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Stochastic Models of Chaotic Systems

*Cecil E. Leith
Global Climate Research Division
Lawrence Livermore National Laboratory
Livermore, CA 94550

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C.E. Leith

Lawrence Livermore National Laboratory, Livermore, CA 94550

Abstract

Nonlinear dynamical systems, although strictly deterministic, often exhibit chaotic behavior which appears to be random. The determination of the probabilistic properties of such systems is, in general, an open problem. Closure approximations for moment expansion methods have been unsatisfactory. More successful has been approximation on the dynamics level by the use of linear stochastic models that attempt to generate the probabilistic properties of the original nonlinear chaotic system as closely as possible. Examples are reviewed of this approach to simple nonlinear systems, to turbulence, and to large-eddy simulation. A stochastic model that simulates the transient energy spectrum of the global atmosphere is developed.

1 Introduction

Many nonlinear dynamical systems exhibit chaotic behavior arising from their sensitive response to small perturbations. Thus, although they are strictly deterministic, they behave much like random systems of limited predictability. The many years of experience in seeking to understand the properties of turbulent flow demonstrate clearly the nature of the problem. The probabilistic properties of such chaotic systems can not, in general, be predicted well theoretically. The nonlinearity of the dynamics causes moment expansion techniques to lead to an infinite hierarchy of moment evolution equations which, for solution, must be closed by some approximation relating higher moments to lower. Unfortunately, such closure approximations often lead to unrealizable moment solutions, the simplest example of which is the generation of negative variances.

The preferred alternate approach, the subject of this article, is to make the approximation on the level of the dynamical equation by replacing the original nonlinear dynamical system by a model dynamical system in which the nonlinear interaction of a mode with all others in the system is simulated by linear damping and random forcing. The resulting stochastic model is governed by a set of stochastic ordinary differential equations whose parameters

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are chosen to simulate, as well as possible, the probabilistic properties of the original system. These parameters may be and usually are chosen to depend on probabilistic moments of the model system so that the evolution equations for the moments of the stochastic model become nonlinear. But they are readily computable, and they clearly guarantee that the resulting moments are realizable. In fact, linear stochastic models preserve the assumed guassianity of the forcing, and thus they provide at best a gaussian approximation to the probabilistic properties of the original nonlinear system known generally to be non-gaussian. Gaussian probability distributions are completely characterized by their first and second moments so that it suffices to provide model evolution equations for them alone.

A stochastic model for a simple nonlinear oscillator is developed in Section 2 as a demonstration of the general procedure. In this case the phase space of the system is only two-dimensional, and the evolution of the probability distribution function is easily visualized as the evolution of a cloud of phase points.

Stochastic models have been widely used for prediction of the probabilistic properties of turbulent flows. In this case the dynamical phase space is myriad-dimensional, but stochastic models based on eddy damping and eddy forcing prescriptions provide readily computable estimates of second-moment quantities, which are of importance for eddy transport. Such models are discussed in Section 3.

The range of spatial scales excited in a turbulent flow determines the needed phase-space dimension of any dynamical model of it, and this may be so large as to rule out as computationally infeasible the direct numerical simulation of the flow by time integration of the evolution equations. The largest scales are, however, of the greatest interest both for being most special to the flow geometry and for being responsible for the major part of the turbulent transport. For this reason the properties of turbulent flows are often deduced from so-called Large-Eddy Simulation (LES) in which the large scales are explicitly computed but scales smaller than a specified truncation scale are treated with a subgrid-scale (SGS) turbulence model. For several decades such SGS models treated the eddy damping by the small scales of those resolved in the LES model. In recent years such models have been extended to include stochastic backscatter, i.e., the stochastic forcing of the large scales by the small. These developments are discussed in Section 4.

An application of these ideas to the climate problem is given in Section 5. The climate system is made up of many components, in particular, a relatively slowly evolving ocean and a rapidly fluctuating atmosphere. Since the time scales of principle interest for the climate are closer to those of the ocean there is a considerable computational benefit in finding some stochastic model of the atmosphere to drive a deterministic ocean as a replacement for the deterministic but chaotic weather models currently used. A crude first step is described in which the global atmosphere is treated as homogeneous shallow water turbulence characterized by white noise forcing and an eddy diffusion of potential vorticity. A reasonable choice of a length and a time parameter in the model leads to a fair fit to the observed atmospheric transient kinetic energy spectrum and its time-lagged covariance properties.

