluid Between Intersection Planes, In *International Journal of Non-Linear Mechanics*, Vol. 26, pp. 769-775 (1991) (with D. Mansutti).
- Flow of a Fluid-Solid Mixture Between Flat Plates, *Chemical Engineering Science*, Vol. 46, pp. 1713-1723 (1991) (with G. Johnson and M. Massoudi).
- Flow of a Fluid Infused with Solid Particles Through a Pipe, *Chemical Engineering Science*, Vol. 29, pp. 649-661 (1991) (with G. Johnson and M. Massoudi).
- Flow of a Non-Newtonian Fluid Through Axisymmetric Pipes of Varying Cross-Sections, In *International Journal of Non-Linear Mechanics*, Vol. 26, pp. 777-792 (1991) (with S. Kasivisvanathan and P.N. Kaloni).
- On an Inhomogeneous Deformation of a Generalized Neo-Hookean Material, *Journal of Elasticity*, Vol. 28, pp. 165-184 (1992) (with L. Tao).
- A Constitutive Equation for Non-Linear Elastic Materials Which Undergo Deformation Induced Microstructural Changes, *International Journal of Plasticity*, Vol. 8, pp. 385-395 (1992) (with A.S. Wineman).
- Some Inhomogeneous Motions and Deformations within the Context of a Non-Linear Elastic Solid, *International Journal of Engineering Science*, Vol. 30, pp. 919-938 (1992) (with J.P. Zhang).
- Secondary Deformation Due to Axial Shearing of a Cylinder of Non-Circular Cross-Section, *International Journal of Engineering Science*, Vol. 30, pp. 1247-1256 (1992) (with J.P. Zhang).
- Circular Shearing and Torsion of Generalized Neo-Hookean Materials, In *Journal of the Institute of Mathematical Applications*, Vol. 48, pp. 23-37 (1992) (with L. Tao).
- Some Remarks on the Modeling of Fluidized Systems, *AIChE Journal*, Vol. 38, pp. 471-473 (1992) (with M. Massoudi, J. Ekmann and M. Mathur).
- Flow of Viscoelastic Fluids Between Rotating Plates, *Theoretical and Computational Fluid Dynamics*, Vol. 3, pp. 185-206 (1992).
- Inhomogeneous Finite Amplitude Motions in a Neo-Hookean Solid, *Proc. Roy. Irish Acad.*, Vol. 92A, pp. 137-147 (1992) (with M.A. Hayes).
- Wave Propagation Through Solids Infused with Fluids, *International Journal of Engineering Science*, Vol. 30, pp. 1209-1232 (1992) (with L. Tao).
- On Darcy's Law as an Approximation in Mathematical Models of Continua: Application to Ground Water Motion Analysis, In *Mathematical Models and Methods in Applied Sciences*, Vol. 3, pp. 231-248 (1993) (with D. Munaf, D. Lee and A.S. Wineman).
- Lubrication with Binary Mixtures: Liquid-Liquid Emulsion, *Journal of Tribology*, Vol. 115, pp. 46-55 (1993) (with A. Al-Arabi, K. Chamniprasart, and A.Z. Szeri).

- Non-radial flow of an Oldroyd-B Fluid Between Intersecting Planes, In Journal of Non-Newtonian Fluid Mechanics, Vol. 46, pp. 49-67 (1993) (with R. Bhatnagar and G. Gupta)
- Lubrication with Binary Mixtures: Liquid-Liquid Emulsion in EHL Contact, In Journal of Tribology, Vol. 115, pp. 515-522 (1993) (with S.H. Wang, A. Al-Arabi and A.Z. Szeri).
- Non-Similar Flow of a Non-Newtonian Fluid Past a Wedge, In International Journal of Engineering Science, Vol. 31, pp. 637-648 (1993) (with D. Mansutti and G. Pontrelli).
- Flows of Non-Newtonian Fluids Past an Infinite Porous Plate with Suction or Injection, In International Journal of Numerical Methods in Fluids, Vol. 17, pp. 927-943 (1993) (with D. Mansutti and G. Pontrelli).
- On the Mechanics of a Mixture of Fluid and Particulate Solid, ASME-AMD Vol. 162, pp. 249-265 (1993).
- Lubrication with Emulsion in Cold Rolling, In Journal of Tribology, Vol. 115, pp. 523-531 (1993) (with S.H. Wang and A.Z. Szeri).
- Modeling Electro-rheological Materials using Mixture Theory, In Electro-rheological Flows (eds. Signer, Kim and Bajura) ASME-FED, Vol. 164, pp. 1-9 (1993).
- On Free Surface Problems in Fluid Mechanics, Submitted for publication (with F. Baldoni).
- Flow of an Oldroyd-B Fluid Due to a Stretching Sheet, Submitted for publication (with R. Bhatnagar and G. Gupta).
- Flow of a Non-Newtonian Fluid Between Two Inclined Planes, One of Which is Stationary and the Other Moving, In Press, Rheologica Acta (with Yu-Ning Huang and R. Bhatnagar).
- Some Remarks on the Stability of Free Surface Flows, submitted for publication (with L. Tao).
- On Constitutive Equations for Electrorheological Materials, In Press, Continuum Mechanics and Thermodynamics, (with A.S. Wineman).
- Viscoelasticity of Biological Tissues: Theory and Applications to Ligaments and Tendons, submitted for publication (with G.A. Johnson and S.L.-Y. Woo).
- Boundary Layers in Finite Thermoelasticity, In Press, Journal of Elasticity.
- Finite Circumferential Shearing of Non-Linear Solids within the Context of Thermoelasticity, In Press, IMA Journal of Applied Mathematics (with Yu-Ning Huang).
- Existence of Solutions for Squeezing and Opening of Non-Linearly Elastic Wedges, Submitted for publication (with J.B. McLeod).
- On Necessary Condition Conditions for the Secondary Flow of Non-Newtonian Fluid in Straight Tubes, In Press, International Journal of Engineering Science (with Yu-Ning Huang).

Here we carry out a systematic numerical study of the flow of granular materials down an inclined plane using the models that stem from both the continuum theory approach and the kinetic theory approach. We also look at the **existence of solutions, multiplicity and stability of solutions** to the governing equations.

1.2.2 Relation to the DOE Mission

This research work is part of DOE's program in the transportation of granular solids. The goal of this research is to provide a better understanding of the mechanics of the such materials.

1.2.3 Project History

The funding budgeted for this research grant is:
From January 1, 1991 to December, 31, 1994 is \$224,250.00

- A Continuum Theory for the Thermomechanics of Solidification, Submitted for publication (with F. Baldoni).
- Mechanics of Non-Newtonian Fluids, In Recent Developments in Theoretical Fluid Mechanics (eds. G.P. Galdi and J. Nečas), pp. 129-162, Pitman Research Notes in Mathematics Series, Vol. 291, Longmans Scientific and Technical (1993).
- Flow of a Second Grade Fluid over a Porous Elastic Sheet Due to Stretching, In Press, Journal of Mathematical and Physical Sciences (with M. Massoudi and C.E. Maneschy).
- Inhomogenous Deformations in Finite Thermoelasticity, Submitted for publication (with M. Massoudi and C.E. Maneschy).
- On Necessary and Sufficient Conditions for Turbulent Secondary Flows in a Straight Tube, In Press, In Mathematical Models and Methods in Applied Sciences (with Yu-Ning Huang).

Articles which are listed as In Press have been accepted for publication.

1.1 Additional Project Personnel

Mr. R. Gudhe worked on this project for his doctoral dissertation.

1.2 Project Overview

1.2.1 Project Objectives

The mechanics of the flowing granular materials such as coal, agricultural products, fertilizers, dry chemicals, metal ores, etc. have received a great deal of attention as it has relevance to several important technological problems. Despite wide interest and more than five decades of experimental and theoretical investigations, most aspects of the behavior of flowing granular materials are still not well understood. So Experiments have to be devised which quantify and describe the non-linear behavior of the granular materials, and theories developed which can explain the experimentally observed facts.

2. SCIENTIFIC AND TECHNICAL CONTENT

As many models have been suggested for describing the behavior of granular materials, from both continuum and kinetic theory viewpoints, we proposed to investigate the validity and usefulness of representative models from both the continuum and kinetic theory points of view, by determining the prediction of such a theory, in a representative flow, with respect to existence, non-existence, multiplicity and stability of solutions. The continuum model to be investigated is an outgrowth of a model due to Goodman and Cowin (1971, 1972) and the kinetic theory models being those due to Jenkins and Richman (1985) and Boyle and Massoudi (1989). In the following sections we present detailed results regarding the same. Interestingly, we find that the predictions of all the theories, in certain parameter space associated with these models, are qualitatively similar. This ofcourse depends on the values assumed for various material parameters in the models, which as yet are unknown, as reliable experiments have not been carried out as yet for their determination.

2.1 Numerical Study of the Flow of Granular Materials Down an Inclined Plane Using a Continuum Model

A granular material is defined as an assembly of discrete solid components dispersed in space such that the solid constituents are in contact with the near neighbors. The behavior of granular materials governed by interparticle cohesion, friction, and collisions. So any theory attempting to describe the behavior of the flow of granular materials should embody these features and characteristics. For example, granular materials are not solids since they take the shape of the vessel containing them; and they cannot be considered as a fluid as they can be piled into heaps. In fact granular material exhibit behavior like non-Newtonian fluids and non-linearly elastic solids.

The Cauchy stress tensor \mathbf{T} in a granular material seems to depend on the manner in

which the granular material is distributed, *i.e.*, the volume fraction v and possibly also its gradient, and the symmetric part of the velocity gradient D . Thus, we shall assume that:

$$T = f(v, \nabla v, D), \quad (2-1)$$

Using standard arguments in mechanics, it is possible to find restrictions on the form of the above constitutive expression based on the assumption of frame-indifference, isotropy, *etc.* [cf. Truesdell and Noll (1965)]. There could be further restrictions on the form of the constitutive expression because of internal constraints like incompressibility, and conformity with thermodynamics. A constitutive model that predicts the possibility of both of the normal stress-differences and that is properly frame invariant is given by [cf. Goodman and Cowin (1971, 1972), Rajagopal and Massoudi (1990)]:

$$T = \{ \beta_0(v) + \beta_1(v) \nabla v \cdot \nabla v + \beta_2(v) \operatorname{tr} D \} \mathbf{1} + \beta_4(v) \nabla v \otimes \nabla v + \beta_3(v) D, \quad (2-2)$$

where $\beta_0(v)$ is similar to pressure in a compressible fluid and is given by an equation of state, $\beta_2(v)$ is like the second coefficient of viscosity in a compressible fluid, $\beta_1(v)$ and $\beta_4(v)$ are the material parameters connected with the distribution of the granular materials and $\beta_3(v)$ is the viscosity of the granular material. The above model allows for normal-stress differences, a feature observed in granular materials. In general, the material properties β_0 through β_4 are functions of the density (*or* volume fraction v), temperature, and the principal invariants of the stretching tensor D , given by

$$D = \frac{1}{2} [(\nabla u) + (\nabla u)^T], \quad (2-3)$$

where u is the velocity of the particles. In equation (2-2), $\mathbf{1}$ is the identity tensor, ∇ the gradient operator, \otimes indicates the outer (dyadic) product of two vectors, and tr designates the trace of a tensor. Furthermore, v is related to the bulk density of the material ρ , through

$$\rho = \gamma v,$$

where γ is the actual density of the grains at the place x and time t and the field v is called the volume fraction (*or* the volume distribution) and is related to the porosity \bar{n} *or* the void ratio \bar{e} by

$$v = 1 - \bar{n} = \frac{1}{1+e}; \quad \text{with } 0 \leq v < 1.$$

As mentioned earlier that $\beta_0(v)$ plays the role of pressure in a compressible gas, with v now playing the role of a density. Assumption of a form similar to that for ideal gases leads as to conclude that β_0 varies linearly with volume fraction. The works of Walton and Braun (1986) support the assumption that the viscosity is a function of both the solid volume fraction v and the stretching tensor D , and varies as a quadratic function of v , D being held fixed. Following, Rajagopal and Massoudi (1990) we shall assume that the material parameters have the structure

$$\beta_0(v) = k v \quad (2-4)$$

$$\beta_1(v) = \beta_{10} + \beta_{11}v + \beta_{12}v^2$$

$$\beta_2(v) = \beta_{20} + \beta_{21}v + \beta_{22}v^2$$

$$\beta_3(v) = \beta_{30} + \beta_{31}v + \beta_{32}v^2$$

$$\beta_4(v) = \beta_{40} + \beta_{41}v + \beta_{42}v^2 \quad (2-5)$$

The above representation can be viewed as a Taylor series approximation for the material parameters. Such a quadratic dependence, atleast for the viscosity β_3 is borne out on the basis of dynamic simulations of particle interactions [cf. Walton and Braun (1986)]. Further restrictions on the coefficients can be obtained by using the following argument. Since the stress should vanish as $v \rightarrow 0$, we can conclude that

$$\beta_{30} = \beta_{20} = 0$$

The rationale for the structure given above can be found in Rajagopal and Massoudi (1990). Also, Johnson et al. (1991a, b) used this model to study two phase flows. Further, Rajagopal and Massoudi (1990) and Rajagopal et al. (1992) have shown that

$$k < 0 \quad (2-6)$$

as compression should lead to densification of the material.

