16019C

National Laboratories

Multifidelity Uncertainty Quantification in Stochastic Media Transport Problems

Gianluca Geraci and Aaron J. Olson

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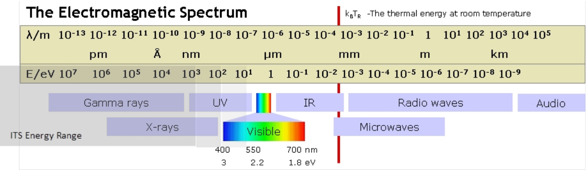
SAND2022-XXXX C

PLAN OF THE TALK

- MOTIVATION AND BACKGROUND
- MF UQ FOR MC RT (W/ STOCHASTIC MEDIA)
- NUMERICAL RESULTS
- CLOSING REMARKS

Motivation and background



From Opensource Handbook of Nanoscience and Nanotechnology

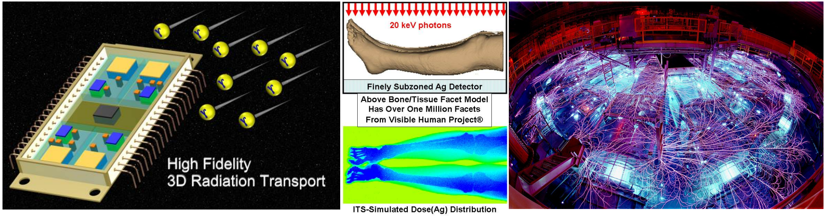


Figure: Courtesy of Brian Franke and Shawn Pautz

High-fidelity state-of-the-art modeling and simulations with HPC

- UQ under **severe** simulations **budget constraints**
- **Significant dimensionality** driven by model complexity

Few considerations on (MF) UQ for MC RT

- MC RT simulations are **expensive** $\rightarrow$ MF sampling UQ to **reduce the computational cost**
- MF UQ approaches **require correlation** among models
- MC RT are *truly* **stochastic solvers** $\rightarrow$ significant correlation can be obtained only by **collecting a large number of particle** histories (for accurate MC RT computations $\mathcal{O}(10^3 - 10^9)$)
- The MC RT **cost increases** with # of particle histories $\rightarrow$ we need to be able to estimate/treat the residual noise

MC RT from the physics/algorithmic perspective:

STEP 1: Use nuclear data to **sample** distance-to-collision event
STEP 2: **Sample** collision events based on cross section values
STEP 3: **Track** particle until it leaves the system
STEP 4: **Evaluate QoIs**, *e.g.* transmittance (# particles passing through the system)

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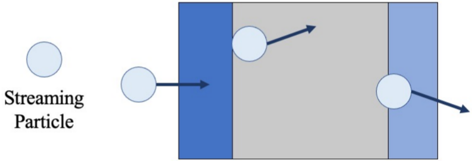

Figure: Courtesy of Kayla Clements.

MC RT from the physics/algorithmic perspective:

Step 1: Use nuclear data to **sample** distance-to-collision event
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MC RT from the UQ perspective:

- **Uncertain parameters**, e.g. cross sections: $\xi \in \Xi \subset \mathbb{R}^d$
- **MC RT (internal) randomness**: $\eta \in H \subset \mathbb{R}^{d'}$
- **Particle histories** are interpreted as elementary events: $f = f(\xi, \eta)$
- **MC RT QoI**: Average of f over the histories for a fixed UQ parameters realization

$$Q(\xi) = \mathbb{E}_\eta \left[f(\xi, \eta) \right] \overset{MC\ RT}{\approx} \frac{1}{N_\eta} \sum_{j=1}^{N_\eta} f\left(\xi, \eta^{(j)}\right) \overset{\text{def}}{=} \tilde{Q}_{N_\eta}(\xi)$$

UQ GOAL: Compute statistics for $Q(\xi)$, *e.g* mean $\mathbb{E}[Q]$ and $\mathbb{V}ar[Q]$, via sampling

Here, we focus on the MF UQ mean estimator[1]

Challenge: $Q(\xi)$ is inaccessible: we can only observe $\tilde{Q}_{N_\eta}(\xi)$

[1] Kayla C. Clements, G. Geraci, and Aaron J. Olson. "A Variance Deconvolution Approach to Sampling Uncertainty Quantification for Monte Carlo Radiation Transport Solvers". In: *Computer Science Research Institute Summer Proceedings 2021*. Technical Report SAND2022-0653R. 2021, pp. 293–307.

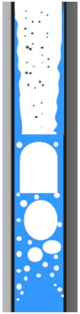
MC RT for stochastic media
Stochastic media in radiation transport

Stochastic media: materials whose internal structure is treated as random

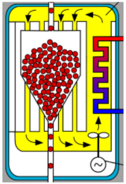
Real-world examples

- Two-phase flow in **Boiling Water Reactor** nuclear power coolant (1)
- **Pebble distribution** in Pebble-Bed nuclear power reactors (2)
- Distribution of **TRISO fuel** particles in Pebble-Bed pebble (3)
- Raleigh-Taylor **instabilities** in Inertial Confinement Fusion reactors
- **Accident scenarios** in various nuclear power reactor cores

1



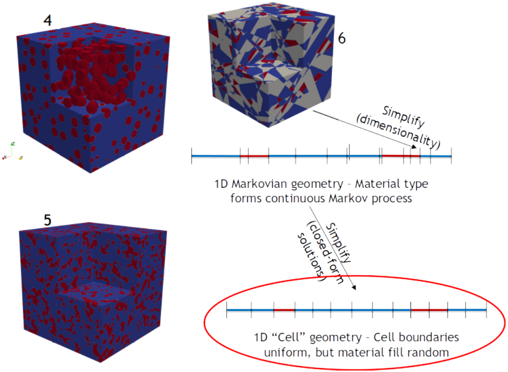
2



3

Numerical approximations

- Spherical inclusions (4)
- Gaussian process (5)
- Markovian/Poisson (6)



MC RT (+ stochastic media) from the UQ perspective:

- **Uncertain parameters**, e.g. cross sections: $\xi \in \Xi \subset \mathbb{R}^d$
- **MC RT (internal) randomness**: $\eta \in H \subset \mathbb{R}^{d'}$
- **Material arrangements/realizations**: $\omega \in \Omega \subset \mathbb{R}^{d''}$
- **Particle histories** are interpreted as elementary events: $f = f(\xi, \omega, \eta)$
- **MC RT QoI**: Average of f over the histories with fixed UQ parameters realization and **material arrangements**

$$Q(\xi, \omega) \;= \mathbb{E}_\eta \left[f(\xi, \omega, \eta) \right] \overset{MC\ RT}{\approx} \frac{1}{N_\eta} \sum_{j=1}^{N_\eta} f\left(\xi, \omega, \eta^{(j)} \right) \overset{\text{def}}{=} \tilde{Q}_{N_\eta}(\xi, \omega)$$

We are interested in the statistics (*i.e.* mean) for the following quantity

$$\mathbb{P}_{\mathbb{E}}(\xi) \overset{\text{def}}{=} \mathbb{E}_\omega \left[Q(\xi, \omega) \right] \longrightarrow \boxed{\mathbb{E}_\xi \left[\mathbb{P}_{\mathbb{E}}(\xi) \right]}$$

MF UQ for MC RT (W/ stochastic media)

MF UQ formulation
Single fidelity MC and notation

We want to compute

$$\mathbb{E}_\xi\left[\mathbb{P}_\mathbb{E}\right] \approx \frac{1}{N_\xi}\sum_{i=1}^{N_\xi}\mathbb{P}_\mathbb{E}(\xi^{(i)})$$

where

$$\mathbb{P}_\mathbb{E}(\xi^{(i)}) \stackrel{\text{def}}{=} \mathbb{E}_\omega\left[Q(\xi^{(i)},\omega)\right] \approx \frac{1}{N_\omega}\sum_{k=1}^{N_\omega}Q(\xi^{(i)},\omega^{(k)})$$

and

$$Q(\xi^{(i)},\omega^{(k)}) \stackrel{\text{def}}{=} \mathbb{E}_\eta\left[f(\xi^{(i)},\omega^{(k)},\eta)\right] \approx \frac{1}{N_\eta}\sum_{j=1}^{N_\eta}f(\xi^{(i)},\omega^{(k)},\eta^{(j)}) \stackrel{\text{def}}{=} \tilde{Q}_{N_\eta}(\xi^{(i)},\omega^{(k)})$$

Let's now use these definitions in reverse order

$$\mathbb{P}_\mathbb{E}(\xi^{(i)}) \approx \frac{1}{N_\omega}\sum_{k=1}^{N_\omega}\tilde{Q}_{N_\eta}(\xi^{(i)},\omega^{(k)}) \stackrel{\text{def}}{=} \tilde{\mathbb{P}}^{\mathbb{E}}_{N_\omega}(\xi^{(i)})$$

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Let's consider a two-model MF UQ estimator

$$\mathbb{E}_\xi\left[\mathbb{P}_\mathbb{E}\right] \approx \left\langle \mathbb{P}_\mathbb{E}^{\mathrm{HF}} \right\rangle_{N_\xi} + \alpha \left(\left\langle \mathbb{P}_\mathbb{E}^{\mathrm{LF}} \right\rangle_{N_\xi} - \left\langle \mathbb{P}_\mathbb{E}^{\mathrm{LF}} \right\rangle_{\tilde{r} N_\xi} \right) \stackrel{\mathrm{def}}{=} \left\langle \mathbb{P}_\mathbb{E} \right\rangle_{N_\xi}^{\mathrm{MF}}$$

The variance of this estimator is[2]

$$\mathbb{V}ar\left[\left\langle \mathbb{P}_\mathbb{E} \right\rangle_{N_\xi}^{\mathrm{MF}}\right] = \mathbb{V}ar\left[\left\langle \mathbb{P}_\mathbb{E}^{\mathrm{HF}} \right\rangle_{N_\xi}\right]\left(1 - \frac{\tilde{r}-1}{\tilde{r}}\tilde{\rho}^2\right),$$

where

$$\tilde{r} = \sqrt{\frac{\tilde{\rho}^2}{1-\tilde{\rho}^2}\frac{\tilde{\mathcal{C}}_{\mathrm{HF}}}{\tilde{\mathcal{C}}_{\mathrm{LF}}}}$$

Remarks:

- The estimator variance $\mathbb{V}ar\left[\left\langle \mathbb{P}_\mathbb{E}^{\mathrm{HF}} \right\rangle_{N_\xi}\right]$ depends on N_ξ, N_ω and N_η
- All quantities denoted by $\tilde{\cdot}$ are **polluted** by the finite number of samples N_ω and N_η

[2] L.W.T. Ng and K. Willcox. "Multifidelity Approaches for Optimization Under Uncertainty". In: *Int. J. Numer. Meth. Engng* 10 (2014), pp. 746–772.

MF UQ formulation
Control variate / MFMC / ACV formulation

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Q: How do we obtain an optimal MF UQ estimator for these problems?