2 Nonlinear Oscillator

Many aspects of the statistical mechanics of nonlinear systems are revealed by the almost trivial example of a nonlinear oscillator. We define this as a system with two degrees of freedom described by the coordinates x and y in phase space evolving according to the equations of motion

$$\dot{x} = -\alpha r y \tag{1}$$
$$\dot{y} = \alpha r x \tag{2}$$

where $r^2 = x^2 + y^2$ and α is a constant. In terms of the phase angle θ for which $x = rcos\theta$, $y = rsin\theta$, the dynamics equations can be written

$$\dot{r} = 0 \tag{3}$$
$$\dot{\theta} = \alpha r \tag{4}$$

and integrated to give

$$r(t) = r(0) = r_o \tag{5}$$
$$\theta(t) = \theta(0) + \alpha r_o t = \theta_o + \alpha r_o t \tag{6}$$

and thus

$$x(t) = r_o cos(\theta_o + \alpha r_o t) \tag{7}$$
$$y(t) = r_o sin(\theta_o + \alpha r_o t) \tag{8}$$

In the x, y phase space of the system the individual motions are quite simple. Each phase path is a circle about the origin traversed counterclockwise at an angular velocity proportional to its radius. The statistical mechanics of this system concerns itself not so much with the individual phase paths as with the behavior of an ensemble of such paths. An ensemble is characterized by a probability distribution function (pdf) $p(r,\theta,t)$ defined such that

$$\int_0^{2\pi} d\theta \int_0^{\infty} r dr\, p(r,\theta,t) = 1. \tag{9}$$

For this system $\partial\dot{x}/\partial x + \partial\dot{y}/\partial y = 0$ and we have a valid Liouville theorem

$$\dot{p}(r,\theta,t) = \partial p(r,\theta,t)/\partial t + \dot{r}\partial p(r,\theta,t)/\partial r + \dot{\theta}\partial p(r,\theta,t)/\partial\theta = 0. \tag{10}$$

The pdf at any time t is related to that at time $t=0$ by

$$p(r,\theta,t) = p(r,\theta - \alpha rt, 0). \tag{11}$$

Let us consider, as a simple example, an ensemble of phase points which at time $t=0$ is distributed along the x-axis with a pdf

$$p(r,\theta,0) = \frac{1}{2}\rho^{-2}\exp(-\frac{1}{2}r^2/\rho^2)[\delta(\theta) + \delta(\theta+\pi)]. \tag{12}$$

At any other time we find

$$p(r,\theta,t) = \frac{1}{2}\rho^{-2}\exp(-\frac{1}{2}r^2/\rho^2)[\delta(\theta - \alpha rt) + \delta(\theta + \pi - \alpha rt)]. \tag{13}$$

and the points are distributed along the arms of a spiral wound counter-clockwise for $t > 0$, clockwise for $t < 0$, and the more tightly wound the greater the magnitude of t, as shown in Fig. 1.

We may use this evolving pdf to compute the evolution of the single-time second moment of x

$$X(t,t) = < x(t)x(t) > \tag{14}$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} r dr\, r^2 cos^2\theta\, p(r,\theta,t) \tag{15}$$

$$= \rho^{-2}\int_0^{\infty} \exp(-\frac{1}{2}r^2/\rho^2) r^3 cos^2(\alpha rt) dr \tag{16}$$

$$= \rho^2[1 - \frac{1}{2}D'''(2\alpha\rho t)] \tag{17}$$

$$= \rho^2[2 - 8(\alpha\rho t)^2 + 16(\alpha\rho t)^4 - \frac{256}{15}(\alpha\rho t)^6 + \ldots] \tag{18}$$

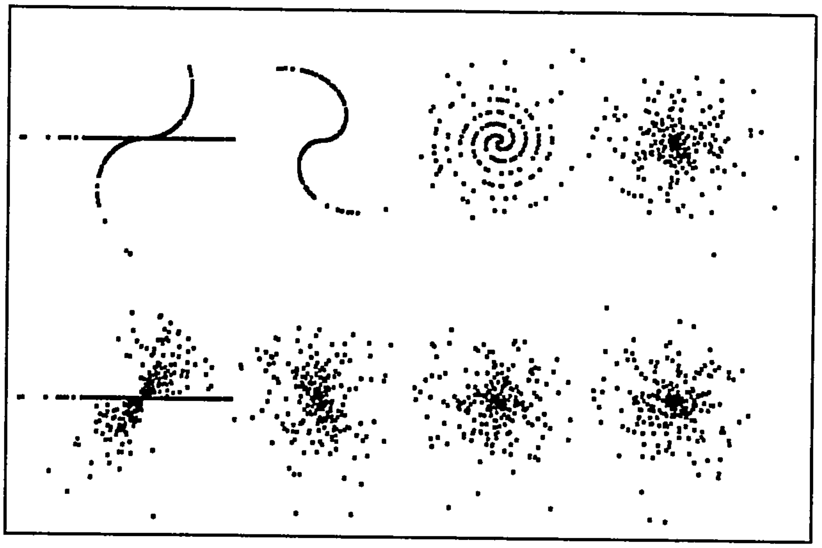

Fig. 1. Evolving ensembles for the nonlinear oscillator, above, and its linear stochastic model, below, shown for times ($0\&\frac{1}{2}$), 1, 10, and 100 from left to right.