Consider the flow of granular materials down an inclined plane modeled by the continuum model proposed by Rajagopal and Massoudi (1990) (cf. Figure 1) due to the

action of gravity. The flow of granular materials down an inclined plane has been studied by several authors [cf. Goodman and (1971), Savage (1979), Hutter, Szidarovszky, and Yakowitz (1986a, b), Johnson and Jackson (1987)]. However, none of the above studies incorporate thermal effects and therefore do not consider the balance of energy. While Hutter et al. (1986a, b) allow for a quantity referred to as "fluctuating energy" and provide an equation for its determination, this equation is not the energy equation. The fluctuating energy represents variations of the velocity from a mean velocity and is like turbulent fluctuations. While the spirit of this approach and that of Hutter et al. (1986a, b) are different, the governing equations for the velocity field could be cast into a form that is similar to theirs. Hutter et al. (1986a, b) show that the existence or non-existence of solutions to their equations depend on the type of boundary conditions that they impose, and to our knowledge this is the first study which investigates the question of how the boundary conditions influence the existence of solutions. Recently, Rajagopal, Troy, and Massoudi (1992) studied questions regarding existence and uniqueness of solutions to the equations governing the flow of granular materials down an inclined plane, in which thermal effects are taken into account and the energy equation is also included. They delineate a range of values for the material parameters, which are assumed to be constant, which ensure **existence of solutions** to the equations under consideration. They also prove rigorously that for certain range of values of the parameters **no solutions exist**, while for others there is **multiplicity of solutions**. In this problem, we consider the steady one-dimensional flow of incompressible granular materials (i.e., $\gamma = \text{constant}$) down an inclined plane, where the angle of inclination is α . But in general one has to study an unsteady two-dimensional problem. Here, we assume that the flow is a fully developed steady flow. Let us further suppose that the inclined plane is maintained at a constant temperature Θ_w , which is at a higher temperature than the temperature of the surrounding environment Θ_∞ , and, as a result, there is transfer of heat. Further, suppose the heat flux vector \mathbf{q} satisfies Fourier's law with constant thermal conductivity, i.e.,

$$\mathbf{q} = -K \nabla \Theta, \quad (2-7)$$

where Θ is the temperature and K is the thermal conductivity, which in general is a function of volume fraction and Θ . At this juncture it would be appropriate to point out that in theories for granular materials based on a kinetic theory approach, the fluctuations in the velocity field give rise to the notion of granular temperature. The convective heat transport, within the context of such theories, is determined by the fluctuations in the velocity field. It is also the conventional wisdom that this mechanism is important for the heat transfer process. In this approach, we have ignored the fluctuations in the velocity

field as the theory does not allow for such velocity fluctuations, and moreover within the context of the continuum theory, the phenomena of heat transfer is incorporated in the energy equation. To include in addition to the energy equation, the notion of granular temperature would be inconsistent with this approach. We feel that this approach is applicable when the packing of the material is reasonably compact and the fluctuations from the mean are not significant. For a fully developed flow, within the context of a continuum theory, wherein the flow is unidirectional as in this case, fluctuations of the velocity normal to the flow direction cannot be incorporated. While this may be a shortcoming of this approach we see that even with the neglect of such fluctuations, heat transfer within the context of the continuum model has a pronounced effect on the nature of the solution [cf. Schlünder (1980, 1982), Wunschmann and Schlünder (1980), Buggisch and Löffelmann (1989)].

For the problem under consideration, the following assumptions are made:

- Steady motion
- Incompressible granular materials, i.e., $\gamma = \text{constant}$
- Negligible radiant heating, i.e., $r = 0$
- The constitutive equation for the stress tensor is given by equation (2-2), and density, velocity, and temperature fields are assumed to be of the form

$$\begin{aligned} v &= v(y) \\ u &= U(y) \\ \Theta &= \Theta(y) \end{aligned} \tag{2-8}$$

We shall consider three cases. The first is the case that β_1 through β_4 and the thermal conductivity K are assumed to be constants. In the second case, it is assumed that β_1 and β_4 are constant, β_2 and β_3 have a quadratic variation in volume fraction and the thermal conductivity K is linear in the volume fraction. In the third case, we consider a purely mechanical problem and assume β_1 through β_4 have a quadratic variation in the volume fraction.

2.1.1 Case I: β_1 through β_4 and thermal conductivity K are constant

Here, we assume β_1 through β_4 and the thermal conductivity K to be constant, with β_0 given by equation (2-4). With the above assumptions, the conservation of mass is identically satisfied and from the balance of linear momentum and balance of energy we end up with three coupled ordinary differential equations. These equations are appropriately non-dimensionalized and are solved numerically using a collocation code COLSYS [cf. Ascher et al. (1981)]. The non-dimensional parameters R_1 , R_2 , R_3 , and R_4 are given by

$$\begin{aligned} R_1 &= \frac{k}{h \gamma g}; & R_2 &= \frac{2(\beta_1 + \beta_4)}{h^3 \gamma g} \\ R_3 &= \frac{\beta_3 u_0}{2 h^2 \gamma g}; & R_4 &= \frac{\beta_3 u_0^2}{2 K (\Theta_w - \Theta_\infty)} \end{aligned} \quad (2-9)$$

It follows from Rajagopal and Massoudi (1990) that R_1 must always be less than zero for the solution to exist and all the other non-dimensional parameters, *i.e.* R_2 , R_3 and R_4 must be greater than zero.

Boundary Conditions

In general, whether it is the kinetic theory approach or the continuum approach, the need for additional boundary conditions arises. In the continuum theories of Goodman and Cowin (1971, 1972) and its modifications [cf. Ahmadi (1982a, b), Passman et al. (1980), and Johnson et al. (1991a, b)], two boundary conditions on the volume fraction are required. In the numerical solution of shearing motion of a fluid-solid flow, Passman et al. (1980) prescribed the values of the volume fraction at the two plates. An alternative way is to use experimental results, if they are available. Later, Johnson et al. (1991a, b) considered this issue and suggested using an integral condition.

In the kinetic theory approach, additional boundary conditions are also necessary for the value of the fluctuating energy [which is related to what is usually referred to as the granular temperature]. There have been many attempts at looking at this issue [cf. Haff (1983, 1986), Hui et al. (1984, 1986), Hutter et al. (1986a, b), and Jenkins (1992)]. The

effect of boundaries on the flow of granular materials has been studied experimentally by Hanes and Inman (1985). Hanes and Inman (1985) performed the experiment by glueing particles on the wall. Craig et al. (1987) have also looked at the effect of boundary conditions.

Whether it is the continuum approach or the kinetic theory approach, slip may occur at the wall, especially when the interstitial fluid is a gas, and therefore the classical adherence boundary condition at the wall no longer applies. In this approach, we follow a procedure similar to that of Hutter et al. (1986a, b) in specifying the slip at the wall. Perhaps, the idea of specifying slip at the wall goes back to Navier¹ [cf. Schowalter (1988)] who introduced a constant ζ_1 to describe slip, where at the wall

$$\zeta_1 u_s = \mu_f \frac{du_f}{dn}, \quad (2-10)$$

where u_s is the slip velocity, u_f the fluid velocity, and n the normal of the wall directed into the fluid. There has been evidence for many years that for flows of some non-Newtonian fluids slip occurs at the wall. In fact, it is possible that the boundary condition is more complex in that the material stick-slip on the boundary. If the shear stress is below a certain value, the material adheres to the boundary while it slips above a critical value of the shear stress. Such a phenomenon has been observed in polymeric materials and is the source of a lot of surface instabilities observed in polymeric extrudates. An attempt to generalize condition (2-10) for non-Newtonian fluids was made by Pearson and Petrie (1964) in the form

$$u_s = f(\tau_w) \tau_w, \quad (2-11)$$

where τ_w is the wall shear stress. Hutter et al. (1986a, b) and Szidarovszky et al. (1987) use a similar relationship to relate the slip velocity, u_s , and the fluctuating energy of the flow, θ , of avalanches on an inclined plane. Specifically, they use

$$u_s = f_1(\tau^2) \tau, \quad (2-12)$$

$$\theta = g_1(u^2) u^2, \quad (2-13)$$

$$\text{where, } \tau = \frac{2}{5} (2 + \zeta_2) \bar{\chi} \kappa \quad (2-14)$$

¹For a review of slip boundary conditions, see Lugt and Schot (1974).

$$\text{with, } \kappa = \frac{1}{2} \frac{du}{dy}$$

$$\bar{\chi} = \frac{2 \gamma \bar{\sigma} (1+e)}{\sqrt{\pi}} \Phi_0(v) \sqrt{\theta}$$

$$\Phi_0(v) = v^2 g_0(v) = \frac{v^2 (2-v)}{2 (1-v)^3}$$

$$\rho = \gamma v$$

$$g_0(v) = \frac{1}{1-v} + \frac{3v}{2(1-v)^2} + \frac{v^2}{2(1-v)^3} \quad (2-15)$$

where $\bar{\sigma}$ is the particle diameter, e is the coefficient of restitution, and f_1 and g_1 depend on the surface roughness. Hutter et al. (1986a, b) suggest the following forms for f_1 and g_1

$$f_1(x^2) = \hat{C}(x^2)^{\frac{n-1}{2}} \quad (2-16)$$

$$g_1(x^2) = \bar{C}(x^2)^{\frac{m-1}{2}} \quad (2-17)$$

with sliding coefficients \hat{C} and \bar{C} and power-law exponents n and m . The case $\hat{C} \rightarrow 0$ corresponds to the classical no-slip boundary condition. Szidarovszky et al. (1987) provide an alternative boundary condition for (2-13) in the form of

$$\hat{a} \theta + \bar{\chi} \frac{d\theta}{dy} = 0, \quad (2-18)$$

which indicates that the gradient of the fluctuating energy at the wall is proportional to the fluctuating energy.

It is assumed that the slip velocity is proportional to the stress vector at the wall. That is

$$\mathbf{u}_s = \hat{f}_3((\mathbf{T}\mathbf{n})_x, (\mathbf{T}\mathbf{n})_y) \quad (2-19)$$

where \mathbf{T} is the stress tensor, \mathbf{n} is a unit normal vector, and \hat{f}_3 in general could be a function of surface roughness, volume fraction, shear rate, etc.

For the problem under consideration, Rajagopal et al. (1992), proved rigorously that the equations admit non-unique solutions, one in which the volume fraction increases monotonically, and the other in which it decreases monotonically, from the inclined plane to the free surface. Multiple solutions are depicted for case II and is not given for case I

and case III. By observing the exact solution obtained for the volume fraction [cf. Gudhe et al. (1993)], it can be seen that the volume fraction profile increases monotonically from the inclined plane to the free surface. This solution is not physically expected, but by carrying out the stability analysis, it is possible to check whether this solution is stable or not. A parametric study of the numerical solutions of volume fraction, velocity, and temperature profiles is carried out and the results are presented in the form of graphs [cf. Gudhe et al. (1993)]. Figures 2 through 5 depict the variation of the volume fraction, velocity and temperature for various values of the non-dimensional parameters, and are self-explanatory.