Roadmap

STEP 1: Understand $\mathbb{V}ar\left[\left\langle \mathbb{P}_{\mathbb{E}}^{\mathrm{HF}} \right\rangle_{N_\xi}\right]$ dependence on N_ξ, N_ω and N_η

STEP 2: Take into account $\tilde{\rho}^2$ dependence on N_ω and N_η

STEP 3: Introduce cost models $\tilde{\mathcal{C}}_{\mathrm{HF}}$ and $\tilde{\mathcal{C}}_{\mathrm{LF}}$

STEP 4: Take into account $\tilde{r}$ dependence on N_ω and N_η

STEP 5: Quantify MF UQ estimator efficiency w.r.t MC

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$$\mathbb{V}ar\left[\tilde{\mathbb{P}}^{\mathbb{E}}_{N_\omega}\right] = \underbrace{\mathbb{V}ar_\xi\left[\mathbb{P}_{\mathbb{E}}\right]}_{\text{Parametric variance}} + \underbrace{\mathbb{E}_\xi\left[\frac{\mathbb{V}ar_\omega\left[Q(\xi,\omega)\right]}{N_\omega}\right]}_{\text{Stochastic media}} + \underbrace{\mathbb{E}_\xi\left[\frac{\mathbb{E}_\omega\left[\sigma^2_\eta(\xi,\omega)\right]}{N_\eta N_\omega}\right]}_{\text{MC RT}}$$

Remarks[3]

- **Polluted** quantities are the only ones we can **directly measure**
- **Variance deconvolution:** remove noise from polluted variance $\mathbb{V}ar\left[\tilde{\mathbb{P}}^{\mathbb{E}}_{N_\omega}\right]$
- $\mathbb{V}ar_\omega\left[Q(\xi,\omega)\right]$ is also inaccessible, this requires another variance deconvolution
- MC RT contribution is

$$\sigma^2_\eta(\xi,\omega) = \mathbb{V}ar_\eta\left[f(\xi,\omega,\eta)\right]$$

[3] G. Geraci and Aaron J. Olson. "Deconvolution strategies for efficient parametric variance estimation in stochastic media transport problems". In: *ANS Transactions* (2022), pp. 279–282.

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Link between the polluted correlation $\tilde{\rho}^2$ and ρ^2

$$\tilde{\rho}^2 = \frac{\rho^2}{1 + \tau \rho^2}$$

where

$$\tau = \mathbb{V}ar_\xi \left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}} \right] \gamma_{\mathrm{LF}}(N_\omega^{\mathrm{LF}}, N_\eta^{\mathrm{LF}}) + \mathbb{V}ar_\xi \left[\mathbb{P}_{\mathbb{E}}^{\mathrm{LF}} \right] \gamma_{\mathrm{HF}}(N_\omega^{\mathrm{HF}}, N_\eta^{\mathrm{HF}}) + \gamma_{\mathrm{HF}}(N_\omega^{\mathrm{HF}}, N_\eta^{\mathrm{HF}}) \gamma_{\mathrm{LF}}(N_\omega^{\mathrm{LF}}, N_\eta^{\mathrm{LF}})$$

represents the **stochastic solver noise effect**, which decreases ρ^2

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Let's introduce a **cost model**

- We account for the **re-sampling cost**, *i.e.* sampling more time the same configuration is less expensive
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  - Stochastic media arrangement $\mathcal{C}_\omega$
  - Particle histories $\mathcal{C}_\eta$
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Roadmap

STEP 1: Understand $\mathbb{V}ar\left[\left\langle \mathbb{P}_{\mathbb{E}}^{\mathrm{HF}} \right\rangle_{N_\xi}\right]$ dependence on N_ξ, N_ω and N_η

STEP 2: Take into account $\tilde{\rho}^2$ dependence on N_ω and N_η

STEP 3: Introduce cost models $\tilde{\mathcal{C}}_{\mathrm{HF}}$ and $\tilde{\mathcal{C}}_{\mathrm{LF}}$

STEP 4: Take into account $\tilde{r}$ dependence on N_ω and N_η

STEP 5: Quantify MF UQ estimator efficiency w.r.t MC

Optimal **oversampling ratio:** $\tilde{r} = \sqrt{\frac{\tilde{\rho}^2}{1-\tilde{\rho}^2}\frac{\tilde{\mathcal{C}}_{\mathrm{HF}}}{\tilde{\mathcal{C}}_{\mathrm{LF}}}}$

Let's introduce the **polluted terms** $\tilde{\rho}^2$, $\tilde{\mathcal{C}}_{\mathrm{HF}}$ and $\tilde{\mathcal{C}}_{\mathrm{LF}}$ (generalization of[4])

- $\tilde{\mathcal{C}}_{\mathrm{HF}} = \mathcal{C}_{\xi}^{\mathrm{HF}} + \mathcal{C}_{\omega}^{\mathrm{HF}} N_{\omega}^{\mathrm{HF}} + \mathcal{C}_{\eta}^{\mathrm{HF}} N_{\omega}^{\mathrm{HF}} N_{\eta}^{\mathrm{HF}} = \mathcal{C}_{\xi}^{\mathrm{HF}} + \mathcal{C}_{\omega,\eta}^{\mathrm{HF}}(N_{\omega}^{\mathrm{HF}}, N_{\eta}^{\mathrm{HF}})$
- $\tilde{\mathcal{C}}_{\mathrm{LF}} = \mathcal{C}_{\xi}^{\mathrm{LF}} + \mathcal{C}_{\omega}^{\mathrm{LF}} N_{\omega}^{\mathrm{LF}} + \mathcal{C}_{\eta}^{\mathrm{LF}} N_{\omega}^{\mathrm{LF}} N_{\eta}^{\mathrm{LF}} = \mathcal{C}_{\xi}^{\mathrm{LF}} + \mathcal{C}_{\omega,\eta}^{\mathrm{LF}}(N_{\omega}^{\mathrm{LF}}, N_{\eta}^{\mathrm{LF}})$

$$\tilde{r} = \underbrace{\sqrt{\frac{\rho^2}{1-\rho^2}\frac{\mathcal{C}_{\mathrm{HF}}}{\mathcal{C}_{\mathrm{LF}}}}}_{r:\ \textbf{unpolluted oversampling ratio}} \underbrace{\sqrt{\frac{1-\rho^2}{1-\rho^2+\tau\rho^2}\,\frac{1-\frac{\mathcal{C}_{\omega}^{\mathrm{HF}}+\mathcal{C}_{\eta}^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}}+\frac{\mathcal{C}_{\omega,\eta}^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}}}{1-\frac{\mathcal{C}_{\omega}^{\mathrm{LF}}+\mathcal{C}_{\eta}^{\mathrm{LF}}}{\mathcal{C}_{\mathrm{LF}}}+\frac{\mathcal{C}_{\omega,\eta}^{\mathrm{LF}}}{\mathcal{C}_{\mathrm{LF}}}}}}_{R(N_{\omega}^{\mathrm{HF}},N_{\eta}^{\mathrm{HF}},N_{\omega}^{\mathrm{LF}},N_{\eta}^{\mathrm{LF}}):\ \textbf{stochastic solver contribution}}$$

[4] G. Geraci, L.P. Swiler, and B.J. Debusschere. "Multifidelity UQ Sampling for Stochastic Simulations". In: *16th U.S. National Congress on Computational Mechanics* (2021).

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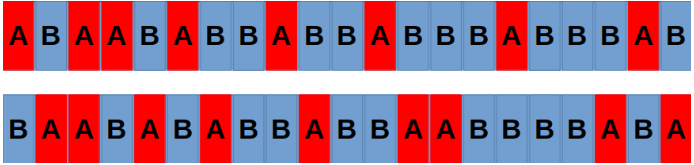
Numerical results



- 1D slab, neutral particle, **absorption-only** mono-energetic steady state radiation transport
- Normally incident beam with unitary magnitude
- **Random cross sections** $(m = A, B)$: $\Sigma_{t,m}(\xi_m) = \Sigma^0_{t,m} + \Sigma^\Delta_{t,m}\xi_m$, where $\xi_A, \xi_B \sim \mathcal{U}(-1, 1)$
- The **QoI** is the **transmittance**

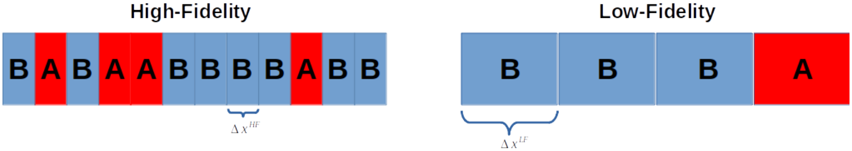
Analytical solution:

Transmittance: $T(\xi, \omega) = \exp\left[-\tau(\xi, \omega)\right],$ where

Slab optical thickness: $\tau(\xi, \omega) = \Delta x \left(N_A(\omega)\Sigma_{t,A}(\xi_A) + N_B(\omega)\Sigma_{t,B}(\xi_B)\right)$

NOTE: $N_A(\omega) \sim \mathcal{B}(N_{tot}, P_A)$, where $N_A(\omega) + N_B(\omega) = N_{tot}$

Multifidelity analysis scenario



Material	$\Sigma_{t,m}^{0}$ [cm^{-1}]	$\Sigma_{t,m}^{\Delta}$ [cm^{-1}]	p_m
A	1.0	0.95	0.3
B	0.4	0.25	0.7

Table: Material properties

Model	N_{tot}	Δx [cm]	$\mathcal{C}_\xi$	$\mathcal{C}_\omega$	$\mathcal{C}_\eta$	$\mathcal{C}$
HF	50	1.0	1.0	0.5	0.01	1.51
LF	10	5.0	0.02	0.01	0.001	0.031

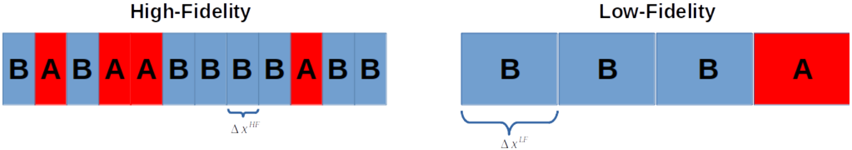
Table: Model configuration – $\mathcal{C}_{\mathrm{HF}}/\mathcal{C}_{\mathrm{LF}} = 48.71$

Two analysis scenarios:

- First – **HF dataset assigned**, *i.e.* N_ω^{HF} and N_η^{HF} are assigned
- Second – **Stochastic media** configurations **assigned** for both models, *i.e.* N_ω^{HF} and N_ω^{LF} are assigned