where

$$D(s) = \exp(-\frac{1}{2}s^2)\int_0^s \exp(\frac{1}{2}t^2)dt = \int_0^\infty \sin(sr)\exp(-\frac{1}{2}r^2)dr \tag{19}$$

is related to Dawson's integral. The second moment $X(t,t)$, for $\alpha = \rho = 1$, is shown as a function of time t by the curve labeled NL in Fig. 2. Similarly, the second moment of y is given by

$$Y(t,t) = < y(t)y(t) > \tag{20}$$

$$= \rho^2[1 + \frac{1}{2}D'''(2\alpha\rho t)] \tag{21}$$

$$= \rho^2[8(\alpha\rho t)^2 - 16(\alpha\rho t)^4 + \frac{256}{15}(\alpha\rho t)^6 + \ldots] \tag{22}$$

As t becomes large, X and Y approach ρ^2 as if determined by the normal equipartition probability distribution

$$\bar{p}(r,\theta,t) = (2\pi\rho^2)^{-1}\exp(-\frac{1}{2}r^2/\rho^2). \tag{23}$$

It is tempting to say from an examination of Fig. 1 that the ensemble approaches a stationary equipartition distribution and that in some sense as $t \to \infty$, $p(r,\theta,t) \to \bar{p}(r,\theta,t)$. This is so only to the extent that we ignore the increasingly fine structure of the tightly wound spiral, a fine structure that has

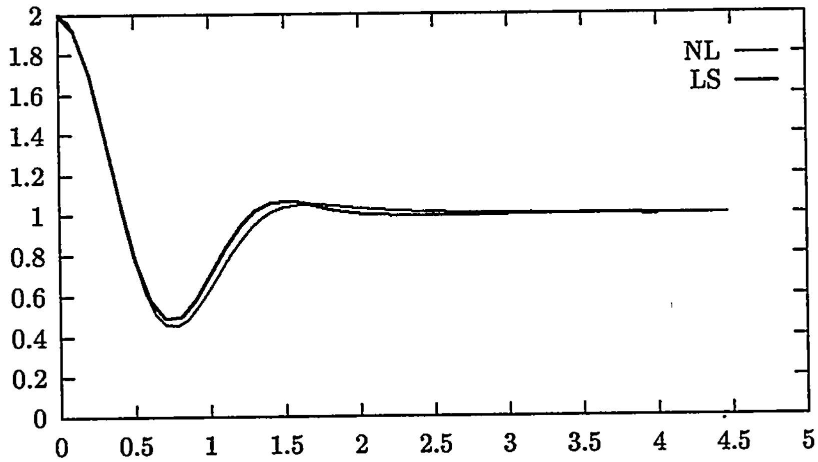

Fig. 2. Variance, X, as a function of time t for the nonlinear oscillator (NL) and its linear stochastic model (LS).

vanishing influence on the calculation of second moments. As soon, however, as we ignore the fine structure we have lost time reversibility, for when the direction of time is reversed the spiral of $p(r,\theta,t)$ unwinds but the equipartition distribution $\bar{p}(r,\theta,t)$ remains unchanged. In approximations that we shall be describing later, we shall want to give up the detail required for reversibility and be content with an irreversible approach to equilibrium.

An ensemble with the initial probability distribution $\bar{p}(r,\theta,0)$ is evidently stationary. For it two-time moments are functions of the time difference alone and are readily computed. For example, we have

$$X(t_1,t_2) = < x(t_1)x(t_2) > \tag{24}$$

$$= -\frac{1}{2}D'''(\alpha\rho t_2 - \alpha\rho t_1) \tag{25}$$

or

$$X(t,t+\tau) = -\frac{1}{2}D'''(\alpha\rho\tau) \tag{26}$$

$$= 1 - 2(\alpha\rho\tau)^2 + (\alpha\rho\tau)^4 - \frac{4}{15}(\alpha\rho\tau)^6 + \dots \tag{27}$$

Although the statistical mechanics of the nonlinear oscillator is quite simple, for more general nonlinear systems the dimensionality of the phase space is so large as to make explicit phase space calculations impractical, and we seek approximations provided by stochastic models. A stochastic model for an original system is an alternate system with differing dynamics equations chosen in such a way as, first, to induce as nearly as possible the statistical mechanics of the original system, and, second, to have its statistical mechanics computable. These two requirements evidently tend to be contradictory, and any stochastic model must represent therefore a compromise.

The nonlinear oscillator needs no stochastic model since its statistical mechanics is already computable, but, as illustration, we shall construct one, a random linear oscillator. The dynamics equations for the random linear oscillator are

$$\dot{x} = -\omega y \tag{28}$$
$$\dot{y} = \omega x \tag{29}$$

or

$$\dot{r} = 0 \tag{30}$$
$$\dot{\theta} = \omega \tag{31}$$

where ω is a time independent but random frequency chosen from a normal probability distribution

$$w(\omega) = (2\pi)^{-\frac{1}{2}}\sigma^{-1}\exp[-\frac{1}{2}(\omega - \bar{\omega})^2/\sigma^2] \tag{32}$$

having mean $\bar{\omega}$ and variance σ^2. We have introduced a new random variable into the system and averages are to be computed over the ω distribution as well as that of x and y. When we compute thus the evolving second moment of x for the model starting from the same initial pdf as for the nonlinear oscillator we find

$$X(t,t) = \rho^2[1 + \cos 2\bar{\omega}t \exp(-2\sigma^2 t^2)] \tag{33}$$
$$= \rho^2[2 - 2(\sigma^2 + \bar{\omega}^2)t^2 + (2\sigma^4 + 4\sigma^2\bar{\omega}^2 + (2/3)\bar{\omega}^4)t^4 + \ldots] \tag{34}$$

