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# $[x, p] = i\hbar$ ?

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## DISCLAIMER

Abstract

Heisenberg's commutation relation for position x and momentum p, and its validity for relativistic harmonic oscillators are examined, using the techniques of Lie algebra and dual-bosonic representation of x, p and the Hamiltonian H. A modification with $[x, p] = i\hbar(\mp 1 + H/m_0c^2)$ is proposed for a particle and an anti-particle in a harmonic potential. For a $2\times2$ matrix representation for x, p and H operators, the quantized eigenenergy E is given by $(E - m_0c^2)/\hbar\omega \doteq 3/2, 5/2, 7/2, \ldots$, where 1/2 is not allowed.

The non-commutability between any pair of conjugate canonical variables such as $[x, p] = i\hbar$ is the most fundamental relation for Heisenberg's matrix approach to quantum theory.[1] It is completely equivalent to the wave function representation of Schrödinger[2] where the momentum p needs to be replaced by $-i\hbar\partial/\partial x$ and the Hamiltonian H by $i\hbar\partial/\partial t$. The presence of such a commutation relation leads to Heisenberg's uncertainty principle proclaiming that the position and the momentum of a particle cannot be measured simultaneously with complete accuracy.

In this work the Heisenberg's commutation relations and their validity for relativistic harmonic oscillators will be examined. Harmonic potential is one of the simplest interactions between particles. In addition to the applications to vibrational motion of isolated molecules or collective motion of the lattice in condensed phases, one can transform many other kinds of physical system in terms of harmonic model. For example, the quantization of electromagnetic waves leads to representation of photons in terms of harmonic oscillators. In the study of hadron dynamics and quark confinement, harmonic potentials have also been used.[3] Because the velocities of quarks may not be slow as compared to the speed of light, the ordinary non-relativistic harmonic model has to be modified.[4] In a recent study, Feynman et al.[3] proposed a potential that replaces $x^2$ by a Lorentz-covariant form $\mathbf{x}^2 - c^2t^2$ (or $x^\mu x_\mu$). Thus, the wave functions possess a term with a Gaussian time-dependence as well as the ordinary Hermite polynomials. A similar harmonic potential was also used by Kim and Wigner[5] in their study of relativistic phase-space representation of quantum mechanics.[6] In a more recent report, Aldaya et al.[7] proposed a different approach using generalized Hermite polynomials as wavefunctions for a relativistic quantum harmonic oscillator. In this work, we would like to extend such a model and examine its consequences using a simpler alternative. More powerful and general techniques using Lie algebra and group theory will be used, and there is no need to invoke generalized Hermite polynomials.

In the non-relativistic quantum theory, the operators for the coordinate x, the momentum p and the Hamiltonian H of a harmonic oscillator satisfy the following commutation relations

$$[H, x] = -i\hbar \frac{p}{m_0}, \quad [H, p] = i\hbar m_0 \omega^2 x$$

$$[x, p] = i\hbar \ . \tag{1}$$

In this work we propose a generalized commutation rules using three operators $K_x$, $K_y$ and $K_z$ to represent x, p and H as

$$K_x = \frac{x}{x_0}, \quad K_y = \frac{p}{p_0}, \quad K_z = \frac{A}{A_c} \ , \tag{2}$$

where $x_0$ and $p_0$ are the natural units for length and momentum. The operator A is related to H which will be determined later. The operators $K_i$'s satisfy

$$[K_z, K_x] = iK_y \ , \quad [K_z, K_y] = -iK_x$$
$$[K_x, K_y] = -\ iK_z \ , \tag{3}$$

where the commutation relation between $K_x$ and $K_y$ has a negative sign. They are different from the generators $S_x$, $S_y$ and $S_z$ for the ordinary SU(2) group. With

Schwinger[8] and Wigner[9] techniques one can use two kinds of boson creation and annihilation operators ($a_1^+$, $a_1$, $a_2^+$ and $a_2$) to represent angular momentum. These boson operators commute with each other if they belong to a different species and $[a_i, a_j^+] = \delta_{ij}$. We shall extend such a technique also to the operator $K_i$ as

$$K_+ = a_1^+ a_2^+, \quad K_- = a_1 a_2$$
$$K_z = \frac{1}{4}\{a_1, a_1^+\} + \frac{1}{4}\{a_2, a_2^+\} \tag{4.a}$$