1D MC RT FOR STOCHASTIC MEDIA WITH UNCERTAIN PROPERTIES

MULTIFIDELITY ANALYSIS SCENARIO



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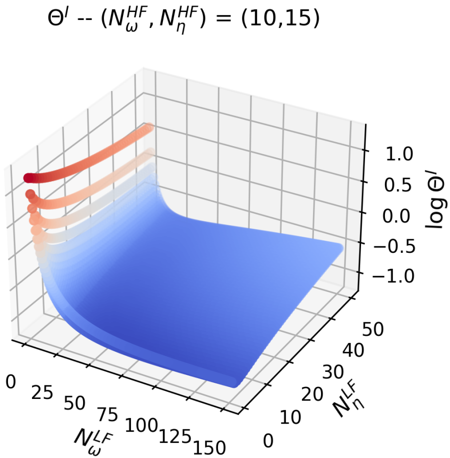
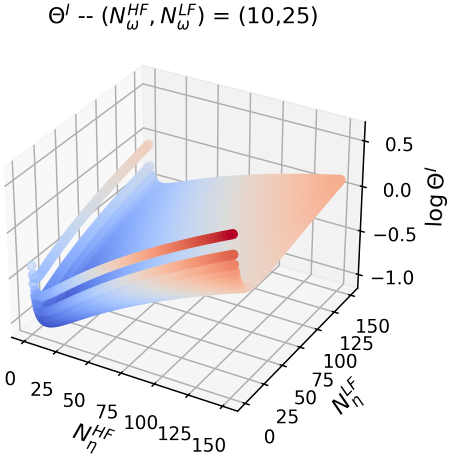
Cost reduction compared to MC (3D view)

Scenario 1: HF dataset assigned – $N_\omega^{\mathrm{HF}} = 10$ and $N_\eta^{\mathrm{HF}} = 15$

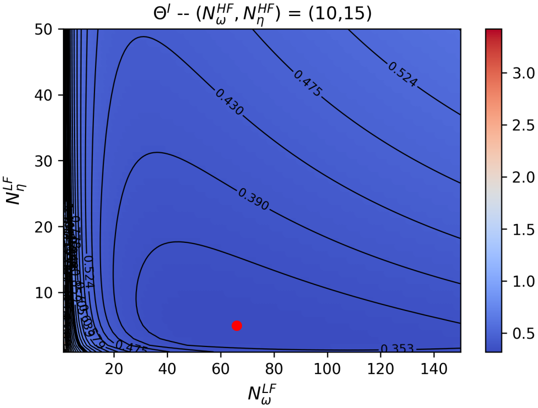


Scenario 2: Stoch media assigned – $N_\omega^{\mathrm{HF}} = 10$ and $N_\omega^{\mathrm{LF}} = 25$

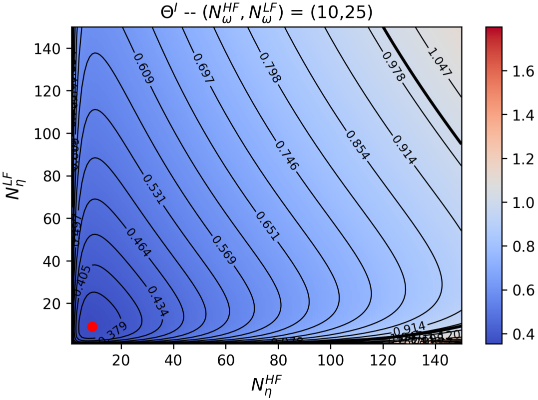


COST REDUCTION COMPARED TO MC (CONTOUR PLOT)

Scenario 1: HF dataset assigned – $N_\omega^{\mathrm{HF}} = 10$ **and** $N_\eta^{\mathrm{HF}} = 15$



Scenario 2: Stoch media assigned – $N_\omega^{\mathrm{HF}} = 10$ **and** $N_\omega^{\mathrm{LF}} = 25$



Closing remarks

Summary

- Radiation transport methods can **benefit from MF UQ**
- Deploying **MF UQ for stochastic solvers** is more **challenging** than for deterministic solvers
- Stochastic **noise needs to be optimally controlled** to preserve MF UQ performance

Talk's contributions

- Formulation for **MF UQ applied to MC RT** problems
- **Stochastic media** effect explicitly included
- Cost model extended to include **re-sampling cost**
- **Verification** exact solution for MF UQ with **binary mixing**

Next steps

- Extension to a realistic deployment (**from pilot samples** to statistics estimation)
- Extension to **multiple models** (*e.g.* leveraging Approximate Control Variate[5], *etc.*)
- Extension to **high-order statistics**

[5] A. Gorodetsky et al. "A generalized approximate control variate framework for multifidelity uncertainty quantification". In: *Journal of Computational Physics* 408 (2020).

THANKS!

Acknowledgements

- Kayla Clements (Oregon State University and SNL)
- Bryan Reuter and Tim Wildey (SNL)

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G. Geraci and Aaron J. Olson. “Deconvolution strategies for efficient parametric variance estimation in stochastic media transport problems”. In: *ANS Transactions* (2022), pp. 279–282.

G. Geraci, L.P. Swiler, and B.J. Debusschere. “Multifidelity UQ Sampling for Stochastic Simulations”. In: *16th U.S. National Congress on Computational Mechanics* (2021).

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BACKUP SLIDES

$$\mathbb{V}ar\left[\left\langle \mathbb{P}_{\mathbb{E}}^{\mathrm{HF}} \right\rangle_{N_\xi}\right] = \mathbb{V}ar\left[\frac{1}{N_\xi}\sum_{i=1}^{N_\xi} \tilde{\mathbb{P}}_{N_\omega}^{\mathbb{E},\mathrm{HF}}(\xi^{(i)})\right] = \frac{1}{N_\xi}\mathbb{V}ar\left[\tilde{\mathbb{P}}_{N_\omega}^{\mathbb{E},\mathrm{HF}}\right]$$

Law-of-total variance

$$\mathbb{V}ar\left[\tilde{\mathbb{P}}_{N_\omega}^{\mathbb{E},\mathrm{HF}}\right] = \mathbb{V}ar_\xi\left[\mathbb{E}_{\eta,\omega}\left[\tilde{\mathbb{P}}_{N_\omega}^{\mathbb{E},\mathrm{HF}}\right]\right] + \mathbb{E}_\xi\left[\mathbb{V}ar_{\eta,\omega}\left[\tilde{\mathbb{P}}_{N_\omega}^{\mathbb{E},\mathrm{HF}}\right]\right]$$

By applying, once again, the law-of-total variance we get

$$\mathbb{V}ar_{\eta,\omega}\left[\tilde{\mathbb{P}}_{N_\omega}^{\mathbb{E}}\right] = \frac{\mathbb{V}ar_\omega\left[Q(\xi,\omega)\right]}{N_\omega} + \frac{\mathbb{E}_\omega\left[\sigma_\eta^2(\xi,\omega)\right]}{N_\omega N_\eta}$$

which leads to the final estimator variance

$$\mathbb{V}ar\left[\tilde{\mathbb{P}}_{N_\omega}^{\mathbb{E}}\right] = \mathbb{V}ar_\xi\left[\mathbb{P}_{\mathbb{E}}\right] + \mathbb{E}_\xi\left[\frac{\mathbb{V}ar_\omega\left[Q(\xi,\omega)\right]}{N_\omega}\right] + \mathbb{E}_\xi\left[\frac{\mathbb{E}_\omega\left[\sigma_{RT,N_\eta}^2(\xi,\omega)\right]}{N_\omega}\right]$$

$$\mathbb{V}ar\left[\left\langle \mathbb{P}^{\text{HF}}_{\mathbb{E}} \right\rangle_{N_\xi}\right] = \mathbb{V}ar\left[\frac{1}{N_\xi}\sum_{i=1}^{N_\xi} \tilde{\mathbb{P}}^{\mathbb{E},\text{HF}}_{N_\omega}(\xi^{(i)})\right] = \frac{1}{N_\xi}\mathbb{V}ar\left[\tilde{\mathbb{P}}^{\mathbb{E},\text{HF}}_{N_\omega}\right]$$

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MF UQ formulation
Polluted correlation (Additional details)

$$\tilde{\rho}^2 = \frac{\mathrm{Cov}^2\left(\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}, \tilde{\mathbb{P}}^{\mathbb{E},\mathrm{LF}}_{N_\omega}\right)}{\mathbb{V}ar\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}\right] \mathbb{V}ar\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{LF}}_{N_\omega}\right]}$$

Properties:

- **Law-of-total covariance**: $\mathrm{Cov}\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}, \tilde{\mathbb{P}}^{\mathbb{E},\mathrm{LF}}_{N_\omega}\right] = \mathbb{E}_\xi\left[\mathrm{Cov}_{\omega,\eta}\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}, \tilde{\mathbb{P}}^{\mathbb{E},\mathrm{LF}}_{N_\omega}\right]\right] + \mathrm{Cov}_\xi\left[\mathbb{E}_{\xi,\omega}\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}\right], \mathbb{E}_{\xi,\omega}\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{LF}}_{N_\omega}\right]\right]$
- All **estimators are unbiased**, *e.g.* $\mathbb{E}_\omega\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}\right] = \mathbb{P}^{\mathrm{HF}}_{\mathbb{E}}$
- ω and η for HF and LF variables are independent

$$\tilde{\rho}^2 = \frac{\left(\mathrm{Cov}_\xi\left[\mathbb{P}^{\mathrm{HF}}_{\mathbb{E}}, \mathbb{P}^{\mathrm{LF}}_{\mathbb{E}}\right]\right)^2}{\left(\mathbb{V}ar_\xi\left[\mathbb{P}^{\mathrm{HF}}_{\mathbb{E}}\right] + \gamma_{\mathrm{HF}}(N^{\mathrm{HF}}_\omega, N^{\mathrm{HF}}_\eta)\right)\left(\mathbb{V}ar_\xi\left[\mathbb{P}^{\mathrm{LF}}_{\mathbb{E}}\right] + \gamma_{\mathrm{LF}}(N^{\mathrm{LF}}_\omega, N^{\mathrm{LF}}_\eta)\right)}$$

$$\gamma_{\mathrm{HF}} = \mathbb{E}_\xi\left[\frac{\mathbb{V}ar_\omega\left[Q^{\mathrm{HF}}(\xi,\omega)\right]}{N^{\mathrm{HF}}_\omega}\right] + \mathbb{E}_\xi\left[\frac{\mathbb{E}_\omega\left[\sigma^2_\eta(\xi,\omega)\right]}{N^{\mathrm{HF}}_\omega N^{\mathrm{HF}}_\eta}\right]$$

$$\gamma_{\mathrm{LF}} = \mathbb{E}_\xi\left[\frac{\mathbb{V}ar_\omega\left[Q^{\mathrm{LF}}(\xi,\omega)\right]}{N^{\mathrm{LF}}_\omega}\right] + \mathbb{E}_\xi\left[\frac{\mathbb{E}_\omega\left[\sigma^2_\eta(\xi,\omega)\right]}{N^{\mathrm{LF}}_\omega N^{\mathrm{LF}}_\eta}\right]$$

$$\tilde{\rho}^2 = \frac{\mathrm{Cov}^2\left(\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}, \tilde{\mathbb{P}}^{\mathbb{E},\mathrm{LF}}_{N_\omega}\right)}{\mathbb{V}ar\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}\right] \mathbb{V}ar\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{LF}}_{N_\omega}\right]}$$