Should we choose

$$\bar{\omega}^2 = 2(3)^{\frac{1}{2}}\alpha^2\rho^2. \tag{35}$$
$$\sigma^2 = (4 - 2(3)^{\frac{1}{2}})\alpha^2\rho^2 \tag{36}$$

then we shall have

$$\sigma^2 + \bar{\omega}^2 = 4\alpha^2\rho^2 \tag{37}$$

$$2\sigma^4 + 4\sigma^2\bar{\omega}^2 + (2/3)\bar{\omega}^4 = 16\alpha^4\rho^4 \tag{38}$$

and the evolution of $X(t,t)$ for the random linear approximation will match that of the original nonlinear system through terms in t^4. The resulting expression for $X(t,t)$, with $\alpha = \rho = 1$, given by the linear stochastic model equation (33), is plotted as the curve labeled LS in Fig. 2.

We may also compute

$$Y(t,t) = \rho^2[1 - \cos 2\bar{\omega}t \exp(-2\sigma^2 t^2)] \tag{39}$$

and observe that for the model as for the original system as t becomes large both X and Y approach ρ^2. In both the model and the original system we have $X + Y = 2\rho^2$ independent of time.

3 Turbulence Models

Turbulence deals with the statistical properties of solutions of the Navier-Stokes equation of fluid flow taken here as incompressible

$$(\partial/\partial t - \nu\nabla^2)v_i(\mathbf{x},t) = -v_j(\mathbf{x},t)\frac{\partial}{\partial x_j}v_i(\mathbf{x},t) - \partial p/\partial x_i. \tag{40}$$

Here unit density is assumed, and the scalar pressure p is determined by the incompressibility condition, $\partial v_i(\mathbf{x},t)/\partial x_i = 0$. The wavevector space transform of the Navier-Stokes equation is given by

$$[d/dt + \nu k^2]u_i(\mathbf{k},t) = -\frac{1}{2}iP_{ijk}(\mathbf{k})\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j(\mathbf{p},t)u_k(\mathbf{q},t) \tag{41}$$

with the incompressibility condition $k_i u_i(\mathbf{k},t) = 0$ and with

$$P_{ijk}(\mathbf{k}) = k_j P_{ik}(\mathbf{k}) + k_k P_{ij}(\mathbf{k}), \tag{42}$$

$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2, k = |\mathbf{k}|. \tag{43}$$

For the purposes of the present discussion we shall recognize that the essential aspect of these equations is that they are quadratically nonlinear. We shall therefore replace them with a highly symbolic equation

$$du/dt = uu. \tag{44}$$

We now proceed with a conventional expansion for the evolution equations of moments for which the first three are written symbolically as:

$$d < u > /dt = < uu > \tag{45}$$
$$d < uu > /dt = < uuu > \tag{46}$$
$$d < uuu > /dt = < uuuu > . \tag{47}$$

This leads to an infinite sequence of equations for increasingly higher moments with an even higher moment always appearing on the right hand side. Thus is posed the traditional closure problem of using some approximation of some high moment in terms of lower ones in order to terminate this sequence.

For a normal or gaussian probability distribution function fourth moments can be written in terms of products of second moments, but, in general, there is something left over called the cumulant. Thus symbolically, for example, we have

$$< uuuu > = < uu >< uu > + < uuuu >' \tag{48}$$

where the final term is the fourth cumulant. One of the oldest closure approximations [1,2] is the so-called quasinormal closure obtained by ignoring the fourth cumulant.

Unfortunately, there is usually no constraint on such cumulant discard closures to assure that the evolving moments remain realizable, that is, derivable from a non-negative joint probability distribution function. In particular, the quasinormal closure was found to generate negative energy spectra in some applications [3–5].

Kraichnan [6] recognized that this difficulty would be avoided if the underlying dynamics could be modified in such a way that the statistical properties of the original system were approximated but that the modified model system could be solved for its evolving moments exactly. Since such moments are for a real (although wrong) system, they must be realizable. Kraichnan's initial model was based on the introduction of random factors in the coupling terms, but later stochastic models were based on simpler linear stochastic differential equations of Langevin type [7]. In this case, the symbolic equation (44) is replaced by the symbolic equation

$$du/dt = -\gamma u + f \tag{49}$$

where γ is an eddy damping coefficient and f is a random white gaussian eddy forcing term. These are intended to simulate the nonlinear turbulent interaction of each mode in the dynamical system with all other modes. The stochastic model equation (49) is not used directly, but from it is derived an equation for the evolution of second moments which is taken as an approximation for that of the original turbulence.