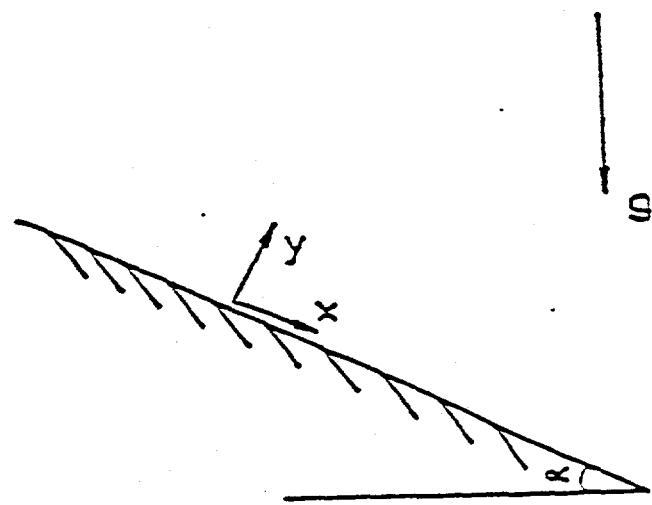


Figure 1. Flow Down An Inclined Plane

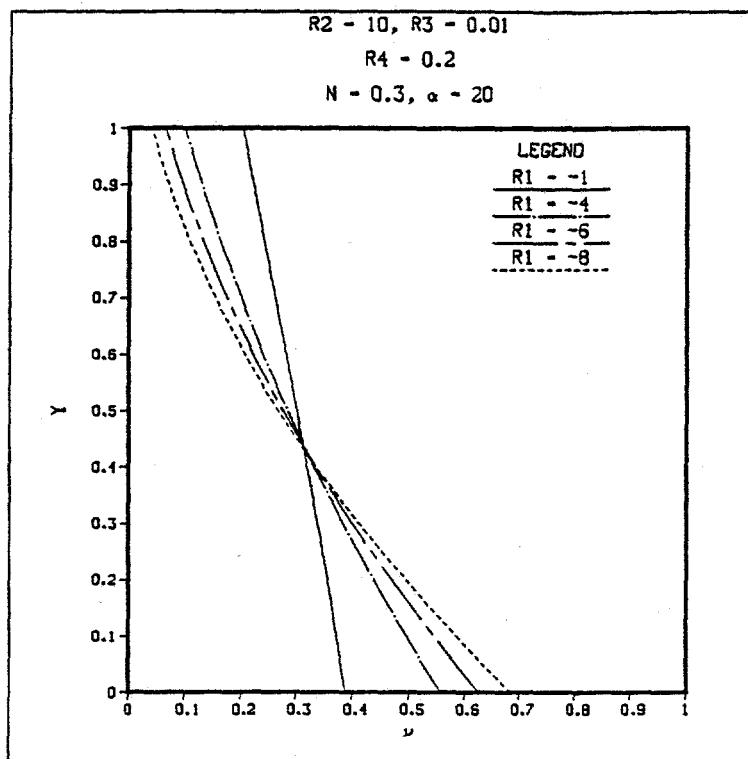


Figure 2. Effect of R_1 on Volume Fraction

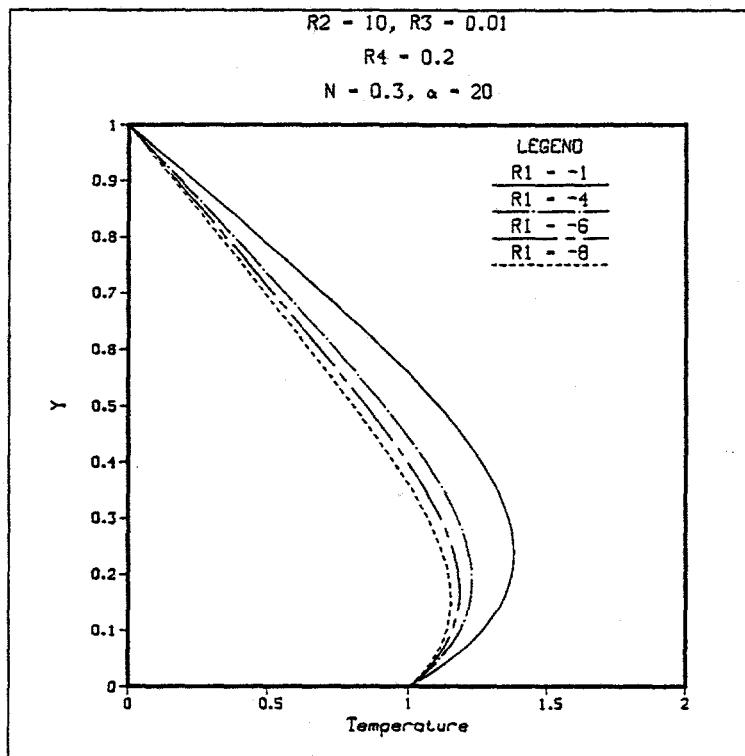


Figure 3. Effect of R_1 on Temperature Profile

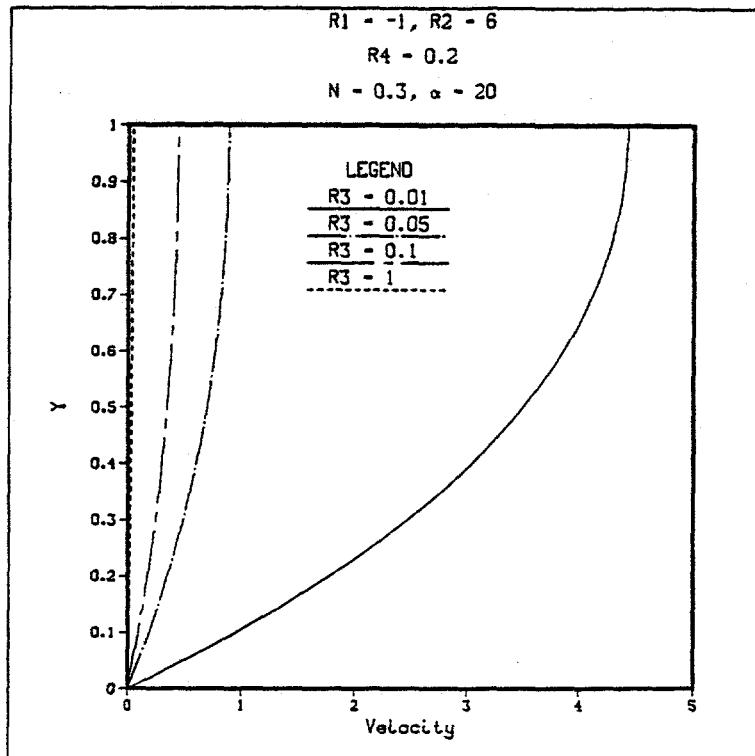


Figure 4. Effect of R_3 on Velocity Profile

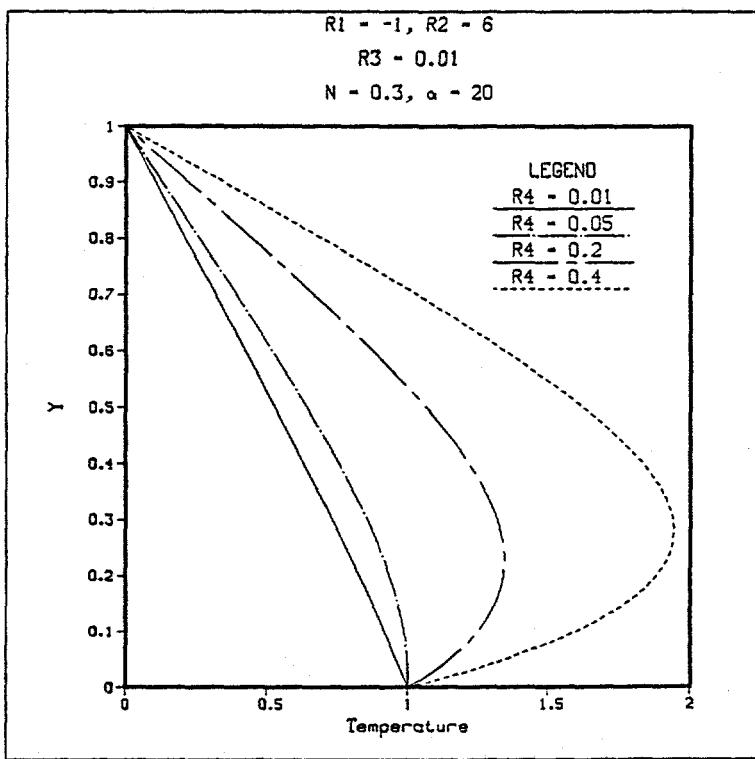


Figure 5. Effect of R_4 on Temperature Profile

2.1.2 Case II: β_3 Varies quadratically in v and the thermal conductivity K varies linearly in v with β_1 , β_2 and β_4 being constant

Here the viscosity of the granular material and its thermal conductivity are allowed to vary with the volume fraction in a manner that is consistent with the physics of the problem. The equations for the conservation of mass, momentum, and energy are solved numerically and we explicitly demonstrate the non-unique solutions, corresponding to the same flow rate of the granular materials. Further, it is assumed that β_1 , β_2 and β_4 are constant. However, the viscosity β_3 is assumed to be of the form [cf. Johnson et al. (1991a, b)]:

$$\beta_3 = \hat{\beta}_3 (v + v^2), \quad \text{where, } \hat{\beta}_3 \text{ is a constant} \quad (2-20)$$

Based on the numerical simulations of Walton and Braun (1986) that suggest a quadratic variation in volume fraction. However, their analysis allows for the viscosity to vary with the shear rate, a feature that is not present in this work. Even so, at fixed shear rate, their simulation implies a quadratic variation in the volume fraction. For K , it is assumed that [cf. Bashir and Goddard (1990) and Batchelor and O'Brien (1977)]:

$$K = K_m (1 + 3 \zeta v), \quad (2-21)$$

$$\text{where, } \zeta = \frac{(\psi_1 - 1)}{(\psi_1 + 2)} \quad (2-22)$$

Here, ψ_1 = ratio of conductivity of the particle to that of the matrix, and K_m is the conductivity of the matrix.

With the above assumptions and the flow field given by equation (2-8), the conservation of mass is identically satisfied and from the balance of linear momentum and balance of energy we end up with three coupled ordinary differential equations. These equations are solved numerically. It follows from Rajagopal and Massoudi (1990) that R_1 must always be less than zero for the solution to exist and all the other non-dimensional parameters, *i.e.* R_2 , R_{21} , A_3 , A_4 and A_5 must be greater than zero. For the problem under consideration, Rajagopal et al. (1992), proved rigorously that the equations admit non-unique solutions, one in which the volume fraction monotonically increases, and the other in which it monotonically decreases, from the inclined plane to

the free surface. Here, a parametric study of the equations is carried out and delineate how the various non-dimensional parameters affect the structure of the solution. For more details we refer to Gudhe et al. (1993).

The non-dimensional parameters R_{21} , A_3 , A_4 , A_5 and \bar{f} are given by

$$\begin{aligned} R_{21} &= \frac{\beta_1}{h^3 \gamma g}; & A_3 &= \frac{\hat{\beta}_3 u_0}{2 h^2 \gamma g}; & A_4 &= \frac{\hat{\beta}_3 u_0^2}{2 K_m (\Theta_w - \Theta_\infty)} \\ A_5 &= \frac{q_f h}{K_m (\Theta_w - \Theta_\infty)}; & \bar{f} &= \frac{\hat{f} h \gamma g}{u_0} \end{aligned} \quad (2-23)$$

In Figure 6, multiple solutions for the volume fraction are depicted, one in which the volume fraction increases from the plane to the free surface and the other which has the opposite character. Figures 7 to 11 depict the variation of velocity and temperature for various values of the non-dimensional numbers.

The existence of multiple solutions leads immediately to the question of stability of these solutions and this is taken up later.