Properties:

- **Law-of-total covariance**: $\mathrm{Cov}\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}, \tilde{\mathbb{P}}^{\mathbb{E},\mathrm{LF}}_{N_\omega}\right] = \mathbb{E}_\xi\left[\mathrm{Cov}_{\omega,\eta}\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}, \tilde{\mathbb{P}}^{\mathbb{E},\mathrm{LF}}_{N_\omega}\right]\right] + \mathrm{Cov}_\xi\left[\mathbb{E}_{\xi,\omega}\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}\right], \mathbb{E}_{\xi,\omega}\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{LF}}_{N_\omega}\right]\right]$
- All **estimators are unbiased**, *e.g.* $\mathbb{E}_\omega\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}\right] = \mathbb{P}^{\mathrm{HF}}_{\mathbb{E}}$
- ω and η for HF and LF variables are independent

$$\tilde{\rho}^2 = \frac{\left(\mathrm{Cov}_\xi\left[\mathbb{P}^{\mathrm{HF}}_{\mathbb{E}}, \mathbb{P}^{\mathrm{LF}}_{\mathbb{E}}\right]\right)^2}{\left(\mathbb{V}ar_\xi\left[\mathbb{P}^{\mathrm{HF}}_{\mathbb{E}}\right] + \gamma_{\mathrm{HF}}(N^{\mathrm{HF}}_\omega, N^{\mathrm{HF}}_\eta)\right)\left(\mathbb{V}ar_\xi\left[\mathbb{P}^{\mathrm{LF}}_{\mathbb{E}}\right] + \gamma_{\mathrm{LF}}(N^{\mathrm{LF}}_\omega, N^{\mathrm{LF}}_\eta)\right)}$$

$$\gamma_{\mathrm{HF}} = \mathbb{E}_\xi\left[\frac{\mathbb{V}ar_\omega\left[Q^{\mathrm{HF}}(\xi,\omega)\right]}{N^{\mathrm{HF}}_\omega}\right] + \mathbb{E}_\xi\left[\frac{\mathbb{E}_\omega\left[\sigma^2_\eta(\xi,\omega)\right]}{N^{\mathrm{HF}}_\omega N^{\mathrm{HF}}_\eta}\right]$$

$$\gamma_{\mathrm{LF}} = \mathbb{E}_\xi\left[\frac{\mathbb{V}ar_\omega\left[Q^{\mathrm{LF}}(\xi,\omega)\right]}{N^{\mathrm{LF}}_\omega}\right] + \mathbb{E}_\xi\left[\frac{\mathbb{E}_\omega\left[\sigma^2_\eta(\xi,\omega)\right]}{N^{\mathrm{LF}}_\omega N^{\mathrm{LF}}_\eta}\right]$$

Cost ratio w.r.t. MC for same estimator variance ε^2_{target}

- MF UQ cost allocation

$$N_\xi^{MF} = \frac{Var\left[\hat{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}\right](N_\omega^{\mathrm{HF}}, N_\eta^{\mathrm{HF}})}{\varepsilon^2_{target}}\left(1 - \frac{rR-1}{rR}\frac{\rho^2}{1+r\rho^2}\right)$$

$$\begin{aligned} C_{tot}^{MF} &= N_\xi^{MF}\tilde{C}_{\mathrm{HF}} + N_\xi^{MF} rR\tilde{C}_{\mathrm{LF}} \\ &= N_\xi^{MF} C_{\mathrm{HF}}\left(\left(1 - \frac{C_\omega^{\mathrm{HF}} + C_\eta^{\mathrm{HF}}}{C_{\mathrm{HF}}} + \frac{C_{\omega,\eta}^{\mathrm{HF}}}{C_{\mathrm{HF}}}\right) + rR\frac{C_{\mathrm{LF}}}{C_{\mathrm{HF}}}\left(1 - \frac{C_\omega^{\mathrm{LF}} + C_\eta^{\mathrm{LF}}}{C_{\mathrm{LF}}} + \frac{C_{\omega,\eta}^{\mathrm{LF}}}{C_{\mathrm{LF}}}\right)\right)\end{aligned}$$

- MC cost allocation

$$N_\xi^{MC} = \frac{Var\left[\hat{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}\right](N_\omega^{\mathrm{HF}}, N_\eta^{\mathrm{HF}})}{\varepsilon^2_{target}}$$

$$C_{tot}^{MC} = N_\xi^{MC}\tilde{C}_{\mathrm{HF}} = N_\xi^{MC} C_{\mathrm{HF}}\left(1 - \frac{C_\omega^{\mathrm{HF}} + C_\eta^{\mathrm{HF}}}{C_{\mathrm{HF}}} + \frac{C_{\omega,\eta}^{\mathrm{HF}}}{C_{\mathrm{HF}}}\right)$$

Cost ratio w.r.t. MC for same estimator variance ε^2_{target}

- **MF UQ** cost allocation

$$N_\xi^{MF} = \frac{\mathbb{V}ar\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}\right](N_\omega^{\mathrm{HF}}, N_\eta^{\mathrm{HF}})}{\varepsilon^2_{target}} \left(1 - \frac{rR-1}{rR}\frac{\rho^2}{1+\tau\rho^2}\right)$$

$$\begin{aligned}\mathcal{C}_{tot}^{MF} &= N_\xi^{MF}\tilde{\mathcal{C}}_{\mathrm{HF}} + N_\xi^{MF} rR\tilde{\mathcal{C}}_{\mathrm{LF}} \\ &= N_\xi^{MF}\mathcal{C}_{\mathrm{HF}}\left(\left(1 - \frac{\mathcal{C}_\omega^{\mathrm{HF}} + \mathcal{C}_\eta^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}}\right) + rR\frac{\mathcal{C}_{\mathrm{LF}}}{\mathcal{C}_{\mathrm{HF}}}\left(1 - \frac{\mathcal{C}_\omega^{\mathrm{LF}} + \mathcal{C}_\eta^{\mathrm{LF}}}{\mathcal{C}_{\mathrm{LF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\mathrm{LF}}}{\mathcal{C}_{\mathrm{LF}}}\right)\right)\end{aligned}$$

- **MC** cost allocation

$$N_\xi^{MC} = \frac{\mathbb{V}ar\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}\right](N_\omega^{\mathrm{HF}}, N_\eta^{\mathrm{HF}})}{\varepsilon^2_{target}}$$

$$\mathcal{C}_{tot}^{MC} = N_\xi^{MC}\tilde{\mathcal{C}}_{\mathrm{HF}} = N_\xi^{MC}\mathcal{C}_{\mathrm{HF}}\left(1 - \frac{\mathcal{C}_\omega^{\mathrm{HF}} + \mathcal{C}_\eta^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}}\right)$$

MF estimator efficiency w.r.t MC (1/2)

Cost ratio w.r.t. MC for same estimator variance ε^2_{target}

- **MF UQ** cost allocation

$$N_\xi^{MF} = \frac{\mathbb{V}ar\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}\right](N^{\mathrm{HF}}_\omega, N^{\mathrm{HF}}_\eta)}{\varepsilon^2_{target}} \left(1 - \frac{rR-1}{rR}\frac{\rho^2}{1+\tau\rho^2}\right)$$

$$\begin{aligned}\mathcal{C}^{MF}_{tot} &= N_\xi^{MF}\tilde{\mathcal{C}}_{\mathrm{HF}} + N_\xi^{MF} rR\tilde{\mathcal{C}}_{\mathrm{LF}} \\ &= N_\xi^{MF}\mathcal{C}_{\mathrm{HF}}\left(\left(1 - \frac{\mathcal{C}^{\mathrm{HF}}_\omega + \mathcal{C}^{\mathrm{HF}}_\eta}{\mathcal{C}_{\mathrm{HF}}} + \frac{\mathcal{C}^{\mathrm{HF}}_{\omega,\eta}}{\mathcal{C}_{\mathrm{HF}}}\right) + rR\frac{\mathcal{C}_{\mathrm{LF}}}{\mathcal{C}_{\mathrm{HF}}}\left(1 - \frac{\mathcal{C}^{\mathrm{LF}}_\omega + \mathcal{C}^{\mathrm{LF}}_\eta}{\mathcal{C}_{\mathrm{LF}}} + \frac{\mathcal{C}^{\mathrm{LF}}_{\omega,\eta}}{\mathcal{C}_{\mathrm{LF}}}\right)\right)\end{aligned}$$

- **MC** cost allocation

$$N_\xi^{MC} = \frac{\mathbb{V}ar\left[\tilde{\mathbb{P}}^{\mathbb{E},\mathrm{HF}}_{N_\omega}\right](N^{\mathrm{HF}}_\omega, N^{\mathrm{HF}}_\eta)}{\varepsilon^2_{target}}$$

$$\mathcal{C}^{MC}_{tot} = N_\xi^{MC}\tilde{\mathcal{C}}_{\mathrm{HF}} = N_\xi^{MC}\mathcal{C}_{\mathrm{HF}}\left(1 - \frac{\mathcal{C}^{\mathrm{HF}}_\omega + \mathcal{C}^{\mathrm{HF}}_\eta}{\mathcal{C}_{\mathrm{HF}}} + \frac{\mathcal{C}^{\mathrm{HF}}_{\omega,\eta}}{\mathcal{C}_{\mathrm{HF}}}\right)$$

First analysis scenario – same averaging for HF

$$\Theta \stackrel{\text{def}}{=} \frac{C_{tot}^{MF}(N_{\xi}^{MF}, N_{\omega}^{\mathrm{HF}}, N_{\eta}^{\mathrm{HF}}, N_{\omega}^{\mathrm{LF}}, N_{\eta}^{\mathrm{LF}})}{C_{tot}^{MC}(N_{\xi}^{MC}, N_{\omega}^{\mathrm{HF}}, N_{\eta}^{\mathrm{HF}})} = \left(1 - \frac{rR-1}{rR}\tilde{\rho}^2\right)\left(1 + rR\frac{\mathcal{C}_{\mathrm{LF}}}{\mathcal{C}_{\mathrm{HF}}}\frac{1 - \frac{\mathcal{C}_{\omega}^{\mathrm{LF}} + \mathcal{C}_{\eta}^{\mathrm{LF}}}{\mathcal{C}_{\mathrm{LF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\mathrm{LF}}}{\mathcal{C}_{\mathrm{LF}}}}{1 - \frac{\mathcal{C}_{\omega}^{\mathrm{HF}} + \mathcal{C}_{\eta}^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}}}\right)$$