A number of stochastic models for turbulence have been constructed by using different formulations of the eddy damping and random forcing terms in equation (49) or in a generalized Langevin equation in which the damping term is an eddy viscoelasticity and the forcing is gaussian but not white, symbolically,

$$du(t)/dt = -\int_0^\infty \gamma(t-s)u(s)ds + f \tag{50}$$

The details of such models will not be discussed here; they are available in published texts [8,9]. However, some general remarks will be made.

Stochastic models of this type, also called two-point or spectral closure models, have been applied primarily to statistically homogeneous and isotropic turbulent flows, i.e., those for which the statistical properties are invariant under spatial translation and rotation. Such symmetries simplify considerably the representation of and the evolution equations for the moments. In some applications the isotropy constraint has been dropped with a large increase in numerical complexity. It is not feasible to drop the homogeneity constraint without leading to such a large increase in numerical complexity that the model can not compete with a direct numerical simulation (DNS) achieved by integration of the original Navier-Stokes equation or with a large-eddy simulation (LES) to be discussed in the next section.

None of the existing stochastic models has a firm theoretical base. Various perturbation expansion techniques used to derive them have had uncontrollable uncertainties. Instead model builders have tried to mimic in the model as many of the known properties of the original turbulent system as possible. Some of these are straightforward and important such as the conservation of known quadratic integrals of the motion such as kinetic energy and, in two dimensions, enstrophy. A more subtle property is the existence of an artificial stationary gaussian statistical state of turbulence toward which a turbulent flow will tend when viscous dissipation is removed and some fictional barrier at high wavenumber prevents cascade. This situation can not be produced in any laboratory flows but can be produced in numerical simulations. The resulting state corresponds in classical statistical mechanics to thermodynamic equilibrium with equipartition of an integral of the motion such as energy across the available dynamical modes of the system. A natural requirement of stochastic models of turbulence is that they also tend to the same equipartition solution in the same artificial situation. In addition, if one prepares an initial state of turbulence which differs from an equipartition state but is gaussian, then it will not be stationary but its initial first time derivatives will vanish and its second time derivatives are computable exactly from the Navier-Stokes equations governing the original turbulent flow. Again this serves as a useful constraint on the corresponding behavior of the stochastic model. Finally, it should be remembered that these stochastic models are linear and driven by gaussian forcing, and thus can at best produce a close gaussian approximation to the statistical properties of turbulence which are known to be non-gaussian.

4 Large-Eddy Simulation

For the practical prediction of the properties of an inhomogeneous turbulent flow one is forced to consider direct numerical simulation (DNS) or large-eddy simulation (LES). For DNS it is, of course, necessary in the calculation to resolve scales of motion down to the smallest scale at which molecular viscosity is important. In many practical applications, this leads to a range of scales far too great for a feasible numerical integration of the Navier-Stokes equations. Of course, DNS when feasible does not require any closure approximation or turbulence model. For this reason DNS has played an important role in providing an experimental basis for the testing of turbulence models at moderate Reynolds numbers where the range of excited scales is small enough to be treated numerically.

The goal of LES is more modest in that it carries out an explicit calculation of the evolution of only the larger scales of motion in a flow. These larger scales are more peculiar to the particular application and in general are known to be primarily responsible for the eddy transport properties of the flow which are of the greatest interest. But now the Navier-Stokes equations must be modified by replacing the molecular viscosity term by some sort of terms that simulate as well as possible the effect of the unresolved scales of turbulent flow on the larger scales that are being explicitly computed. Such terms define a so-called subgrid-scale (SGS) model that characterizes the LES.

Consider, for example, the problem of the numerical simulation of an isotropic homogeneous turbulent fluid whose Reynolds number is so high that it exhibits a clearly defined energy-cascading inertial range satisfying the Kolmogorov law for the energy spectrum

$$E(k) = K\epsilon^{2/3}k^{-5/3} \tag{51}$$

where $K \approx 1.7$ is the non-dimensional Kolmogorov coefficient, and ϵ is the energy cascade rate, assumed to be independent of the wavenumber k. This law is derived on purely dimensional grounds considering the dimensions of the factors involved, namely: $E[L^3T^{-2}]$, $\epsilon[L^2T^{-3}]$, and $k[L^{-1}]$. At the high wavenumber end this inertial range spectrum is terminated by a viscous dissipation range, and again dimensional scaling arguments can be used to identify the so-called Kolmogorov wavenumber k_K at which viscosity becomes important. This obviously depends on the viscosity coefficient ν with dimension $[L^2T^{-1}]$, and is given by

$$k_K = \epsilon^{1/4}\nu^{-3/4}. \tag{52}$$

The simplest solution to the subgrid-scale modeling problem is to increase the viscosity coefficient so that the corresponding Kolmogorov wavenumber is resolved. Let k_* be the limiting resolvable wavenumber. (In general, the subscript $_*$ will denote a resolution dependent quantity.) Then inversion of equation (52) gives the required artificially increased viscosity coefficient

$$\nu_* = \beta\epsilon^{1/3}k_*^{-4/3} \tag{53}$$

where the non-dimensional coefficient $\beta \approx 1$ has been introduced for flexibility. If we know an expected value of ϵ for the flow considered, then equation (53) provides an estimate for the artificial viscosity coefficient ν_*. In practice, the coefficient β is chosen to be as small as possible without leading to erratic behavior in the smallest scales of the flow. The resulting artificial viscosity is linear; that is, the coefficient is independent of the details of the flow itself.