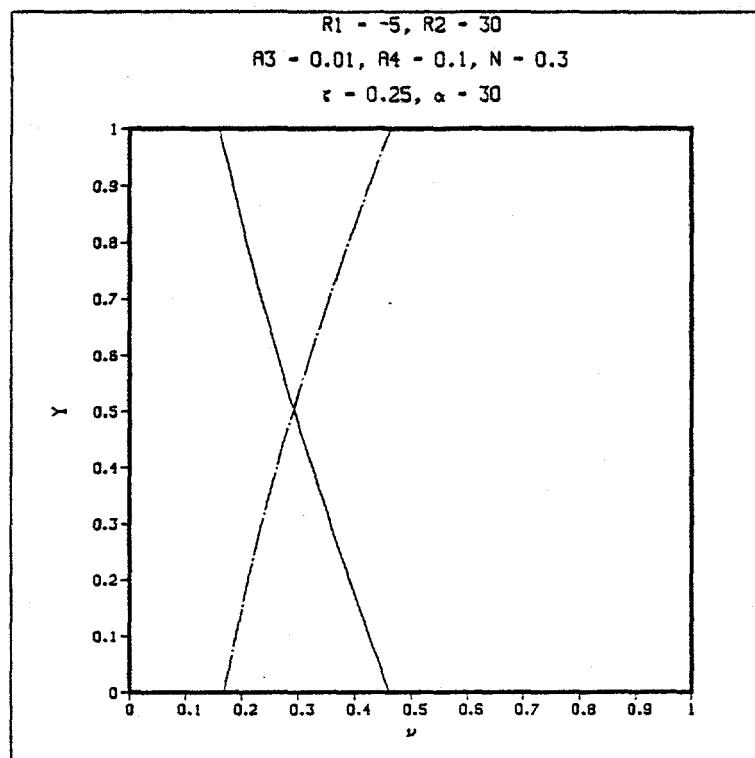


Figure 6. Multiple Solutions for the Volume Fraction Profile

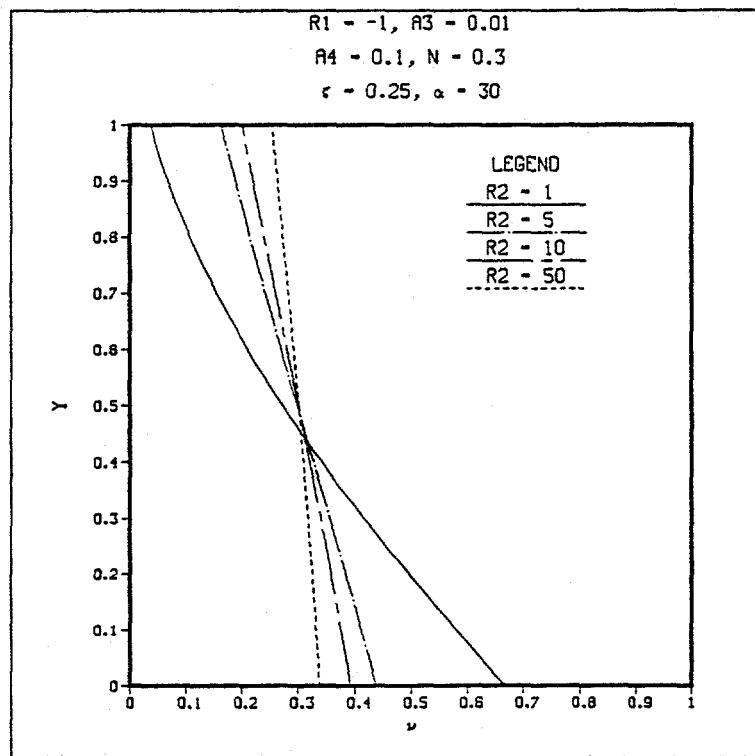


Figure 7. Effect of R_2 on the Volume Fraction

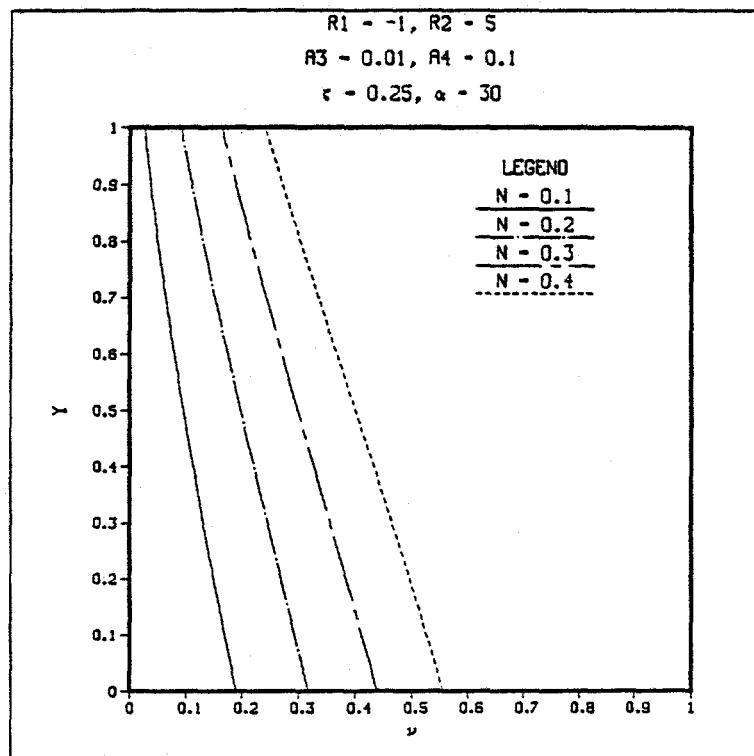


Figure 8. Effect of N on the Volume Fraction

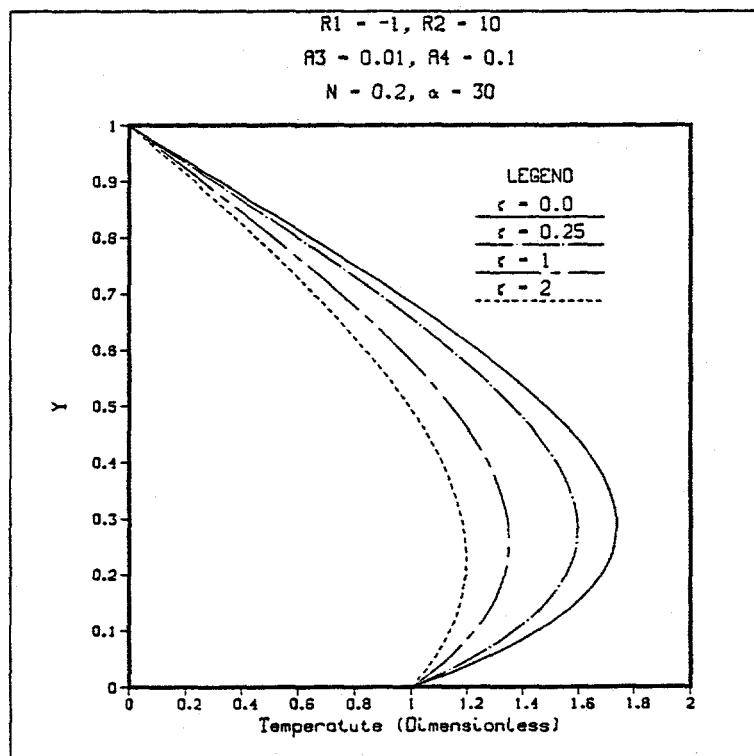


Figure 9. Effect of ζ on the Temperature Profile

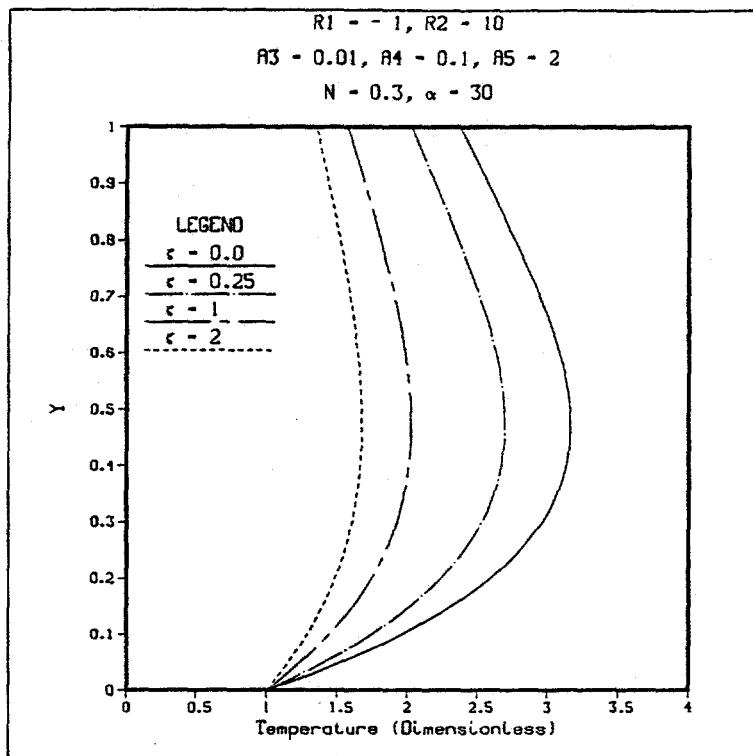


Figure 10. Effect of ζ on the Temperature Profile with Heat Transfer at the Surface

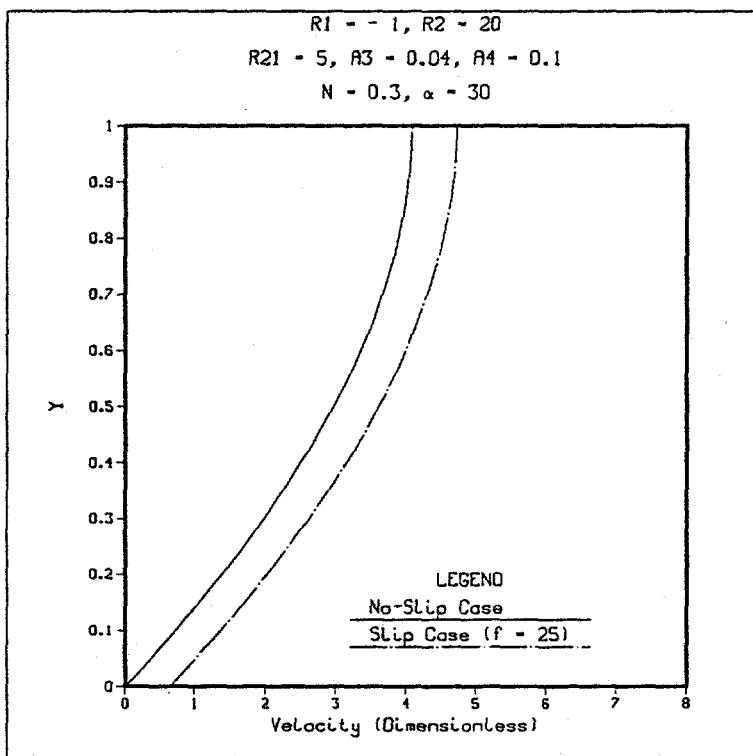


Figure 11. Effect of No-Slip and Slip on the Velocity Profile

2.1.3 Case III: β_1 through β_4 vary quadratically in ν

It is assumed that β_0 and β_3 are given by equations (2-4) and (2-10) with β_1 , β_2 and β_4 to be quadratic in volume fraction are given by

$$\beta_1 = \hat{\beta}_1 (1 + \nu + \nu^2), \quad \text{where, } \hat{\beta}_1 \text{ is a constant} \quad (2-24)$$

$$\beta_2 = \hat{\beta}_2 (1 + \nu + \nu^2), \quad \text{where, } \hat{\beta}_2 \text{ is a constant} \quad (2-25)$$

$$\beta_4 = \hat{\beta}_4 (1 + \nu + \nu^2), \quad \text{where, } \hat{\beta}_4 \text{ is a constant} \quad (2-26)$$

In this case, we consider the purely mechanical problem. With the above assumptions and the flow field given by (2-8)_{1,2}, the conservation of mass is identically satisfied and from the balance of linear momentum we end up with two coupled ordinary differential equations. These equations are solved numerically. It follows from Rajagopal and Massoudi (1990) that R_1 must always be less than zero for the solution to exist.

2.1.4 Comparision of case I, II and III results

The volume fraction, velocity and temperature profiles are compared for case I, II and III. In Figure 12, the volume fraction profiles are shown for case I and III for the same values of non-dimensional paramters and for case II it is same as Case I. The velocity profiles are shown for Case I, II and III in Figure 13. Notice, that the velocity depends upon the form assumed for β_1 through β_4 . Finally, in Figure 14 the temperature profiles are shown for case I and II. We notice that significant changes in the temperature profile can be effected depending upon the form assumed for the thermal conductivity of the particles.