Second analysis scenario – $N_{\omega}^{\mathrm{HF}} = 1$ and $N_{\eta}^{\mathrm{HF}} = 1$ for MC

$$\Theta^{I} \stackrel{\text{def}}{=} \frac{C_{tot}^{MF}(N_{\xi}^{MF}, N_{\omega}^{\mathrm{HF}}, N_{\eta}^{\mathrm{HF}}, N_{\omega}^{\mathrm{LF}}, N_{\eta}^{\mathrm{LF}})}{C_{tot}^{MC}(N_{\xi}^{MC}, N_{\omega}^{\mathrm{HF}} = 1, N_{\eta}^{\mathrm{HF}} = 1)}$$

$$= \frac{\mathbb{V}ar\left[\tilde{\mathbb{P}}_{N_{\omega}}^{\mathbb{E},\mathrm{HF}}\right](N_{\omega}^{\mathrm{HF}}, N_{\eta}^{\mathrm{HF}})}{\mathbb{V}ar\left[\tilde{\mathbb{P}}_{N_{\omega}}^{\mathbb{E},\mathrm{HF}}\right](N_{\omega}^{\mathrm{HF}} = 1, N_{\eta}^{\mathrm{HF}} = 1)}\left(1 - \frac{rR-1}{rR}\tilde{\rho}^2\right)\left(1 - \frac{\mathcal{C}_{\omega}^{\mathrm{HF}} + \mathcal{C}_{\eta}^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}}\right)\left(1 + rR\frac{\mathcal{C}_{\mathrm{LF}}}{\mathcal{C}_{\mathrm{HF}}}\frac{1 - \frac{\mathcal{C}_{\omega}^{\mathrm{HF}} + \mathcal{C}_{\eta}^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}}}{1 - \frac{\mathcal{C}_{\omega}^{\mathrm{LF}} + \mathcal{C}_{\eta}^{\mathrm{LF}}}{\mathcal{C}_{\mathrm{LF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\mathrm{LF}}}{\mathcal{C}_{\mathrm{LF}}}}\right)$$

$$= \frac{\mathbb{V}ar_{\xi}\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}\right] + \mathbb{E}_{\xi}\left[\frac{\mathbb{V}ar_{\omega}\left[Q^{HF}(\xi,\omega)\right]}{N_{\omega}^{\mathrm{HF}}}\right] + \mathbb{E}_{\xi}\left[\frac{\mathbb{E}_{\omega}\left[\sigma_{\eta}^{2}(\xi,\omega)\right]}{N_{\omega}^{\mathrm{HF}}N_{\eta}^{\mathrm{HF}}}\right]}{\mathbb{V}ar_{\xi}\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}\right] + \mathbb{E}_{\xi}\left[\mathbb{V}ar_{\omega}\left[Q^{HF}(\xi,\omega)\right]\right] + \mathbb{E}_{\xi}\left[\mathbb{E}_{\omega}\left[\sigma_{\eta}^{2}(\xi,\omega)\right]\right]}\left(1 - \frac{\mathcal{C}_{\omega}^{\mathrm{HF}} + \mathcal{C}_{\eta}^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}}\right)\Theta$$

First analysis scenario – same averaging for HF

$$\Theta \stackrel{\text{def}}{=} \frac{C_{tot}^{MF}(N_\xi^{MF}, N_\omega^{\text{HF}}, N_\eta^{\text{HF}}, N_\omega^{\text{LF}}, N_\eta^{\text{LF}})}{C_{tot}^{MC}(N_\xi^{MC}, N_\omega^{\text{HF}}, N_\eta^{\text{HF}})} = \left(1 - \frac{rR-1}{rR}\tilde{\rho}^2\right)\left(1 + rR\frac{\mathcal{C}_{\text{LF}}}{\mathcal{C}_{\text{HF}}}\frac{1 - \frac{\mathcal{C}_\omega^{\text{LF}} + \mathcal{C}_\eta^{\text{LF}}}{\mathcal{C}_{\text{LF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\text{LF}}}{\mathcal{C}_{\text{LF}}}}{1 - \frac{\mathcal{C}_\omega^{\text{HF}} + \mathcal{C}_\eta^{\text{HF}}}{\mathcal{C}_{\text{HF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\text{HF}}}{\mathcal{C}_{\text{HF}}}}\right)$$

Second analysis scenario – $N_\omega^{\text{HF}} = 1$ and $N_\eta^{\text{HF}} = 1$ for MC

$$\Theta^I \stackrel{\text{def}}{=} \frac{C_{tot}^{MF}(N_\xi^{MF}, N_\omega^{\text{HF}}, N_\eta^{\text{HF}}, N_\omega^{\text{LF}}, N_\eta^{\text{LF}})}{C_{tot}^{MC}(N_\xi^{MC}, N_\omega^{\text{HF}} = 1, N_\eta^{\text{HF}} = 1)}$$

$$= \frac{\mathbb{V}ar\left[\tilde{\mathbb{P}}_{N_\omega}^{\mathbb{E},\text{HF}}\right](N_\omega^{\text{HF}}, N_\eta^{\text{HF}})}{\mathbb{V}ar\left[\tilde{\mathbb{P}}_{N_\omega}^{\mathbb{E},\text{HF}}\right](N_\omega^{\text{HF}} = 1, N_\eta^{\text{HF}} = 1)}\left(1 - \frac{rR-1}{rR}\tilde{\rho}^2\right)\left(1 - \frac{\mathcal{C}_\omega^{\text{HF}} + \mathcal{C}_\eta^{\text{HF}}}{\mathcal{C}_{\text{HF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\text{HF}}}{\mathcal{C}_{\text{HF}}}\right)\left(1 + rR\frac{\mathcal{C}_{\text{LF}}}{\mathcal{C}_{\text{HF}}}\frac{1 - \frac{\mathcal{C}_\omega^{\text{HF}} + \mathcal{C}_\eta^{\text{HF}}}{\mathcal{C}_{\text{HF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\text{HF}}}{\mathcal{C}_{\text{HF}}}}{1 - \frac{\mathcal{C}_\omega^{\text{LF}} + \mathcal{C}_\eta^{\text{LF}}}{\mathcal{C}_{\text{LF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\text{LF}}}{\mathcal{C}_{\text{LF}}}}\right)$$

$$= \frac{\mathbb{V}ar_\xi\left[\mathbb{P}_{\mathbb{E}}^{\text{HF}}\right] + \mathbb{E}_\xi\left[\frac{\mathbb{V}ar_\omega\left[Q^{HF}(\xi,\omega)\right]}{N_\omega^{\text{HF}}}\right] + \mathbb{E}_\xi\left[\frac{\mathbb{E}_\omega\left[\sigma_\eta^2(\xi,\omega)\right]}{N_\omega^{\text{HF}} N_\eta^{\text{HF}}}\right]}{\mathbb{V}ar_\xi\left[\mathbb{P}_{\mathbb{E}}^{\text{HF}}\right] + \mathbb{E}_\xi\left[\mathbb{V}ar_\omega\left[Q^{HF}(\xi,\omega)\right]\right] + \mathbb{E}_\xi\left[\mathbb{E}_\omega\left[\sigma_\eta^2(\xi,\omega)\right]\right]}\left(1 - \frac{\mathcal{C}_\omega^{\text{HF}} + \mathcal{C}_\eta^{\text{HF}}}{\mathcal{C}_{\text{HF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\text{HF}}}{\mathcal{C}_{\text{HF}}}\right)\Theta$$

1D MC RT for stochastic media with uncertain properties

Summary of the UQ exact solution (1/2)

- Derived all terms needed for **closed-form variance** $\mathbb{V}ar_{\xi}\left[\mathbb{P}_{\mathbb{E}}\right]$
- MF UQ analysis required $\mathrm{Cov}_{\xi}\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}, \mathbb{P}_{\mathbb{E}}^{\mathrm{LF}}\right]$ (to evaluate the correlation)

Summary of the Cov derivation

- **QoI in closed-form** as function of $N_A(\omega)$

$$\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}} = \mathbb{E}_{\omega}\left[T^{\mathrm{HF}}\right] = \exp\left[-N_{tot}^{\mathrm{HF}}\Delta x^{\mathrm{HF}}\Sigma_{t,B}(\xi_B)\right]\mathbb{E}_{\omega}\left[N_A^{\mathrm{HF}}(\omega)\Delta x^{\mathrm{HF}}\overbrace{\left(\Sigma_{t,A}^{0}-\Sigma_{t,B}^{0}+\Sigma_{t,A}^{\Delta}\xi_A-\Sigma_{t,B}^{\Delta}\xi_B\right)}^{F(\xi_A,\xi_B)=F_0+F_{\Delta}(\xi_A,\xi_B)}\right]$$

- Leverage the probability mass function of $N_A(\omega)$

$$\mathbb{E}_{\omega}\left[N_A(\omega)\Delta x^{\mathrm{HF}}F(\xi_A,\xi_B)\right] = \sum_{x=0}^{N_{tot}^{\mathrm{HF}}}\frac{N_{tot}^{\mathrm{HF}}!}{x!(N_{tot}^{\mathrm{HF}}-x)!}p_A^{x}(1-p_A)^{N_{tot}^{\mathrm{HF}}-x}\exp\left[-x\Delta x^{\mathrm{HF}}F(\xi_A,\xi_B)\right]$$

$$= \sum_{x=0}^{N_{tot}} B_{\omega}(x)\exp\left[-x\Delta x^{\mathrm{HF}}F_{\Delta}(\xi_A,\xi_B)\right]$$

- **Evaluate statistics**, *e.g.* the expected value

$$\mathbb{E}_{\xi}\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}\right] = \exp\left[-N_{tot}\Delta x\Sigma_{t,B}^{0}\right]\left(\sum_{x=0}^{N_{tot}^{\mathrm{HF}}} B_{\omega}(x)(x)\frac{\sinh\left[x\Delta x^{\mathrm{HF}}\Sigma_{t,A}^{\Delta}\right]}{x\Delta x^{\mathrm{HF}}\Sigma_{t,A}^{\Delta}}\frac{\sinh\left[N_{tot}^{\mathrm{HF}}\Delta x^{\mathrm{HF}}\Sigma_{t,B}^{\Delta}\left(1-\frac{x}{N_{tot}^{\mathrm{HF}}}\right)\Delta x^{\mathrm{HF}}\Sigma_{t,A}^{\Delta}\right]}{N_{tot}^{\mathrm{HF}}\Delta x^{\mathrm{HF}}\Sigma_{t,B}^{\Delta}\left(1-\frac{x}{N_{tot}^{\mathrm{HF}}}\right)\Delta x^{\mathrm{HF}}\Sigma_{t,A}^{\Delta}}\right),$$

Summary of the UQ exact solution (1/2)

- Derived all terms needed for **closed-form variance** $\mathbb{V}ar_{\xi}\left[\mathbb{P}_{\mathbb{E}}\right]$
- MF UQ analysis required $\mathrm{Cov}_{\xi}\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}, \mathbb{P}_{\mathbb{E}}^{\mathrm{LF}}\right]$ (to evaluate the correlation)

Summary of the Cov **derivation**

- **QoI in closed-form** as function of $N_A(\omega)$

$$\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}} = \mathbb{E}_{\omega}\left[T^{\mathrm{HF}}\right] = \exp\left[-N_{tot}^{\mathrm{HF}}\overset{\mathrm{HF}}{\Delta x}\,\Sigma_{t,B}(\xi_B)\right]\mathbb{E}_{\omega}\left[N_A^{\mathrm{HF}}(\omega)\overset{\mathrm{HF}}{\Delta x}\overbrace{\left(\Sigma_{t,A}^{0} - \Sigma_{t,B}^{0} + \Sigma_{t,A}^{\Delta}\xi_A - \Sigma_{t,B}^{\Delta}\xi_B\right)}^{F(\xi_A,\xi_B)=F_0+F_{\Delta}(\xi_A,\xi_B)}\right]$$