A more satisfactory procedure is to deduce ϵ from the flow itself and indeed in a space mesh representation to attempt to compute a local value of ϵ appropriate to each mesh interval. This leads in turn to a local value of ν_* and a nonlinear artificial viscosity.

This problem arose in the early days of numerical weather prediction where megameter length scales are of interest while viscous dissipation occurs at millimeter scales. Smagorinsky[10] generalized a technique that had been used for the numerical treatment of shocks in compressible fluids[11] to generate a nonlinear eddy viscosity depending on the locally computed shear. It adjusts itself to remove kinetic energy at a resolved scale that would otherwise attempt to cascade to unresolvable scales and would lead to erratic behavior in the numerical simulation.

The form of the Smagorinsky eddy viscosity is chosen by dimensional scaling arguments. An eddy viscosity coefficient, ν_*, has dimensions L^2T^{-1}. Since the turbulence is considered to consist of unresolved scales of the fluid flow, its characteristic length scale λ_* is given by the resolution length scale, say, the grid spacing. Its time scale is taken as that given by the resolved local strain rate, S_*, to be defined below. The dimension of S_* is T^{-1}, so that the eddy viscosity coefficient becomes

$$\nu_* = (C_S\lambda_*)^2 S_* \tag{54}$$

where $C_S \approx 0.2$ is the non-dimensional Smagorinsky coefficient chosen empirically to make things work.

The deviatoric stain rate tensor for the velocity u_i in Cartesian coordinates x_i is given by

$$S_{ik} = \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\frac{\partial u_j}{\partial x_j}\delta_{ik}. \tag{55}$$

The mean local strain rate introduced above is defined as $S = [S_{ik}\partial u_i/\partial x_k]^{\frac{1}{2}}$.

The Smagorinsky formulation given by equation (54) is consistent with the inertial range formulation of equation (53) as can be seen from the following dimensional analysis.

Whatever the local value of ν_* the rate of energy dissipation will be computed as

$$\epsilon = \nu_* S_*^2 \tag{56}$$

where S_*^2 is the resolved local squared rate of strain in finite difference approximation for which we may set $k_* = \pi/\lambda_*$. If we substitute equation (56) into equation (53) we find

$$\nu_* = \beta^{3/2} S_* k_*^{-2} \tag{57}$$
$$= \gamma S_* \lambda_*^2, \tag{58}$$

which is obviously equivalent to equation (54) if the non-dimensional constants are set to $\gamma = \beta^{3/2}/\pi^2 = C_S^2$.

A test of the validity of this analysis would be the independence of ϵ, a flow-dependent quantity, and λ_*, a mesh-dependent quantity. That is, we might hope that as the mesh was refined and λ_* became smaller the average value of ϵ would remain unchanged. From equations (56) and (58) we find

$$\epsilon = \gamma S_*^3 \lambda_*^2 \tag{59}$$

and the required independence of ϵ and λ_* would arise only because with smaller λ_* the finite difference estimate of S_* will increase as $\lambda_*^{-2/3}$ owing to finer scales of motion being explicitly computed.

We see then that the Smagorinsky eddy viscosity defined by equation (54) is consistent with the idea that the truncation wavenumber k_* lies within the energy-cascading inertial range of three-dimensional isotropic turbulence. In its original application, however, this is a strange idea since it is now realized that the truncation wavenumber for numerical models of the global atmosphere are more likely to lie within an enstrophy-cascading range of quasi-two-dimensional turbulence. We turn then to two-dimensional turbulence analysis to develop a more suitable formulation.

In two-dimensional incompressible flow there are new constraints arising from the conservation of vorticity. In particular the enstrophy (defined as one-half the squared vorticity) is conserved in inviscid flow. Associated with this integral there is an inertial range through which enstrophy is cascaded at a constant rate η to be removed at sufficiently high wavenumber by dissipative processes.

The dimensional scaling arguments applied above to an energy-cascading range that depended on ϵ with dimension L^2T^{-3} may now be translated to an enstrophy-cascading inertial range that depends on η with dimension T^{-3}. The corresponding energy spectrum, of dimension L^3T^{-2}, becomes

$$E = A\eta^{2/3}k^{-3} \tag{60}$$

with some new non-dimensional coefficient A. A more careful analysis has led Kraichnan[12] to introduce an additional non-dimensional logarithmic factor which will be ignored here. The artificial viscosity becomes

$$\nu_* = \beta\eta^{1/3}k_*^{-2} \tag{61}$$

We estimate the local enstrophy dissipation rate as

$$\eta = \nu_*|\nabla_*\omega|^2 \tag{62}$$

and thus the artificial viscosity as

$$\nu_* = \gamma|\nabla_*\omega|\lambda_*^3 \tag{63}$$

where $|\nabla_* \omega|$ is the finite difference approximation to the magnitude of the gradient of the vorticity ω.