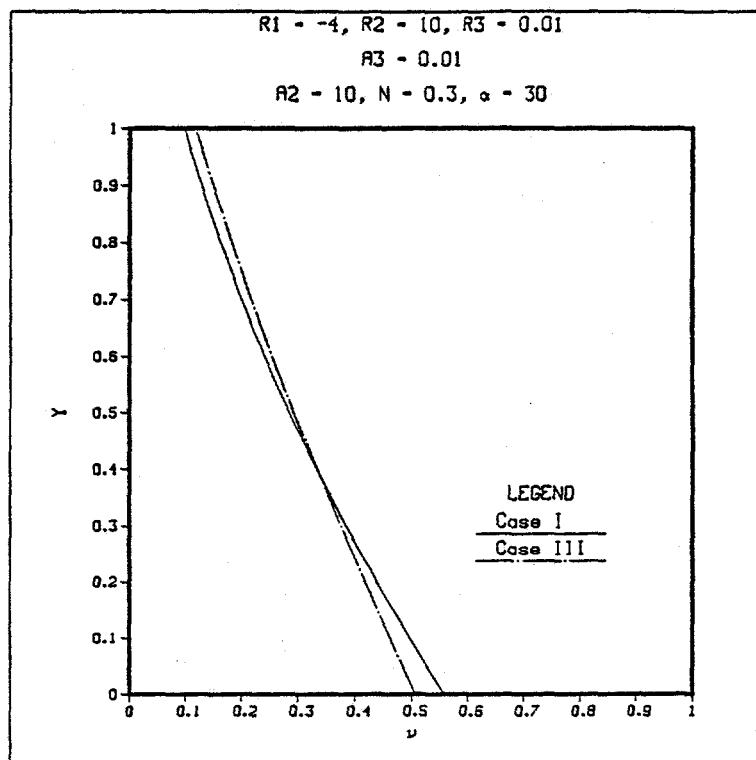


Figure 12. Comparison of Volume fraction Profiles

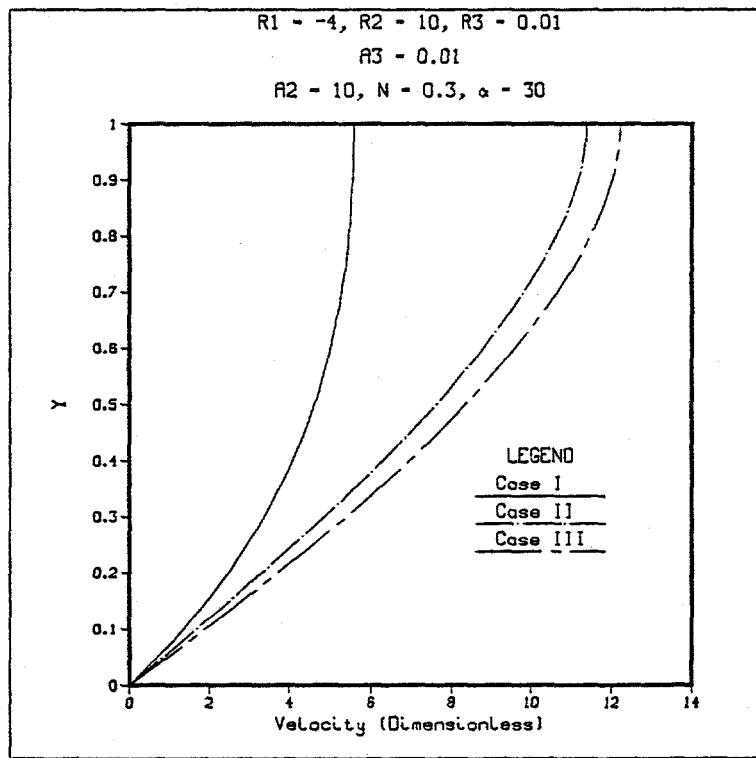


Figure 13. Comparison of Velocity Profiles

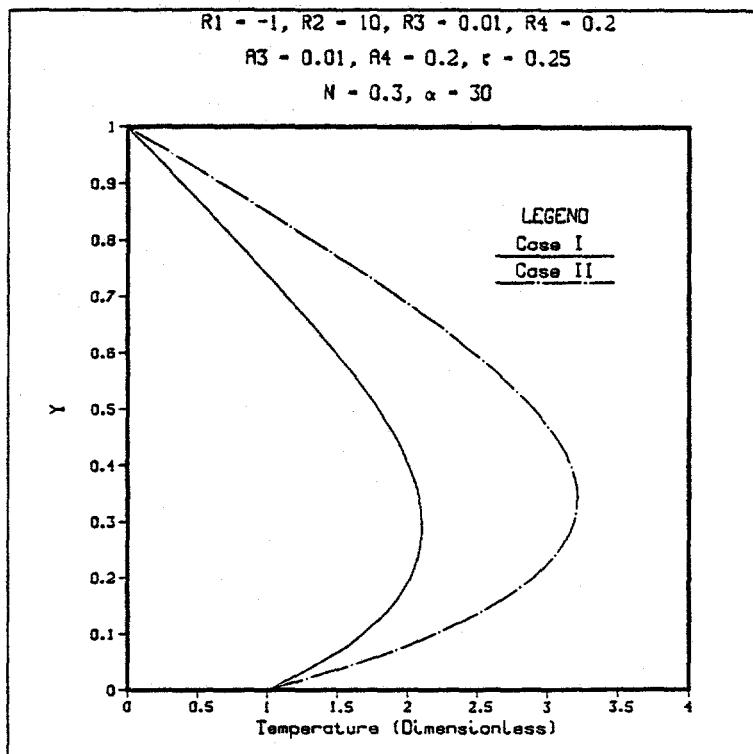


Figure 14. Comparison of Temperature Profiles

2.2 Numerical Study of the Flow of Granular Materials Down an Inclined Plane Using Kinetic Theory Approach

Once again we consider the flow of granular materials down an inclined plane (cf. Figure 1) due to the action of gravity.

2.2.1 Case A: Jenkins and Richman constitutive equations

Richman and Marciniec (1990) studied the above problem using the constitutive equations proposed by Jenkins and Richman (1985a) by making approximation that the field variable could be replaced by the average value. This simplified the problem and made it possible for Richman and Marciniec (1990) to obtain exact solutions. Here, the full equations for the flow of granular materials down an inclined plane are derived. The constitutive equation for the Cauchy stress tensor \mathbf{T} is given by [cf. Richman and Marciniec (1990)]

$$\mathbf{T} = -\{4 \rho G F \theta\} \mathbf{1} + 2 \mu E \hat{\mathbf{D}} \quad (2-27)$$

$$\text{Where, } F(v) = 1 + \frac{1}{4G} \quad (2-28)$$

$$E(v) = 1 + \pi \frac{(1+5/8G)^2}{12} \quad (2-29)$$

$$G(v) = \frac{v(2-v)}{2(1-v)^3} \quad (2-30)$$

$$\mu = \frac{8 \sigma \rho G \theta^{1/2}}{5 \pi^{1/2}} \quad (2-31)$$

In the above equation \mathbf{T} denotes the Cauchy stress, v the volume fraction of the solid, $\hat{\mathbf{D}}$ denotes the deviatoric part of stretching tensor \mathbf{D} associated with the solid motion and θ the granular temperature.

For the problem under consideration, it is assumed that the volume fraction, velocity and granular temperature to be of the form

$$\begin{aligned} v &= v(y) \\ u &= U(y)i \\ \theta &= \theta(y) \end{aligned} \quad (2-32)$$

With the above flow field the conservation of mass is automatically satisfied. From the balance of linear momentum and pseudo-energy we have three coupled ordinary differential equations. These equations are solved numerically. A parametric study of the numerical solutions of volume fraction, velocity and granular temperature profiles is carried out and the results are presented in the form of graphs. The value of β is chosen from the predictions of the boundary conditions for the energy flux, from the solution of Richman and Marciniec (1990).

The manner in which the volume fraction, velocity and granular temperature change with the angle of inclination (α) and Q are shown in Figures 15, 16 and 17 when $e = 0.95$ and $e_w = 0.8$ ($\hat{r} = .5$, and $\tilde{r} = .414$). Notice, that the volume fraction decreases from the surface of the plane to the free surface, the velocity increases from the plane to the free surface and the granular temperature increases from the surface of the plane to the free surface. Similarly, in the second case profiles of granular temperature are shown in

Figures 18 when $e = 0.8$ and $e_w = 0.95$ ($\hat{r} = .5$, and $\bar{r} = .414$). Here, it is observed that the granular temperature decreases from the surface of the inclined plane to the free surface. We notice that the analytical solution of Richman and Marcineic (1990) agrees reasonably well with the solutions to the full equations.

See Figure 15 v is not equal to zero at $y/\beta = 0$, and notice that the solutions due to Richman and Marcineic (1990) for the volume fraction always attain the value zero at the free surface due to the prescription of the boundary condition. The solution to the full equations on the other hand do not go to zero. This in general would lead to difficulties with regard to the satisfying null normal tractions on the boundary. However, it is observed that corresponding to the solutions, the stresses are very small, of the order of 10^{-3} , which suggests the possibility that it may be feasible to patch up a thin boundary layer at the surface within which the stresses may decrease to zero. However, within the context of the equations and boundary conditions, there is no alternative but to accept the value that is a consequence of the solution to the boundary value problem. It is also possible that allowing β to be determined by the solution rather than prescribing it, from the Richman and Marcineic (1990) analysis might lead to the vanishing of the normal stress at the free boundary.