- Leverage the probability mass function of $N_A(\omega)$

$$\mathbb{E}_{\omega}\left[N_A(\omega)\overset{\mathrm{HF}}{\Delta x}F(\xi_A,\xi_B)\right] = \sum_{x=0}^{N_{tot}^{\mathrm{HF}}}\frac{N_{tot}^{\mathrm{HF}}!}{x!(N_{tot}^{\mathrm{HF}}-x)!}p_A^{x}(1-p_A)^{N_{tot}^{\mathrm{HF}}-x}\exp\left[-x\overset{\mathrm{HF}}{\Delta x}F(\xi_A,\xi_B)\right]$$

$$= \sum_{x=0}^{N_{tot}} B_{\omega}(x)\exp\left[-x\overset{\mathrm{HF}}{\Delta x}F_{\Delta}(\xi_A,\xi_B)\right]$$

- **Evaluate statistics**, *e.g.* the expected value

$$\mathbb{E}_{\xi}\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}\right] = \exp\left[-N_{tot}\Delta x\Sigma_{t,B}^{0}\right]\left(\sum_{x=0}^{N_{tot}^{\mathrm{HF}}} B_{\omega}(x)(x)\frac{\sinh\left[x\overset{\mathrm{HF}}{\Delta x}\Sigma_{t,A}^{\Delta}\right]}{x\overset{\mathrm{HF}}{\Delta x}\Sigma_{t,A}^{\Delta}}\frac{\sinh\left[N_{tot}^{\mathrm{HF}}\overset{\mathrm{HF}}{\Delta x}\Sigma_{t,B}^{\Delta}\left(1-\frac{x}{N_{tot}^{\mathrm{HF}}}\right)\overset{\mathrm{HF}}{\Delta x}\Sigma_{t,A}^{\Delta}\right]}{N_{tot}^{\mathrm{HF}}\overset{\mathrm{HF}}{\Delta x}\Sigma_{t,B}^{\Delta}\left(1-\frac{x}{N_{tot}^{\mathrm{HF}}}\right)\overset{\mathrm{HF}}{\Delta x}\Sigma_{t,A}^{\Delta}}\right),$$

Summary of the UQ exact solution (1/2)

- Derived all terms needed for **closed-form variance** $\mathbb{V}ar_{\xi}\left[\mathbb{P}_{\mathbb{E}}\right]$
- MF UQ analysis required $\mathrm{Cov}_{\xi}\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}, \mathbb{P}_{\mathbb{E}}^{\mathrm{LF}}\right]$ (to evaluate the correlation)

Summary of the Cov derivation

- **QoI in closed-form** as function of $N_A(\omega)$

$$\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}} = \mathbb{E}_{\omega}\left[T^{\mathrm{HF}}\right] = \exp\left[-N_{tot}^{\mathrm{HF}} \overset{\mathrm{HF}}{\Delta x}\, \Sigma_{t,B}(\xi_B)\right] \mathbb{E}_{\omega}\left[N_A^{\mathrm{HF}}(\omega) \overset{\mathrm{HF}}{\Delta x} \overbrace{\left(\Sigma_{t,A}^{0} - \Sigma_{t,B}^{0} + \Sigma_{t,A}^{\Delta}\xi_A - \Sigma_{t,B}^{\Delta}\xi_B\right)}^{F(\xi_A,\xi_B)=F_0+F_{\Delta}(\xi_A,\xi_B)}\right]$$

- Leverage the probability mass function of $N_A(\omega)$

$$\mathbb{E}_{\omega}\left[N_A(\omega) \overset{\mathrm{HF}}{\Delta x} F(\xi_A, \xi_B)\right] = \sum_{x=0}^{N_{tot}^{\mathrm{HF}}} \frac{N_{tot}^{\mathrm{HF}}!}{x!(N_{tot}^{\mathrm{HF}} - x)!} p_A^x (1 - p_A)^{N_{tot}^{\mathrm{HF}} - x} \exp\left[-x \overset{\mathrm{HF}}{\Delta x} F(\xi_A, \xi_B)\right]$$
$$= \sum_{x=0}^{N_{tot}} B_{\omega}(x) \exp\left[-x \overset{\mathrm{HF}}{\Delta x} F_{\Delta}(\xi_A, \xi_B)\right]$$

- **Evaluate statistics**, *e.g.* the expected value

$$\mathbb{E}_{\xi}\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}\right] = \exp\left[-N_{tot}\Delta x \Sigma_{t,B}^{0}\right] \left(\sum_{x=0}^{N_{tot}^{\mathrm{HF}}} B_{\omega}(x)(x) \frac{\sinh\left[x \overset{\mathrm{HF}}{\Delta x} \Sigma_{t,A}^{\Delta}\right]}{x \overset{\mathrm{HF}}{\Delta x} \Sigma_{t,A}^{\Delta}} \frac{\sinh\left[N_{tot}^{\mathrm{HF}} \overset{\mathrm{HF}}{\Delta x} \Sigma_{t,B}^{\Delta} \left(1 - \frac{x}{N_{tot}^{\mathrm{HF}}}\right) \overset{\mathrm{HF}}{\Delta x} \Sigma_{t,A}^{\Delta}\right]}{N_{tot}^{\mathrm{HF}} \overset{\mathrm{HF}}{\Delta x} \Sigma_{t,B}^{\Delta} \left(1 - \frac{x}{N_{tot}^{\mathrm{HF}}}\right) \overset{\mathrm{HF}}{\Delta x} \Sigma_{t,A}^{\Delta}}\right),$$

- Derived all terms needed for **closed-form variance** $\mathbb{V}ar_{\xi}\left[\mathbb{P}_{\mathbb{E}}\right]$
- MF UQ analysis required $\mathrm{Cov}_{\xi}\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}, \mathbb{P}_{\mathbb{E}}^{\mathrm{LF}}\right]$ (to evaluate the correlation)

Summary of the Cov derivation

- **QoI in closed-form** as function of $N_A(\omega)$

$$\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}} = \mathbb{E}_{\omega}\left[T^{\mathrm{HF}}\right] = \exp\left[-N_{tot}^{\mathrm{HF}} \overset{\mathrm{HF}}{\Delta x} \Sigma_{t,B}(\xi_B)\right] \mathbb{E}_{\omega}\left[N_A^{\mathrm{HF}}(\omega) \overset{\mathrm{HF}}{\Delta x} \overbrace{\left(\Sigma_{t,A}^{0} - \Sigma_{t,B}^{0} + \Sigma_{t,A}^{\Delta}\xi_A - \Sigma_{t,B}^{\Delta}\xi_B\right)}^{F(\xi_A,\xi_B)=F_0+F_{\Delta}(\xi_A,\xi_B)}\right]$$

- Leverage the probability mass function of $N_A(\omega)$

$$\mathbb{E}_{\omega}\left[N_A(\omega)\overset{\mathrm{HF}}{\Delta x} F(\xi_A,\xi_B)\right] = \sum_{x=0}^{N_{tot}^{\mathrm{HF}}} \frac{N_{tot}^{\mathrm{HF}}!}{x!(N_{tot}^{\mathrm{HF}}-x)!} p_A^{x}\left(1-p_A\right)^{N_{tot}^{\mathrm{HF}}-x} \exp\left[-x\overset{\mathrm{HF}}{\Delta x} F(\xi_A,\xi_B)\right]$$

$$= \sum_{x=0}^{N_{tot}} B_{\omega}(x) \exp\left[-x\overset{\mathrm{HF}}{\Delta x} F_{\Delta}(\xi_A,\xi_B)\right]$$

- **Evaluate statistics**, *e.g.* the expected value

$$\mathbb{E}_{\xi}\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}\right] = \exp\left[-N_{tot}\Delta x \Sigma_{t,B}^{0}\right] \left(\sum_{x=0}^{N_{tot}^{\mathrm{HF}}} B_{\omega}(x)(x) \frac{\sinh\left[x\overset{\mathrm{HF}}{\Delta x}\Sigma_{t,A}^{\Delta}\right]}{x\overset{\mathrm{HF}}{\Delta x}\Sigma_{t,A}^{\Delta}} \frac{\sinh\left[N_{tot}^{\mathrm{HF}}\overset{\mathrm{HF}}{\Delta x}\Sigma_{t,B}^{\Delta}\left(1-\frac{x}{N_{tot}^{\mathrm{HF}}}\right)\overset{\mathrm{HF}}{\Delta x}\Sigma_{t,A}^{\Delta}\right]}{N_{tot}^{\mathrm{HF}}\overset{\mathrm{HF}}{\Delta x}\Sigma_{t,B}^{\Delta}\left(1-\frac{x}{N_{tot}^{\mathrm{HF}}}\right)\overset{\mathrm{HF}}{\Delta x}\Sigma_{t,A}^{\Delta}}\right),$$

Summary of the UQ exact solution (2/2)

$$\mathrm{Cov}_\xi\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}, \mathbb{P}_{\mathbb{E}}^{\mathrm{LF}}\right] = \mathbb{E}_\xi\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}\mathbb{P}_{\mathbb{E}}^{\mathrm{LF}}\right] - \mathbb{E}_\xi\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}\right]\mathbb{E}_\xi\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{LF}}\right]$$

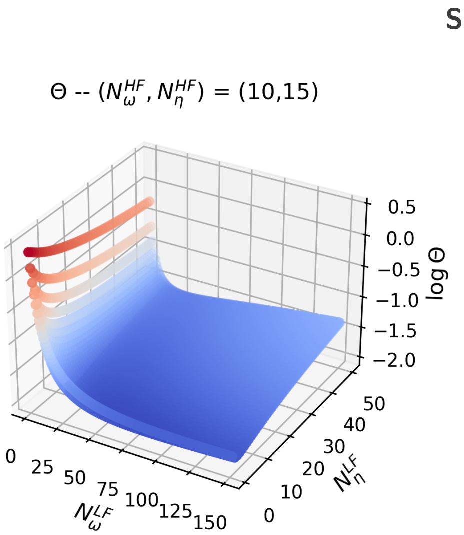
$$\mathbb{E}_\xi\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}\mathbb{P}_{\mathbb{E}}^{\mathrm{LF}}\right] = \exp\left[-\Sigma_{t,B}^{0}\left(N_{tot}^{\mathrm{HF}}\overset{\mathrm{HF}}{\Delta x} + N_{tot}^{\mathrm{LF}}\overset{\mathrm{LF}}{\Delta x}\right)\right]\left(\sum_{x=0}^{N_{tot}^{\mathrm{LF}}} B_\omega^{\mathrm{HF}}(x)B_\omega^{\mathrm{LF}}(x)\frac{\sinh\left[x\left(\overset{\mathrm{HF}}{\Delta x}+\overset{\mathrm{LF}}{\Delta x}\right)\Sigma_{t,A}^{\Delta}\right]}{x\left(\overset{\mathrm{HF}}{\Delta x}+\overset{\mathrm{LF}}{\Delta x}\right)\Sigma_{t,A}^{\Delta}}\right.$$

$$\cdot\frac{\sinh\left[\left(\left(x-N_{tot}^{\mathrm{HF}}\right)\overset{\mathrm{HF}}{\Delta x}+\left(x-N_{tot}^{\mathrm{LF}}\right)\overset{\mathrm{LF}}{\Delta x}\right)\Sigma_{t,B}^{\Delta}\right]}{\left(\left(x-N_{tot}^{\mathrm{HF}}\right)\overset{\mathrm{HF}}{\Delta x}+\left(x-N_{tot}^{\mathrm{LF}}\right)\overset{\mathrm{LF}}{\Delta x}\right)\Sigma_{t,B}^{\Delta}}$$