The eddy damping or viscous effect of unresolved scales of motion included in SGS turbulence models is only a part of what is needed as has been pointed out in a review of LES techniques by Mason[13]. The nonlinear interaction of unresolved and resolved scales of motion also induces a forcing of the larger scales which can only be treated as random. Such so-called stochastic backscatter as an eddy forcing supplement to the usual eddy viscosity provides a stochastic model of SGS turbulence, and, of course, makes the LES itself stochastic in its nature. But this latter consequence, disturbing as it may be, is completely consistent with the well-known limits on the predictability of turbulent flows considered as chaotic dynamical systems.

Traditional stochastic models of turbulence have been applied to the problem of predicting suitable formulations of both eddy damping and eddy forcing in SGS turbulence models. Chasnov[14] describes such an application to the problem of the optimal consistent truncation of an energy-cascading inertial range for three-dimensional homogeneous isotropic turbulence.

5 Austausch Model of the Global Atmosphere

It is tempting to try to devise a stochastic atmospheric climate model of the Langevin type, i.e., with random white forcing and specified damping, that mimics all first and second moments as observed in the real atmosphere. The fluctuation dissipation relation would be built into such a model which would thus provide a crude estimate of climate sensitivity. The feasibility of doing so is suggested by the success of a first simple step in which the atmosphere is treated as a homogeneous, isotropic, two-dimensional turbulent fluid with an eddy mixing of potential vorticity.

Define the potential vorticity as

$$q = \Delta\psi - \lambda^2\psi \tag{64}$$

where Δ is the Laplacian operator, ψ is the stream function, and λ is a specified constant deformation wavenumber. Eddy diffusion dynamics for the model is given by

$$\partial q/\partial t = D\Delta q - \alpha q + w \tag{65}$$

where D is an eddy diffusion coefficient, α is an eddy damping rate, and

w is space- and time-white noise forcing. For a particular wavenumber, k, equations (64) and (65) may be written as

$$q_k = -(k^2 + \lambda^2)\psi_k \quad (66)$$
$$\partial q_k/\partial t = -(Dk^2 + \alpha)q_k + w_k \quad (67)$$

The stochastic differential equation (67) of Langevin type generates stationary statistics with the variance Q_k of q_k given by

$$Q_k = A/[2(\alpha + Dk^2)] \quad (68)$$
$$= (A/2D)/(k^2 + \mu^2) \quad (69)$$

and with a newly defined characteristic wavenumber μ. In order to maintain parsimony of parameters it has been found to be adequate to set

$$\mu^2 = \alpha/D = \lambda^2 \quad (70)$$

The stream function variance, Ψ_k, for wavenumber k is given by

$$\Psi_k = (A/2D)/(k^2 + \lambda^2)^3 \quad (71)$$

where A is a constant, and the two-dimensional velocity variance is given by

$$U(k) = k^2\Psi_k \propto k^2/(k^2 + \lambda^2)^3 \quad (72)$$

The isotropic energy spectrum has the shape

$$E(k) \propto kU(k) \propto k^3/(k^2 + \lambda^2)^3 \quad (73)$$

For $x = k/\lambda$, on finds

$$E(k) \propto f(x) = 8(x + x^{-1})^{-3} \quad (74)$$

which has a maximum at $x = 1$. The transient energy spectrum for the global atmosphere is observed to have a maximum at planetary wavenumber $k = \lambda = 8$, and for such a choice of λ equation (74) provides a fair fit as shown in Fig. 3.

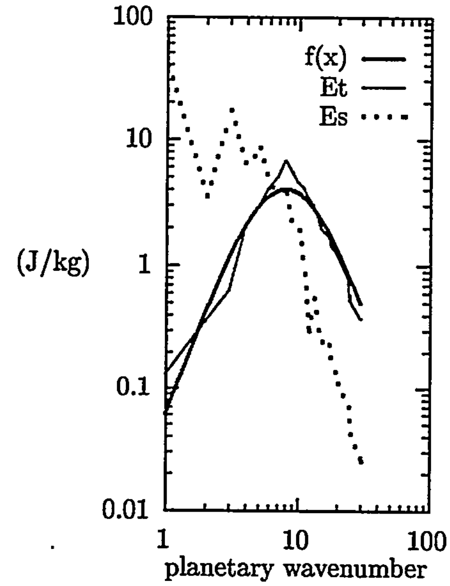

Fig. 3. Fit of f(x) to an observed atmospheric transient kinetic energy spectrum. Et and Es indicate global January transient and stationary spectra, respectively, taken from Boer and Shepherd [15].