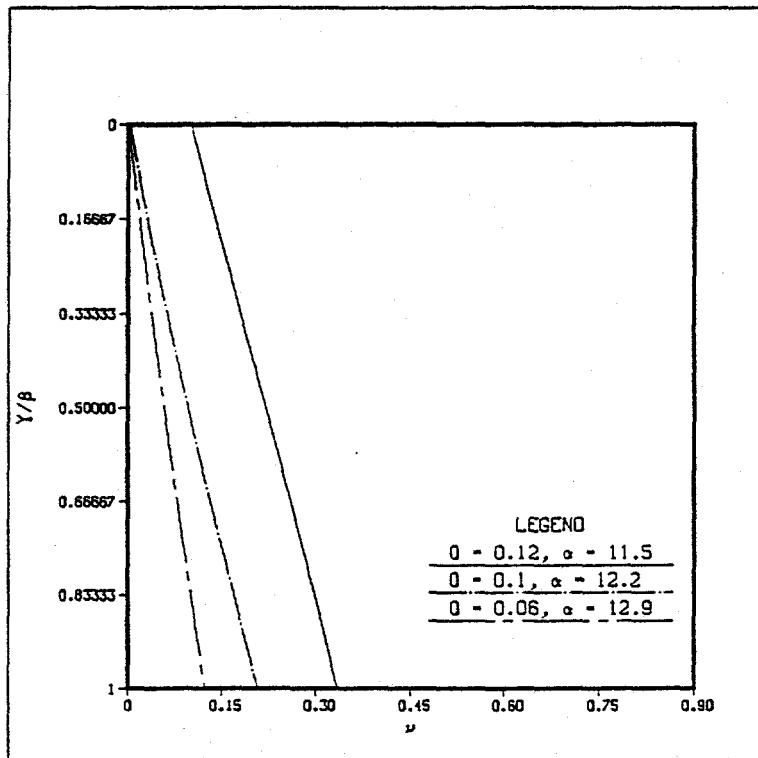


Figure 15. Volume Fraction Profiles With $e = .95$ and $e_w = .8$

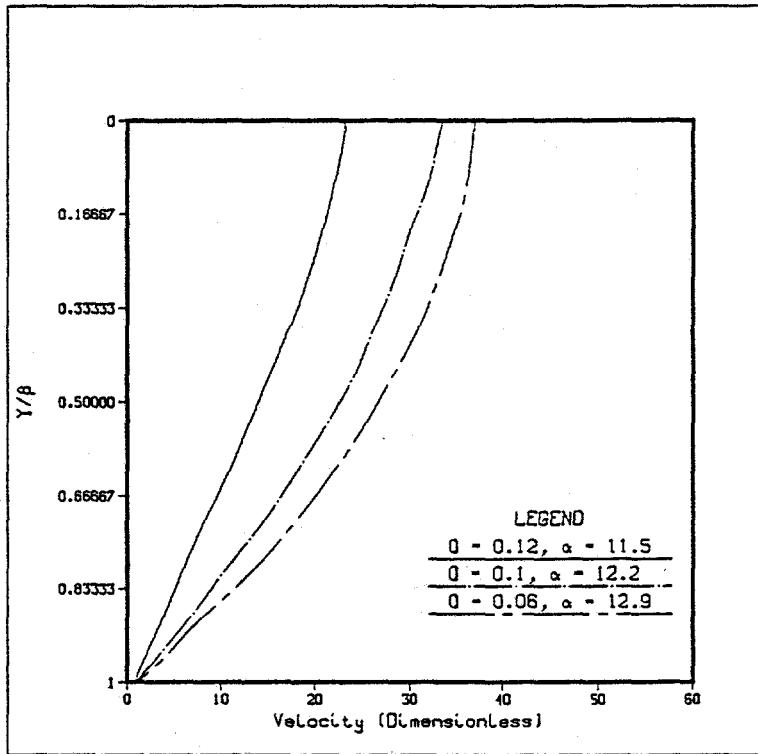


Figure 16. Velocity Profiles With $e = .95$ and $e_w = .8$

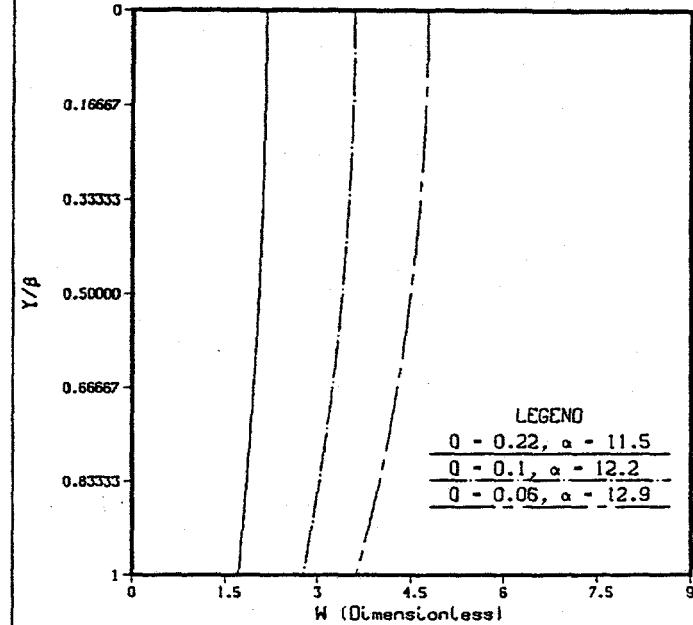


Figure 17. Granular Temperature Profiles With $e = .95$ and $e_w = .8$

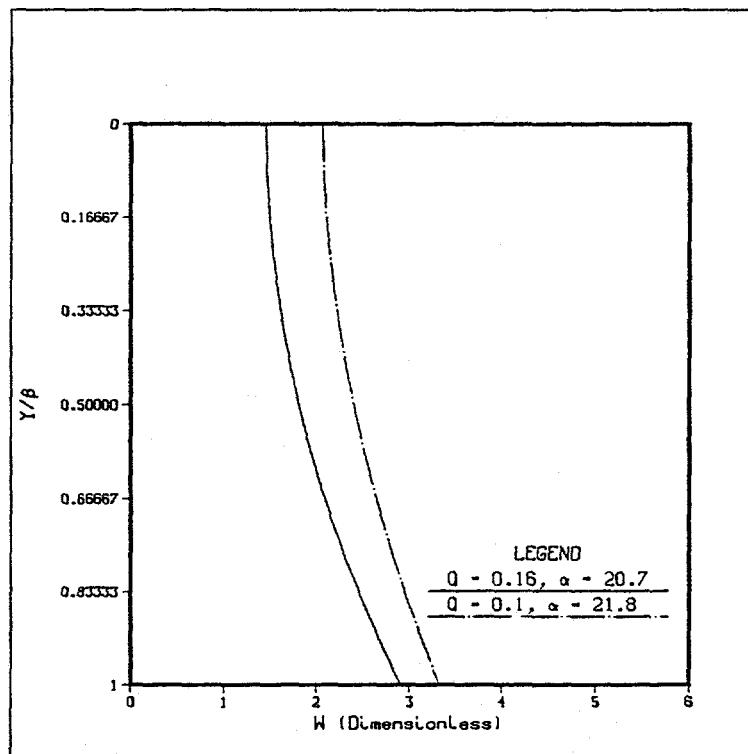


Figure 18. Granular Temperature Profiles With $e = .8$ and $e_w = .95$

2.2.2 Case B: Boyle and Massoudi constitutive equation

Boyle and Massoudi (1989) proposed a model which can exhibit normal-stress effects by including the effects of the gradient of the volume distribution function. The granular stress tensor \mathbf{T} is the sum of the two terms, reflecting that momentum can be transported by the uninterrupted streaming of granules \mathbf{T}_k and by the essentially instantaneous transport from one center to another during a collision \mathbf{T}_c . The granular stress tensor \mathbf{T} follows from that of Lun et al. (1984), but there is an additional contribution to \mathbf{T} from \mathbf{M} (or \mathbf{T}_m) that is given by

$$\mathbf{T}_m = \frac{4}{5} \eta g_0 \rho \theta \nu \frac{V_0^* \sigma^2}{(1 - \nu V_0^*)^2} (2 \mathbf{M} + \text{tr} \mathbf{M} \mathbf{1}) \quad (2-33)$$

The stress tensor for a rapidly sheared granular material is found by summing the above individual contributions, which is given by

$$\begin{aligned} \mathbf{T} = & \left\{ \rho \theta (1 + 4 \eta \nu g_0) + \left(\frac{2 \mu}{3 \eta (2 - \eta) g_0} (1 + \frac{8}{5} \eta \nu g_0) \right. \right. \\ & \left. \left. \{ 1 + \frac{8}{5} \eta (3 \eta - 2) \nu g_0 \} - \frac{3}{5} \mu_b \eta \right) \nabla \cdot \mathbf{u} \right. \\ & \left. + \frac{4}{5} \eta \nu g_0 \rho \theta \frac{V_0^* \sigma^2}{(1 - \nu V_0^*)^2} \text{tr} \mathbf{M} \right\} \mathbf{1} \\ & - \left\{ \frac{2 \mu}{\eta (2 - \eta) g_0} (1 + \frac{8}{5} \eta \nu g_0) [1 + \frac{8}{5} \eta (3 \eta - 2) \nu g_0] + \frac{6}{5} \mu_b \eta \right\} \mathbf{D} \\ & + \frac{8}{5} \eta g_0 \rho \theta \nu \frac{V_0^* \sigma^2}{(1 - \nu V_0^*)^2} \mathbf{M} \end{aligned} \quad (2-34)$$

$$\text{where, } \mu_b = \frac{256 \mu \nu^2 g_0}{5 \pi}$$

$$V_0^* = \frac{6 V_0}{\pi \sigma^3}$$

$$\mathbf{M} = \nabla \mathbf{v} \otimes \nabla \mathbf{v}$$

V_0 represents the volume per neighboring particle.

Rajagopal and Massoudi (1990) proposed a continuum model that incorporates the normal-stress effects caused by gradient of the volume distribution function. In that model β_0 , β_1 , β_2 , β_3 and β_4 are the undetermined coefficients. However, Boyle and Massoudi (1989) have correlated the material parameters with that of the kinetic model (equation (2-34)) proposed by using Enskog's dense gas theory, from which the undetermined coefficients are determined.

The flow of granular materials down an inclined plane is modeled by the constitutive equation for the Cauchy stress given by equation (2-34). we assume that the granular temperature is constant and the volume fraction, velocity to be of the form

$$\begin{aligned} v &= v(y) \\ u &= U(y)i \end{aligned} \quad (2-35)$$

With the above flow field the conservation of mass is automatically satisfied. From the balance of linear momentum we get two coupled ordinary differential equations. These equations are solved numerically using a collocation code COLSYS [cf. Ascher et al. (1981)]. A parametric study for the volume fraction and velocity profiles is carried out and the results are presented in the form of graphs. The appropriate non-dimensional numbers in the governing equation are given by[cf. Gudhe et al. (1993)]

$$\begin{aligned} D_1 &= \frac{E_1}{L \gamma g}; & D_2 &= \frac{E_2 V_0^*}{L \gamma g}; & D_3 &= \frac{3 E_6 V_0^*}{L^3 \gamma g} \\ D_4 &= \frac{6 E_3 u_0}{L^2 \gamma g V_0^*}; & D_5 &= \frac{3 E_4 u_0}{2 L^2 \gamma g}; & D_6 &= \frac{E_7 u_0 V_0^*}{8 L^2 \gamma g} \end{aligned} \quad (2-36)$$

where,

$$\begin{aligned} E_1 &= \gamma \theta, & E_2 &= \gamma \theta \eta \\ E_3 &= \frac{2 \mu}{3 \eta (2-\eta)}, & E_4 &= \frac{16 \mu (3\eta-1)}{15 (2-\eta)} \end{aligned}$$

$$\begin{aligned}
 E_5 &= \mu \eta \left\{ \frac{128(3\eta-2)}{75(2-\eta)} - \frac{768}{25\pi} \right\} \\
 E_6 &= \frac{4}{5} \gamma \theta \eta \bar{\sigma}^2 V_0^{*2} \\
 E_7 &= \mu \eta \left\{ \frac{128(3\eta-2)}{25(2-\eta)} + \frac{1536}{25\pi} \right\}
 \end{aligned} \tag{2-37}$$

The manner in which the volume fraction and velocity profiles change with D_2 and N is shown in Figures 19 and 20 respectively. Notice that the volume fraction decreases from the surface of the plane to the free surface and attains a zero value for the volume fraction at the free surface. Figure 21 shows the effect of D_4 on velocity profiles. Here, the velocity decreases as D_4 increases. Finally, Figure 22 depicts how the velocity profile changes, when D_6 is changed.