and

$$[K_z, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_z \ . \tag{4.b}$$

It can be shown that $K_x$, $K_y$ and $K_z$ commute with $K_z^2 - K_x^2 - K_y^2$ which will be defined as $\mathbf{K}^2$. Thus, $\mathbf{K}^2$ is a Casimir operator of the noncompact Lie group. Unlike the normal spin-½ operators $S_i$'s where $\mathbf{S}^2 = S_z^2+S_x^2+S_y^2$, $\mathbf{K}^2$ has a negative metric in $K_x$ and $K_y$. Using eq. (4), one can show that $K_z = j+½$ where $j = (n_1 + n_2)/2$ is the average of the quantum numbers $n_1$ for $a_1^+a_1$ and $n_2$ for $a_2^+a_2$. In addition, one can show $\mathbf{K}^2 =- K_+K_-+K_z(K_z-1)$. Because $K_+K_- = n_1n_2$, we have found that $\mathbf{K}^2 = m^2 - ¼$ where m is defined as $(n_1-n_2)/2$. In SU(2) group where the eigenstates of $\mathbf{S}^2 = S(S+1)$ and $S_z$ form a base set. Similarly, one can define a quantum state of a relativistic harmonic oscillator in terms of the eigenstate of $\mathbf{K}^2$ and $K_z$. If one defines $|K, j+½\rangle$ as such an eigenstate with the corresponding eigenvalues $K(K+1)$ for $\mathbf{K}^2$ and $j+½$ for $K_z$, one can show $K =-½\pm m$. Both n and m can be a half integer as well as a whole integer. Using the step operators $K_\pm$ in eq. (4) one can transform one eigenstate into another and to map out all possible states. For example, one can show $K_+ |K, j+½\rangle = C_1 |K,$

$j+3/2\rangle$ and $K_- | K, j+½\rangle = C_2 | K, j-½\rangle$ . The proportional constants $C_1$ and $C_2$ can be determined by using $\langle K, j+½ | K_-K_+ | K, j+½\rangle = |C_1|^2$ and $\langle K, j+½ | K_+K_- | K, j+½\rangle = |C_2|^2$. With $\mathbf{K}^2 = -K_+K_- + K_z(K_z-1)$ or $\mathbf{K}^2 = -K_-K_+ + K_z(K_z+1)$ one can show

$$K_+|K,j+\frac{1}{2}\rangle = \sqrt{-K(K+1)+(j+\frac{1}{2})(j+\frac{3}{2})}\,|K,j+\frac{3}{2}\rangle$$

$$K_-|K,j+\frac{1}{2}\rangle = \sqrt{-K(K+1)+(j+\frac{1}{2})(j-\frac{1}{2})}\,|K,j-\frac{1}{2}\rangle \ . \tag{5}$$

These coefficients are different from the ordinary Clebsch-Gordan coefficient.[10] The hyperbolic metric of $K_i$'s leads to a reversed role between $\mathbf{K}^2$ and $K_z$ as compared to the ordinary angular momentum operators $\mathbf{S}^2$ and $S_z$. By applying $K_-$ repetitively on $|K, j+½\rangle$ as illustrated in eq. (5), one can reach the lower bound of j. Because $j = (n_1+n_2)/2$ and $m = (n_1-n_2)/2$ where $n_1$ and $n_2 = 0, 1, 2,...,$ j and m can only be whole integers or half integers at the same time but not a mix of them.

Let us consider the case with $m = \pm 1$. In the first branch with $m = 1$ (or $K = 1/2$ because $K = -½+m$), one can have $(n_1, n_2) = (2, 0), (3, 1), (4, 2), (5, 3), ...$, which corresponds to $j+½ = 3/2, 5/2, 7/2, ...$ Therefore, j must be an positive integer ($\geq 1$) and $j+½$ has a half-integer form ($\geq 3/2$). The lowest boundary state is $|1/2, 3/2\rangle$. Other higher state $|1/2, j+1/2\rangle$ with a larger j can be generated from $|1/2, 3/2\rangle$ by applying $K_+$ j - 1 times. It can be shown $|1/2, j+1/2\rangle = C_j^{-1}K_+^{j-1} |1/2, 3/2\rangle$, where $C_j = [(j^2-1)((j-1)^2-1)...(3^2-1)(2^2-1)]^{½}$. There is no upper boundary for n. In the second branch with $m = -1$ (or $K = -3/2$), one has $(n_1, n_2) = (0, 2), (1, 3), (2, 4), (3, 5) \ldots$, which corresponds to $j+½ = 3/2, 5/2, 7/2, \ldots$ The bottom state is $|-3/2, 3/2\rangle$. In either case with $K = 1/2$ or $-3/2$, one has $\mathbf{K}^2 = K(K+1) = 3/4$. Therefore, for either branches of $m = \pm 1$ ($K = 1/2$ or $-3/2$) one has the following allowed states $|K, 3/2\rangle$, $|K, 5/2\rangle$, $|K, 7/2\rangle$, .., where $K_z = 3/2, 5/2, 7/2, \ldots$ $|K, 1/2\rangle$ is not allowed here. One can use Pauli matrices to represent $K_x$, $K_y$ and $K_z$ as