Consider next the temporal statistics, in particular, the time-lagged height-height correlation. In this model this is proportional to

$$R(\tau) = \pi \int_0^\infty k\Psi(k) \exp[-(\alpha + Dk^2)\tau] dk \tag{75}$$

$$= \pi \int_0^\infty k\Psi(k) \exp[-\alpha(1 + k^2/\lambda^2)\tau] dk \tag{76}$$

$$\propto \int_0^\infty (1 + k^2/\lambda^2)^{-3} \exp[-\alpha(1 + k^2/\lambda^2)\tau] k dk \tag{77}$$

$$\propto \int_1^\infty s^{-3} \exp[-\alpha\tau s] ds = E_3(\alpha\tau) \tag{78}$$

With suitable normalization, we find

$$R(\tau) = 2E_3(\alpha\tau) \tag{79}$$

in terms of the exponential integral, E_3. A good fit to the observed height-height correlation is obtained by choosing the parameter $\alpha = 0.187/\text{day}$ as shown in Fig. 4.

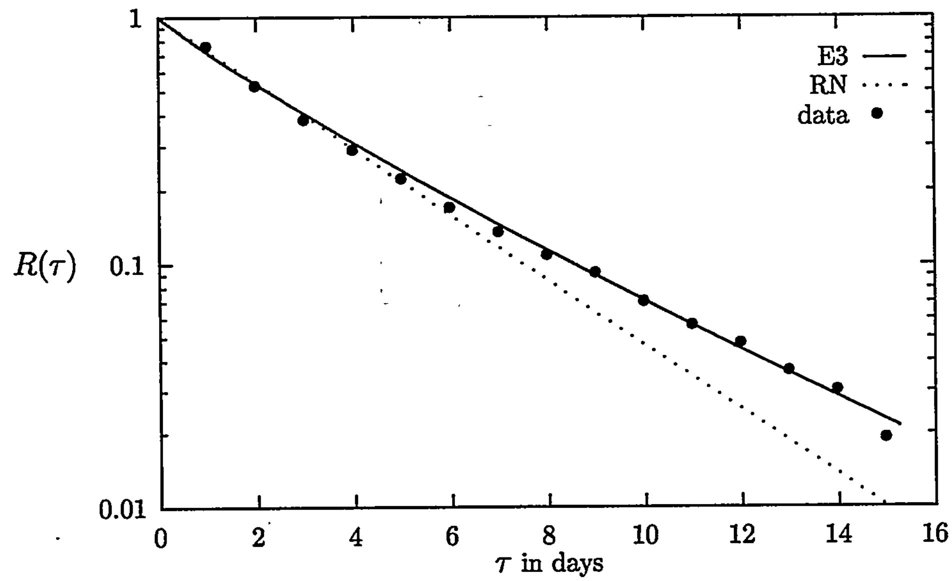

Fig. 4. Fit to height-height lagged time correlation data taken from Lorenz [16]. E3 and RN indicate the fit of the model equation (79) and a simple red noise model, respectively.

Note that the parameter λ is chosen to fit spatial statistics and the parameter α is chosen independently to fit temporal statistics. Note also that in this model the amplitude of the variance depends on the specified strength of the white-noise forcing.

It is clear that such a model is only a simple starting point for the development of stochastic models that take into account the observed three-dimensional mean flow and the inhomogeneous nature of the real climate system. Examination of Fig. 3 shows clearly that the atmospheric mean flow, labeled Es, is far from the vanishing mean flow used in the model.

It is natural to consider the construction of stochastic models of the transient component taking as a starting point the linearized dynamics of the flow about the mean. The linear operator characterizing such dynamics is in general non-normal and has therefore a much more complicated behavior than exhibited by normal operators with traditional spectral analysis. A simple example of this approach is given in a recent paper by Farrell and Ioannou[17].

6 Summary and Conclusion

This paper reviews the general problem of replacing nonlinear chaotic dynamical systems whose probabilistic properties are, in general, not readily computable by linear stochastic dynamical systems that model the original system and have more easily computed probabilistic properties. A trivial example is given of a nonlinear oscillator modeled by a random linear oscillator.

Stochastic models have been widely used to simulate the properties of homogeneous isotropic turbulent flows, and from them have been deduced formulations of stochastic models of the damping and forcing effects of subgrid-scale turbulence on the resolved scales in large-eddy simulations (LES). The properties of two- and three-dimensional turbulence are quite different, and both cases are examined.

Climate is largely defined in terms of the probabilistic properties of the weather fluctuations in the global atmosphere. A stochastic model of these global average properties is developed in terms of the eddy diffusion of potential vorticity with specified random forcing and damping. The choice of two parameters in the model, one determining spatial and the other temporal probabilistic properties, leads to a fair fit to the observations.

Although stochastic models lack rigor in their theoretical foundations, they appear to have considerable pragmatic value, and they should therefore be considered as engineering models that mimic the probabilistic properties of chaotic nonlinear dynamical systems. In constructing such models one matches those properties of the original system that are known and hopes for the best for the rest.

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