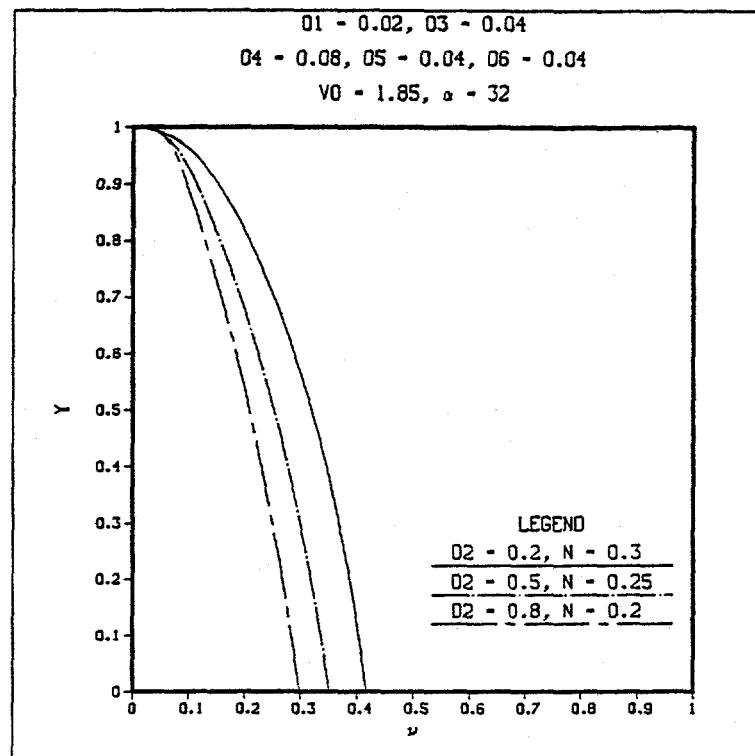


Figure 19. Effect of D_2 and N on Volume Fraction Profiles

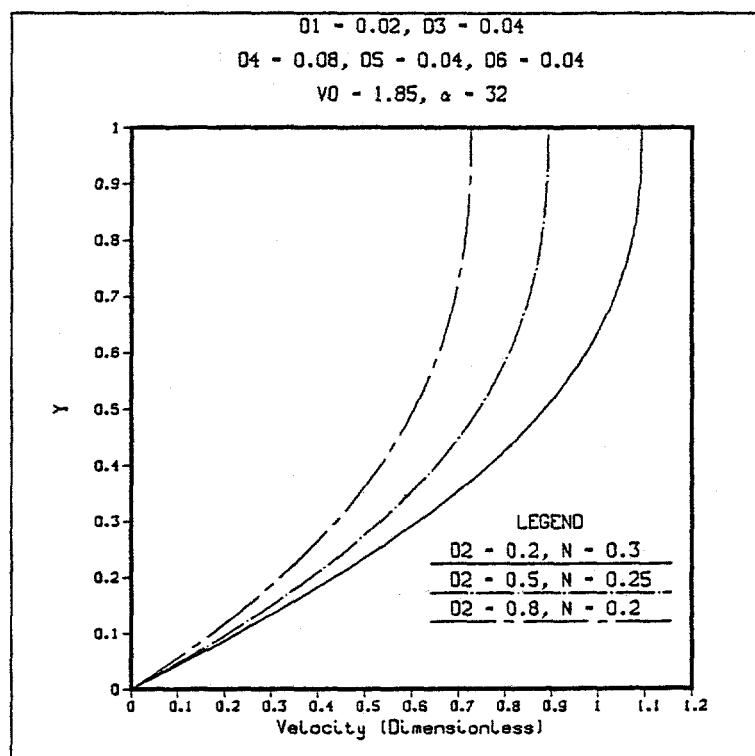


Figure 20. Effect of D_2 and N on Velocity Profiles

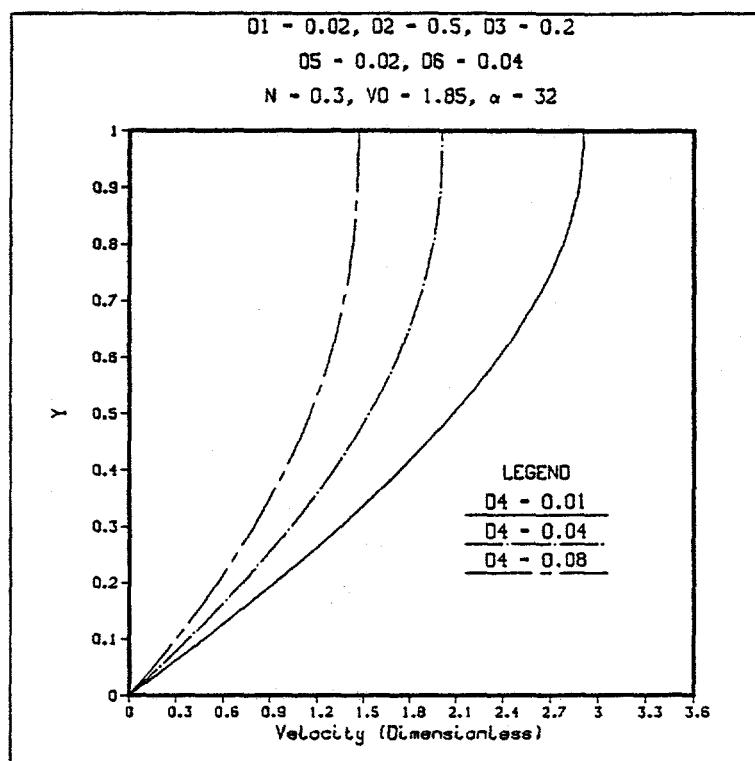


Figure 21. Effect of D_4 on Velocity Profiles

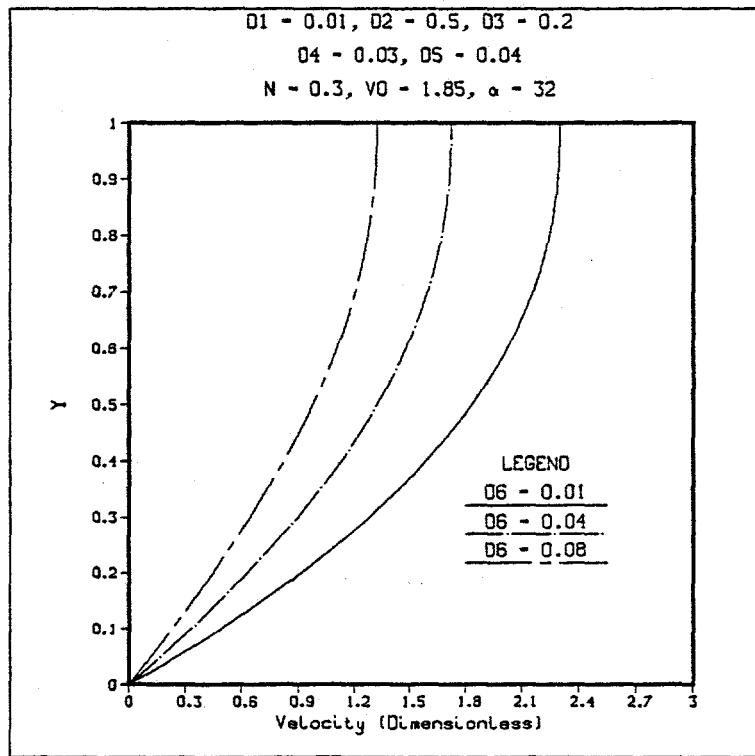


Figure 22. Effect of D_6 on Velocity Profiles

2.3 Linearized Stability

If the solution to the governing equations of motion is disturbed, then the solution is asymptotically stable if that disturbance eventually decays to zero and unstable if the disturbance grows in amplitude in such a way that the solution departs from initial state or reaches some constant value, yielding a new solution. Once the stable or unstable states are classified for the governing equations, then the locus which separates the two classes of states is defined as marginal stability or neutral stability curve. The numerical solutions obtained in the preceding chapters are the basic flows. The linearized stability of these previously obtained numerical solutions will be investigated now.

Linearized stability is concerned with the behavior of infinitesimal disturbances only. The results of this type can only give conclusive information about instability. Linearized stability does not yield sufficient conditions for stability, that is, if the solution is unstable to small disturbances then it will be unstable to finite disturbances, while on the otherhand, if the solution is stable to small disturbances it is not necessarily stable to finite disturbances. The locus which separates both the stable and the unstable regions is called the marginal stability of the system. A marginal state is a state of neutral stability. If $s = 0$ at marginal stability, there is said to be exchange of stabilities, where the disturbance is expressed in the usual normal mode [cf. equation (3-41)].

The states of marginal stability can be of two kinds. In the first kind, the amplitudes of a small disturbance can grow aperiodically, and in the second kind they can grow by oscillations of increasing amplitude. In the former case, the transition from stable to unstable flow takes place through a marginal state thereby exhibiting a stationary pattern of motions, then one says that the principle of the exchange of stabilities is valid. In the latter case, the transition takes place via a marginal state exhibiting oscillatory motions with a certain definite characteristic frequency, then we have the case of overstability.

2.3.1 Stability analysis for the continuum theory model

The linearized stability analysis will be illustrated here only for Case II of the previously obtained numerical solutions. Consider solutions which consist of the basic flow plus an infinitesimal perturbation of the form

$$\hat{v} = v + \epsilon v_1 \quad (2-38)$$

$$u = U + \epsilon u_1 \quad (2-39)$$

$$V = \epsilon u_2$$

where \hat{v} and U correspond to the basic solution of the governing equations and v_1, u_1, u_2 represent the disturbance. The basic flow is assumed to have the form

$$\begin{aligned} v &= v(y) \\ U &= U(y)i \end{aligned} \quad (2-40)$$

Now substituting equations (2-38) and (2-39) into the conservation of mass and balance of linear momentum, the equations corresponding to the basic flow *i.e.* at order zero are same as Case II. In general, in addition to perturbing the velocity field and the volume fraction, it is necessary to perturb the free surface. The basic solution will not hold in all of the perturbed domain. In the case of the flow of the Navier-Stokes fluid down an inclined plane, since the basic solution is known explicitly, it is extended into the perturbed domain as though the basic solution holds in all of the perturbed domain. It is expected that the error made by doing this is small enough to be neglected if the perturbation of the domain is small. However, if the solution to the basic flow is numerical, then such an extension is not unique. Here, we do not allow for the perturbation of the free surface. Such a stability study is incomplete in that it addresses only the disturbances corresponding to a special case. Ofcourse, a small disturbance without such a constraint may prove to be unstable, nullifying the instability predictions. It is assumed that the disturbances are spatially periodic. That is the perturbed quantities have the form

$$v_1 = v_p(\bar{y}) e^{st} e^{i\sigma\bar{x}} \quad (2-41)$$

$$\bar{u}_1 = U_{px}(\bar{y}) e^{st} e^{i\sigma\bar{x}} \quad (2-42)$$

$$\bar{u}_2 = U_{py}(\bar{y}) e^{st} e^{i\sigma\bar{x}} \quad (2-43)$$

Where v_p is the amplitude of the volume distribution function,

U_{px} and U_{py} are the amplitudes of the perturbed velocity,

i is the imaginary number such that $i^2 = -1$,

σ is the wave number (real) and

$$s = \zeta + i \omega.$$

It would be appropiate at this stage to observe that there is no equivalent of Squire's

theorem for parallel flows of granular materials and in general we should study three dimensional disturbances. However, the equations, even for the class of disturbances (2-41) through (2-43), are so complicated that at this stage we wish to restrict ourself to this case. We then substitute (2-41) through (2-43) into the governing equations and boundary conditions, and set $s = 0$ as we are interested in the marginal stability curve. Also, in the present problem the free surface is fixed, *i.e.* same as the basic solution domain ($\bar{y} = 1$). The final equations are

$$M_1 \frac{d^2 U_{px}}{d\bar{y}^2} + (M_2 + i M_3) \frac{d^2 U_{py}}{d\bar{y}^2} + (M_4 + i M_5) \frac{dU_{px}}{d\bar{y}} + (M_6 + i M_7) \frac{dU_{py}}{d\bar{y}} + (M_8 + i M_9) U_{px} + (M_{10} + i M_{11}) U_{py} = 0 \quad (2-44)$$

$$i M_{12} \frac{d^3 U_{py}}{d\bar{y}^3} + (M_{13} + i M_{14}) \frac{d^2 U_{py}}{d\bar{y}^2} + (M_{15} + i M_{16}) \frac{dU_{py}}{d\bar{y}} + M_{17} \frac{d^2 U_{px}}{d\bar{y}^2} + (M_{18} + i M_{19}) \frac{dU_{px}}{d\bar{y}} + (M_{20} + i M_{21}) U_{py} + (M_{22} + i M_{23}) U_{px} = 0 \quad (2-45)$$

subjected to the boundary conditions

$$U_{px} = f \left(i C_1 \frac{d^2 U_{py}}{d\bar{y}^2} + (C_2 + i C_3) \frac{dU_{py}}{d\bar{y}} + C_4 \frac{dU_{px}}{d\bar{y}} + (C_5 + i C_6) U_{py} + (C_7 + i C_8) U_{px} \right) \quad (2-46)$$

$$\int_0^1 \frac{1}{\bar{u}_0} \left\{ \frac{d(\nu U_{py})}{d\bar{y}} + i \sigma \nu U_{px} \right\} d\bar{y} = 0 \quad (2-47)$$

$$(B_1 + i B_2) \frac{dU_{py}}{d\bar{y}} + B_3 \frac{dU_{px}}{d\bar{y}} + (B_4 + i B_5) U_{py} + (B_6 + i B_7) U_{px} = 0 \quad (2-48)$$

$$i B_8 \frac{d^2 U_{py}}{d\bar{y}^2} + (B_9 + i B_{10}) \frac{dU_{py}}{d\bar{y}} + B_{11} \frac{dU_{px}}{d\bar{y}} + i B_{12} U_{py} + (B_{13} + i B_{14}) U_{px} = 0 \quad (2-49)$$

Equations (2-46)_{1,2} are the boundary conditions at $\bar{y} = 0$ on the inclined plane and equations (2-48) and (2-49) are the boundary conditions at $\bar{y} = 0$ at the free surface. In the above equations σ is the wave number and all the co-efficients depend on the basic solution. In the above the co-efficients $M_1 - M_{23}$, $B_1 - B_{14}$, $C_1 - C_8$ all depend on the basic solution [cf. Gudhe et al. (1993)].

The base solutions equations corresponding to Case II and the stability equations (2-44) and (2-45) subjected to boundary conditions (2-46), (2-47), (2-48) and (2-49) are solved numerically using a collocation code COLSYS [cf. Ascher et al. (1981)] and IMSL routines to obtain the base solution and the marginal stability curves. The non-dimensional parameters A_{31} , R_{22} and Fr that appear in the governing equations are given by

$$A_{31} = \frac{\beta_2 u_0}{h^2 \gamma g}; \quad R_{22} = \frac{\beta_4}{h^3 \gamma g}; \quad Fr = \frac{u_0^2}{h g}, \quad (2-50)$$

The base solution equations admit non-unique solutions for the same values of R_1 and R_2 , one in which the volume fraction monotonically increases and the other in which it monotonically decreases from the inclined plane to the free surface as shown in Figure 6. However, it is observed from the stability analysis that the solution in which the volume fraction monotonically increases from the inclined plane to the free surface is an unstable solution. In Figure 23, $\frac{1}{A_3}$ vs σ (marginal stability curve) is plotted for different values of R_2 for the solution in which the volume fraction decreases monotonically from the inclined plane to the free surface. Similarly, in Figure 24, R_1 vs σ (marginal stability curve) is plotted for different values of A_3 .

It is interesting to note that the marginal stability curves are qualitatively similar to the marginal stability curves obtained by Yih (1964) for classical linearly viscous fluid.