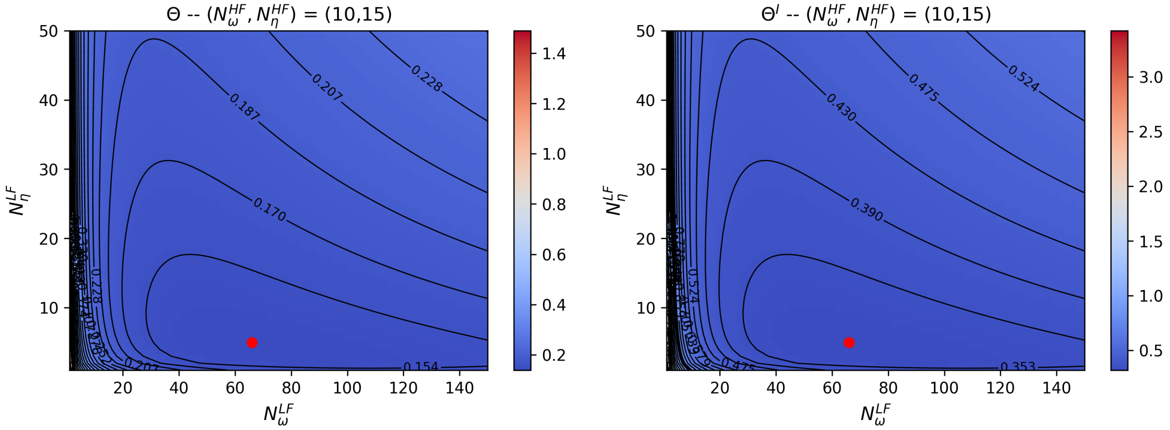
$$+\sum_{x=0}^{N_{tot}^{\mathrm{LF}}}\sum_{\substack{y=0\\ y\neq x}}^{N_{tot}^{\mathrm{LF}}} B_\omega^{\mathrm{HF}}(x)B_\omega^{\mathrm{LF}}(y)\frac{\sinh\left[\left(x\overset{\mathrm{HF}}{\Delta x}+y\overset{\mathrm{LF}}{\Delta x}\right)\Sigma_{t,A}^{\Delta}\right]}{\left(x\overset{\mathrm{HF}}{\Delta x}+y\overset{\mathrm{LF}}{\Delta x}\right)\Sigma_{t,A}^{\Delta}}\frac{\sinh\left[\left(\left(x-N_{tot}^{\mathrm{HF}}\right)\overset{\mathrm{HF}}{\Delta x}+\left(y-N_{tot}^{\mathrm{LF}}\right)\overset{\mathrm{LF}}{\Delta x}\right)\Sigma_{t,B}^{\Delta}\right]}{\left(\left(x-N_{tot}^{\mathrm{HF}}\right)\overset{\mathrm{HF}}{\Delta x}+\left(y-N_{tot}^{\mathrm{LF}}\right)\overset{\mathrm{LF}}{\Delta x}\right)\Sigma_{t,B}^{\Delta}}$$

$$\left.+\sum_{x=0}^{N_{tot}^{\mathrm{LF}}}\sum_{y=N_{tot}^{\mathrm{LF}}+1}^{N_{tot}^{\mathrm{HF}}} B_\omega^{\mathrm{LF}}(x)B_\omega^{\mathrm{HF}}(y)\frac{\sinh\left[\left(x\overset{\mathrm{LF}}{\Delta x}+y\overset{\mathrm{HF}}{\Delta x}\right)\Sigma_{t,A}^{\Delta}\right]}{\left(x\overset{\mathrm{LF}}{\Delta x}+y\overset{\mathrm{HF}}{\Delta x}\right)\Sigma_{t,A}^{\Delta}}\frac{\sinh\left[\left(\left(x-N_{tot}^{\mathrm{LF}}\right)\overset{\mathrm{LF}}{\Delta x}+\left(y-N_{tot}^{\mathrm{HF}}\right)\overset{\mathrm{HF}}{\Delta x}\right)\Sigma_{t,B}^{\Delta}\right]}{\left(\left(x-N_{tot}^{\mathrm{LF}}\right)\overset{\mathrm{LF}}{\Delta x}+\left(y-N_{tot}^{\mathrm{HF}}\right)\overset{\mathrm{HF}}{\Delta x}\right)\Sigma_{t,B}^{\Delta}}\right)$$

Scenario 1: HF dataset assigned – $N_\omega^{\mathrm{HF}} = 10$ and $N_\eta^{\mathrm{HF}} = 15$ – Cost ratio (1/2)

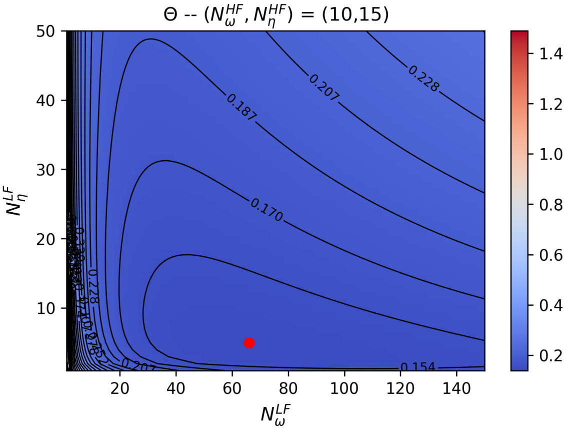


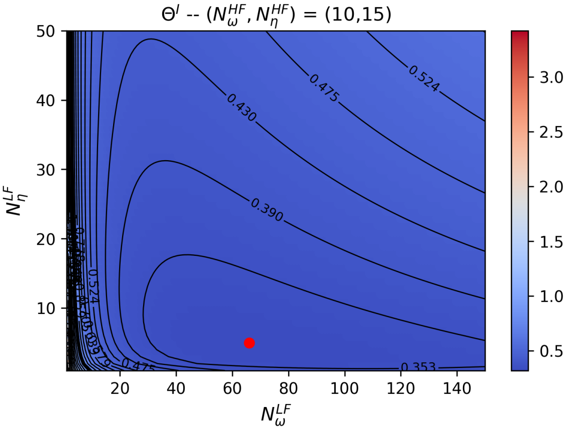
Scenario 1: HF dataset assigned – $N_\omega^{\mathrm{HF}} = 10$ and $N_\eta^{\mathrm{HF}} = 15$ – Cost ratio (1/2)



$$\Theta^I = \underbrace{\frac{\mathbb{V}ar_\xi\left[\mathbb{P}^{\mathrm{HF}}_{\mathbb{E}}\right] + \mathbb{E}_\xi\left[\frac{\mathbb{V}ar_\omega\left[Q^{HF}(\xi,\omega)\right]}{N^{\mathrm{HF}}_\omega}\right] + \mathbb{E}_\xi\left[\frac{\mathbb{E}_\omega\left[\sigma^2_\eta(\xi,\omega)\right]}{N^{\mathrm{HF}}_\omega N^{\mathrm{HF}}_\eta}\right]}{\mathbb{V}ar_\xi\left[\mathbb{P}^{\mathrm{HF}}_{\mathbb{E}}\right] + \mathbb{E}_\xi\left[\mathbb{V}ar_\omega\left[Q^{HF}(\xi,\omega)\right]\right] + \mathbb{E}_\xi\left[\mathbb{E}_\omega\left[\sigma^2_\eta(\xi,\omega)\right]\right]} \left(1 - \frac{\mathcal{C}^{\mathrm{HF}}_\omega + \mathcal{C}^{\mathrm{HF}}_\eta}{\mathcal{C}_{\mathrm{HF}}} + \frac{\mathcal{C}^{\mathrm{HF}}_{\omega,\eta}}{\mathcal{C}_{\mathrm{HF}}}\right)}_{\text{Constant for this test case } (>1)} \Theta$$

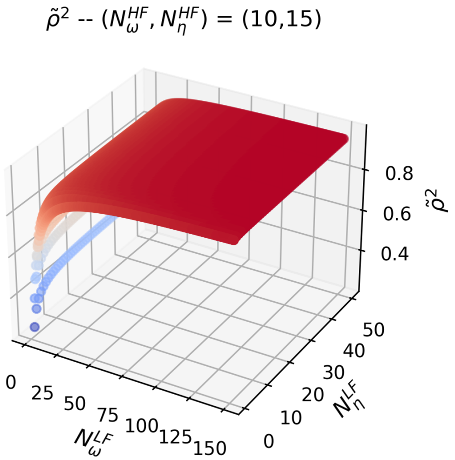
Scenario 1: HF dataset assigned – $N_\omega^{\mathrm{HF}} = 10$ and $N_\eta^{\mathrm{HF}} = 15$ – Cost ratio (1/2)