$$K_x = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K_y = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6}$$

where $K_z$ is hermitean but $K_x$ and $K_y$ are anti-hermitean. It can be shown that these matrices satisfy the Lie algebra defined in eq. (3). In addition, one has $\mathbf{K}^2 = K_z^2 - K_x^2 - K_y^2 = 3/4$ as expected for $|m| = 1$, because $\mathbf{K}^2 = m^2 - ¼$ as derived earlier.

Now we shall seek a general Hamiltonian for a relativistic quantum oscillator from the point of view of group theory and Lie algebra.[11] The Hamiltonian H must be a function of the Casimir operator $\mathbf{K}^2$ and $K_z$, i.e., $H(\mathbf{K}^2, K_z)$. Because $\mathbf{K}^2 = m^2 - ¼$ is a constant for a given m, thus one may choose the following simplest form for H

$$H = H_0 + H_1 K_z \ . \tag{7}$$

If one sets $H_1 = m_0c^2$, $H_2 = \pm m_0c^2$ then $K_z = A/A_0 = H/m_0c^2 \mp 1$. Eq. (4) can be reduced to

$$[H, x] = -i\hbar\frac{p}{m_0}, \quad [H, p] = i\hbar m_0\omega^2 x$$

$$[x, p] = i\hbar(\mp 1+\frac{H}{m_0 c^2}) \ . \qquad (8)$$

If one chooses $K_z = H/m_0c^2 - 1$, one has

$$c^2p^2 + m_0^2c^2\omega^2x^2 = (n^2+3n+\frac{3}{2})\hbar^2\omega^2$$

$$E = (n+\frac{3}{2})\hbar\omega + m_0c^2 , \quad n = 0, 1, 2, \ldots \qquad (9)$$

The energy E in eq. (9) approaches the rest mass energy $m_0c^2$ as $\omega \rightarrow 0$ for a free particle. In this case one has $[x, p] = i\hbar(-1 + H/m_0c^2)$, the commutator has a sign different from the conventional Heisenberg relation as $H/m_0c^2 \rightarrow 0$ in the non-relativistic limit. If one chooses $K_z = H/\alpha + 1$, one has $[x, p] = i\hbar(1+H/m_0c^2)$ as suggested by Aldaya et al.[7] In this case, we have found

$$c^2p^2 + m_0^2c^2\omega^2x^2 = (n^2+3n+\frac{3}{2})\hbar^2\omega^2$$

$$E = (n+\frac{3}{2})\hbar\omega - m_0c^2 , \quad n = 0, 1, 2, \ldots, \qquad (10)$$

where E is the eigenenergy of H. The lowest bound for $E + m_0c^2$ is given by $3\hbar\omega/2$ which is different from the case for a non-relativistic oscillator. Eq. (9) expression is more favorable than eq. (10). Although eq. (10) preserves the same conventional sign in the commutator, however, eq. (10) leads to $E \rightarrow -m_0c^2$ as $\omega \rightarrow 0$. We would like to argue that the sign convention in the commutator is not important. It is only a matter of choice because one can use time-reversal operation to reverse the sign of p and change the definition of a particle versus an anti-particle without altering the basic physics. Thus, $[x, p] = i\hbar(\mp 1 + H/m_0c^2)$ is proposed for a particle ("-" sign) and an anti-particle ("+" sign) in a harmonic potential.

In conclusion, in this work the Heisenberg's commutation relations for relativistic harmonic oscillators were examined. The approach presented here is based on representation of x, p and H operators in terms of dual-bosonic operators. A modification of the commutation relations is given in eqs. (2), (3) and (7). In particular, $[x, p] = i\hbar(\mp 1 + H/m_0c^2)$ is proposed for a particle and an anti-particle in a harmonic potential. For a $2\times2$ matrix representation for x, p and H operators as shown in eq. (6), corresponding to $m = \pm1$, the quantized eigenenergy E is given by $(E - m_0c^2)/\hbar\omega = 3/2, 5/2, 7/2, \ldots$. The lowest bound of 1/2 for a non-relativistic oscillator is not allowed in this case. A similar result has also been obtained by Aldaya et al.[7] by a different method using generalized Hermite polynomials. This method, however, using Lie algebra is simpler and more powerful which can be easily extended to any general case with $|m| \neq 1$.

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