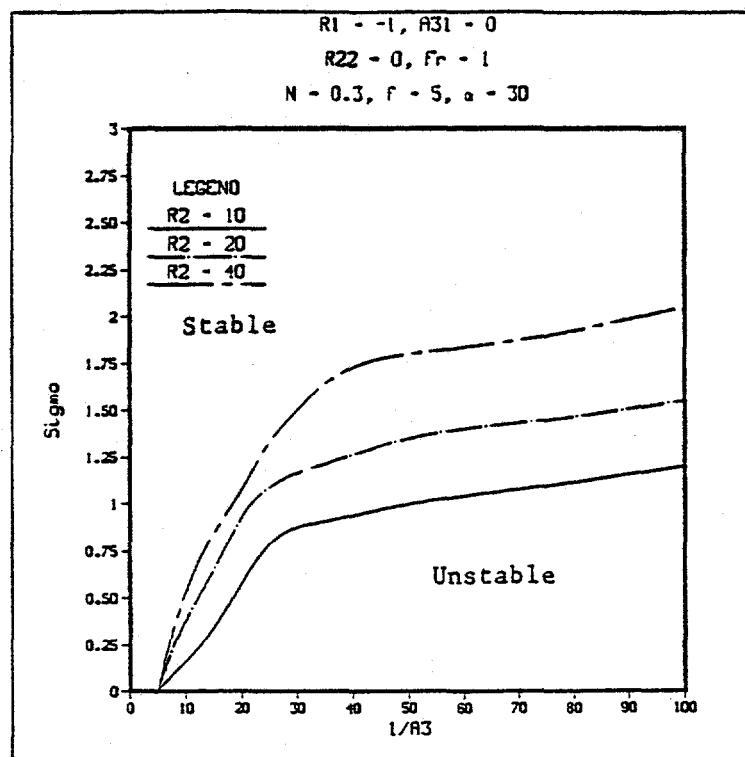


Figure 23. Marginal Stability Curve [$\frac{1}{R_3}$ vs σ]

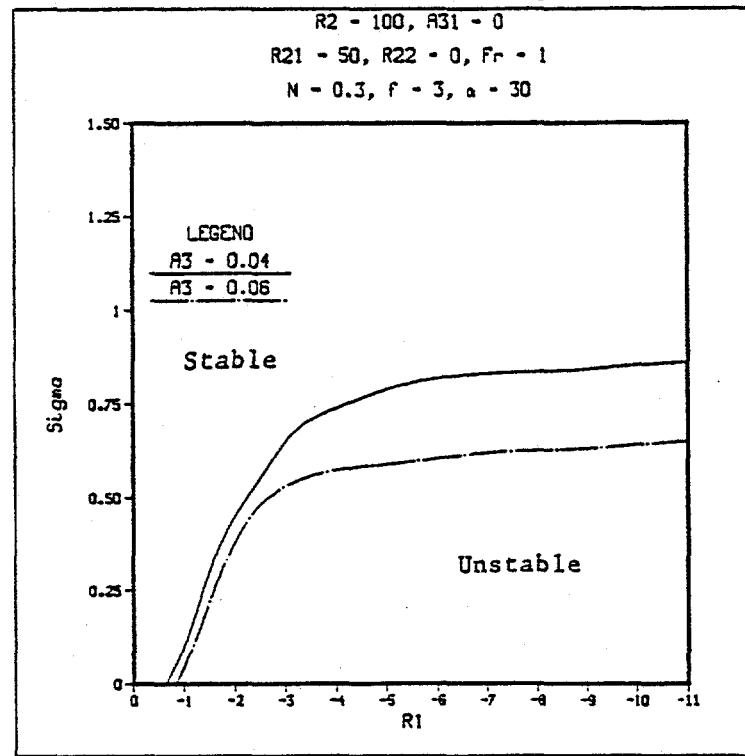


Figure 24. Marginal Stability Curve [R_1 vs σ]

2.3.2 Stability analysis for the kinetic theory model

The linearized stability analysis will be illustrated here for Case A only. Consider solutions which consist of the basic flow plus an infinitesimal perturbation of the form

$$\hat{v} = v + \varepsilon v_1 \quad (2-51)$$

$$u = U + \varepsilon u_1 \quad (2-52)$$

$$V = \varepsilon u_2 \quad (2-53)$$

$$\hat{\theta} = \theta + \varepsilon \theta_1 \quad (2-54)$$

where v , U and θ correspond to the basic solution of the governing equations and v_1 , u_1 , u_2 , θ_1 represent the disturbance. It is assumed that for infinitesimal disturbances, the equations may be linearized *i.e.* the terms of order ε^2 and higher order terms can be neglected. In general, in addition to perturbing the volume fraction, velocity field and granular temperature, it is necessary to perturb the free surface. Ofcourse, the base solution will not hold in all of the perturbed domain. Here also, we do not allow for the perturbation of the free surface. The basic flow is assumed to have the form

$$\begin{aligned} v &= v(y) \\ U &= U(y)i \\ \theta &= \theta(y) \end{aligned} \quad (2-55)$$

Now substituting equations (2-51) through (2-54) into conservation of mass, balance of linear momentum and balance of energy, the equations corresponding to the basic flow *i.e.* of order zero are same as Case A.

Once again, there is no equivalent of Squire's theorem for parrallel flows of granular materials and in general have to study three dimensional disturbances. As we are interested in the marginal stability curve, s is set to zero and we end up with three second order ordinary differential equations. Here, also the free surface is fixed, *i.e.* same as the basic solution domain ($\bar{y} = 0$). But, in the actual problem the perturbed domain will not be the same as the basic solution domain. The final equations are

$$\begin{aligned} P_1 \frac{d^2 U_{px}}{d\bar{y}^2} + i G_7 \frac{d^2 U_{py}}{d\bar{y}^2} + G_8 \frac{dU_{px}}{d\bar{y}} + (G_{10} + i G_9) \frac{dU_{py}}{d\bar{y}} + P_3 \frac{dW_p}{d\bar{y}} \\ + (G_{11} + i G_{12}) U_{px} + (G_{13} + i G_{14}) U_{py} + (P_4 + i G_{15}) W_p = 0 \end{aligned} \quad (2-56)$$

$$(G_{16} + i G_{17}) \frac{d^2 U_{py}}{d\bar{y}^2} + (G_{18} + i G_{19}) \frac{dU_{py}}{d\bar{y}} + (G_{20} + i G_{21}) \frac{dU_{px}}{d\bar{y}} - S_2 \frac{dW_p}{d\bar{y}} + (G_{22} + i G_{23}) U_{py} + G_{24} U_{px} + (G_{25} + i G_{26}) W_p = 0 \quad (2-57)$$

$$G_{27} \frac{d^2 W_p}{d\bar{y}^2} + i G_{30} \frac{d^2 U_{py}}{d\bar{y}^2} + (G_{31} + i G_{32}) \frac{dU_{py}}{d\bar{y}} + G_{33} \frac{dU_{px}}{d\bar{y}} + (G_{28} + i G_{29}) \frac{dW_p}{d\bar{y}} + (G_{34} + i G_{35}) W_p + (G_{36} + i G_{37}) U_{py} + (G_{38} + i G_{39}) U_{px} = 0 \quad (2-58)$$

and the boundary conditions become

$$U_{px} = \sqrt{\frac{\pi}{2}} \tan \alpha \bar{W} \left\{ f_p v_p + f \frac{1}{2 \bar{W}^2} W_p \right\} \quad (2-59)$$

$$U_{py} = 0 \quad (2-59)$$

$$\int_0^{\beta} \left\{ i v \frac{dU_{py}}{d\bar{y}} + i \frac{dv}{d\bar{y}} U_{py} - \sigma v U_{px} \right\} d\bar{y} = 0 \quad (2-60)$$

$$S_1 \frac{dU_{px}}{d\bar{y}} + i \frac{J_1 v}{\sigma \bar{U}} \frac{d\bar{U}}{d\bar{y}} \frac{dU_{py}}{d\bar{y}} + i \left\{ \sigma S_1 + \frac{J_1}{\sigma \bar{U}} \frac{d\bar{U}}{d\bar{y}} \frac{dv}{d\bar{y}} \right\} U_{py} - \frac{J_1 v}{\bar{U}} \frac{d\bar{U}}{d\bar{y}} U_{px} + \frac{S_1}{2 \bar{W}^2} \frac{d\bar{U}}{d\bar{y}} W_p = 0 \quad (2-61)$$

$$2 \bar{W} \frac{dW_p}{d\bar{y}} + \frac{d\bar{W}}{d\bar{y}} W_p = 0 \quad (2-62)$$

$$W_p = 0 \quad (2-63)$$

here, as before the coefficients P^s , G^s , J^s , etc depend on the basic solution. Equations (2-59)_{1,2} are the boundary conditions at $\bar{y} = \beta$ on the inclined plane and equations (2-61), (2-61) and (2-63) are the boundary conditions at $\bar{y} = 0$ at the free surface. In the above equations σ is the wave number and all the co-efficients depend on the base solution.

The system of equations (2-56), (2-57) and (2-58) with the boundary conditions (2-59) through (2-63) are solved numerically to obtain marginal stability curves. For the base solution of Richman and Marciniec (1990) approximate solution is used in the stability analysis. In Figure 25, Q Vs σ (marginal stability curve) is plotted for different

values of angle of inclination (α). We notice that the marginal stability curves are qualitatively similar to those that were obtained for the continuum model.

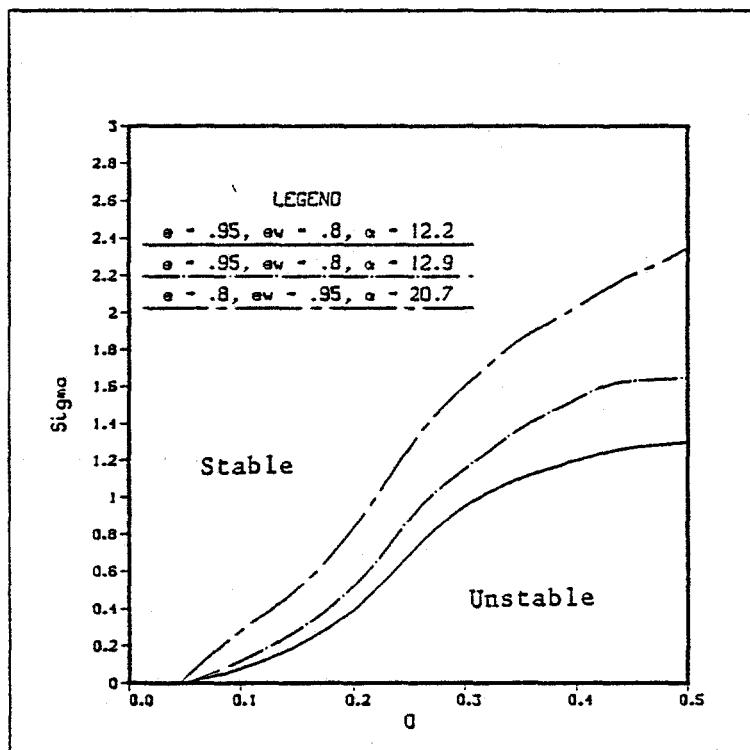


Figure 25. Marginal Stability Curve [Q vs σ]

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2.4 Project Output

The following summarizes accomplishments for the first three years of the project.

During the first year of the project we investigated the existence of solutions to the equations governing the continuum model, that is an outgrowth of the model of Goodman and Cowin (1971, 1972). We also looked at the multiplicity of solutions to the governing equations. In the second year we studied numerically the equation governing the flow of granular materials modeled by the kinetic theory approach. We considered the models due to Richman and Marciniec (1990) and Boyle and Massoudi(1989). During the third year of the grant we investigated the stability of flow for both the continuum model and the kinetic theory model considered in the previous year.

The following publications have so far resulted from this project

- K.R. Rajagopal, G.P. Galdi, and M. Padula, "Stability of Flows of Granular Solids", *Recent Advances in mechanics of Structured Continua*, AMD-Vol. 117, ASME, New York, 1991.
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- Hutter, K., and K.R. Rajagopal, "On Flows of Granular Materials", To appear in Continuum Mechanics and Thermodynamics.
- Yalamanchili,R.C., R. Gudhe, and K.R. Rajagopal, "Cohesionless Granular Materials Flow in a Vertical Channel Under the Action of Gravity, Submitted for publication.
- Rajagopal, K.R., and R. Gudhe, "Stability of the Flow of Granular Materials Down an Inclined Plane Using Continuum Theory Approach", In Preparation.