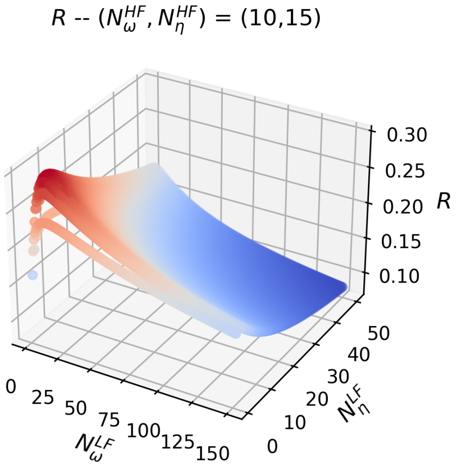




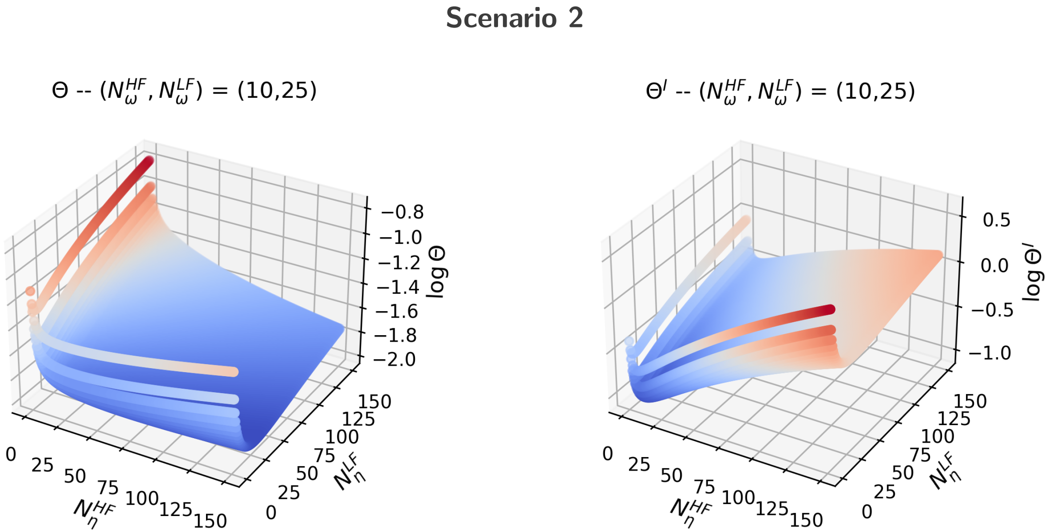
$$\Theta^I = \underbrace{\frac{\mathbb{V}ar_\xi\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}\right] + \mathbb{E}_\xi\left[\frac{\mathbb{V}ar_\omega\left[Q^{HF}(\xi,\omega)\right]}{N_\omega^{\mathrm{HF}}}\right] + \mathbb{E}_\xi\left[\frac{\mathbb{E}_\omega\left[\sigma_\eta^2(\xi,\omega)\right]}{N_\omega^{\mathrm{HF}} N_\eta^{\mathrm{HF}}}\right]}{\mathbb{V}ar_\xi\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}\right] + \mathbb{E}_\xi\left[\mathbb{V}ar_\omega\left[Q^{HF}(\xi,\omega)\right]\right] + \mathbb{E}_\xi\left[\mathbb{E}_\omega\left[\sigma_\eta^2(\xi,\omega)\right]\right]}\left(1 - \frac{\mathcal{C}_\omega^{\mathrm{HF}} + \mathcal{C}_\eta^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}}\right)}_{\text{Constant for this test case } (>1)} \Theta$$

Scenario 1: HF dataset assigned – $N_{\omega}^{\mathrm{HF}} = 10$ and $N_{\eta}^{\mathrm{HF}} = 15$ – Correlation/Allocation



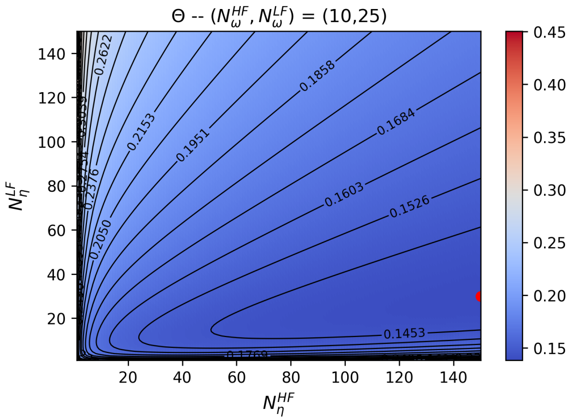


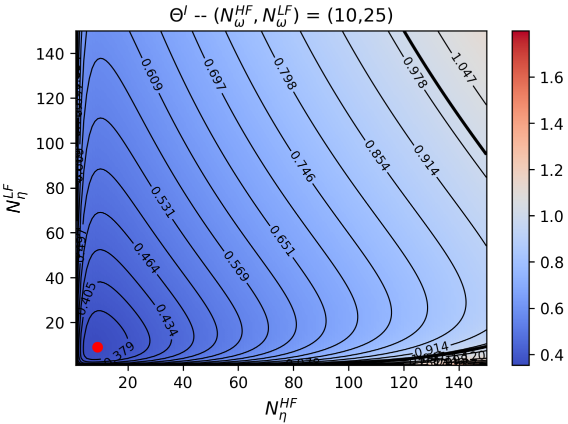
$r \approx 54.32$


Scenario 2
Θ -- (N_ω^{HF}, N_ω^{LF}) = (10,25)
log Θ
−0.8
−1.0
−1.2
−1.4
−1.6
−1.8
−2.0
N_η^{HF}
0
25
50
75
100
125
150
N_η^{LF}
0
25
50
75
100
125
150
Θ^I -- (N_ω^{HF}, N_ω^{LF}) = (10,25)
log Θ^I
0.5
0.0
−0.5
−1.0
N_η^{HF}
0
25
50
75
100
125
150
N_η^{LF}
0
25
50
75
100
125
150

Numerical results

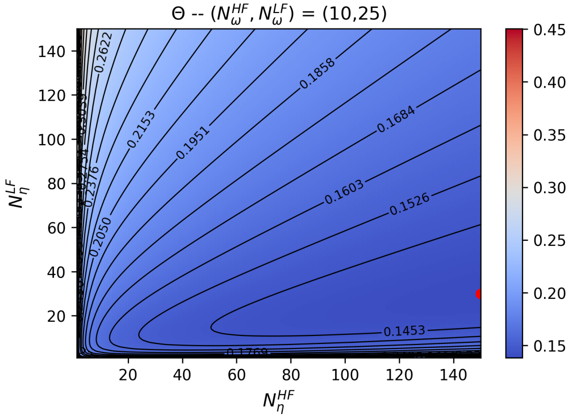
Scenario 2: Stochastic media assigned – $N_{\omega}^{\mathrm{HF}} = 10$ and $N_{\omega}^{\mathrm{LF}} = 25$ – Cost ratio (1/2)

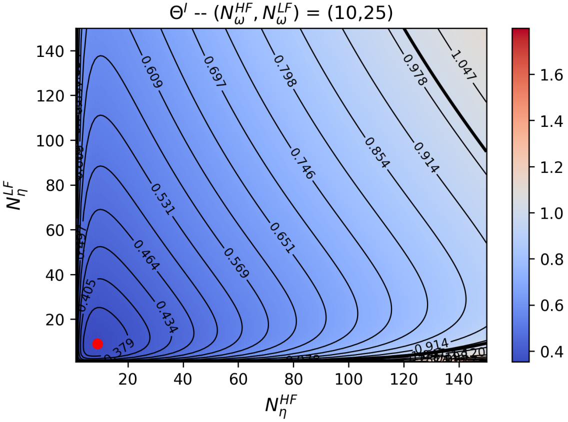




$$\Theta^I = \underbrace{\frac{\mathbb{V}ar_\xi\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}\right] + \mathbb{E}_\xi\left[\frac{\mathbb{V}ar_\omega\left[Q^{HF}(\xi,\omega)\right]}{N_\omega^{\mathrm{HF}}}\right] + \mathbb{E}_\xi\left[\frac{\mathbb{E}_\omega\left[\sigma_\eta^2(\xi,\omega)\right]}{N_\omega^{\mathrm{HF}} N_\eta^{\mathrm{HF}}}\right]}{\mathbb{V}ar_\xi\left[\mathbb{P}_{\mathbb{E}}^{\mathrm{HF}}\right] + \mathbb{E}_\xi\left[\mathbb{V}ar_\omega\left[Q^{HF}(\xi,\omega)\right]\right] + \mathbb{E}_\xi\left[\mathbb{E}_\omega\left[\sigma_\eta^2(\xi,\omega)\right]\right]} \left(1 - \frac{\mathcal{C}_\omega^{\mathrm{HF}} + \mathcal{C}_\eta^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\mathrm{HF}}}{\mathcal{C}_{\mathrm{HF}}}\right)}_{\text{Increases with } N_\eta^{\mathrm{HF}}} \Theta$$

Scenario 2: Stochastic media assigned – $N_{\omega}^{\text{HF}} = 10$ and $N_{\omega}^{\text{LF}} = 25$ – Cost ratio (1/2)

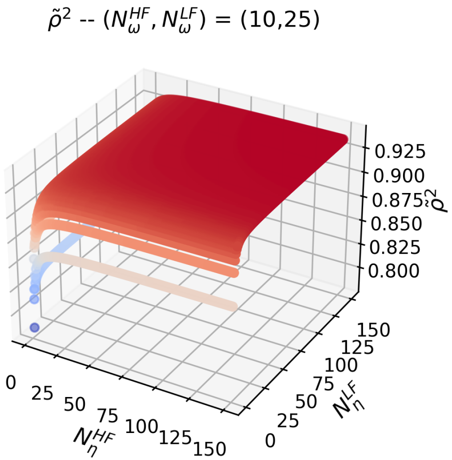


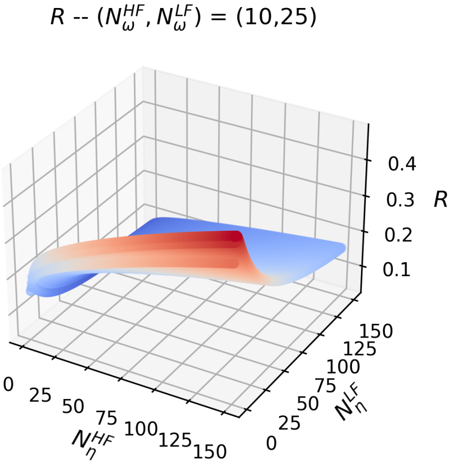


$$\Theta^I = \underbrace{\frac{\mathbb{V}ar_{\xi}\left[\mathbb{P}_{\mathbb{E}}^{\text{HF}}\right] + \mathbb{E}_{\xi}\left[\frac{\mathbb{V}ar_{\omega}\left[Q^{HF}(\xi,\omega)\right]}{N_{\omega}^{\text{HF}}}\right] + \mathbb{E}_{\xi}\left[\frac{\mathbb{E}_{\omega}\left[\sigma_{\eta}^{2}(\xi,\omega)\right]}{N_{\omega}^{\text{HF}}N_{\eta}^{\text{HF}}}\right]}{\mathbb{V}ar_{\xi}\left[\mathbb{P}_{\mathbb{E}}^{\text{HF}}\right] + \mathbb{E}_{\xi}\left[\mathbb{V}ar_{\omega}\left[Q^{HF}(\xi,\omega)\right]\right] + \mathbb{E}_{\xi}\left[\mathbb{E}_{\omega}\left[\sigma_{\eta}^{2}(\xi,\omega)\right]\right]}\left(1 - \frac{\mathcal{C}_{\omega}^{\text{HF}} + \mathcal{C}_{\eta}^{\text{HF}}}{\mathcal{C}_{\text{HF}}} + \frac{\mathcal{C}_{\omega,\eta}^{\text{HF}}}{\mathcal{C}_{\text{HF}}}\right)}_{\text{Increases with } N_{\eta}^{\text{HF}}} \Theta$$

Numerical results

Scenario 2: Stochastic media assigned – $N_\omega^{\text{HF}} = 10$ and $N_\omega^{\text{LF}} = 25$ – Correlation/Allocation





$r \approx 54.32$