SAND2022-14513C

Sandia National Laboratories

Exceptional service in the national interest

Harmonic Balance Method for Large-Scale Models with Krylov Subspace Recycling

Robert J. Kuether and Andrew Steyer

Sandia National Laboratories

International Modal Analysis Conference (IMAC) XLI

Austin, Texas, USA

February 13-16, 2023

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology & Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525.

Motivation

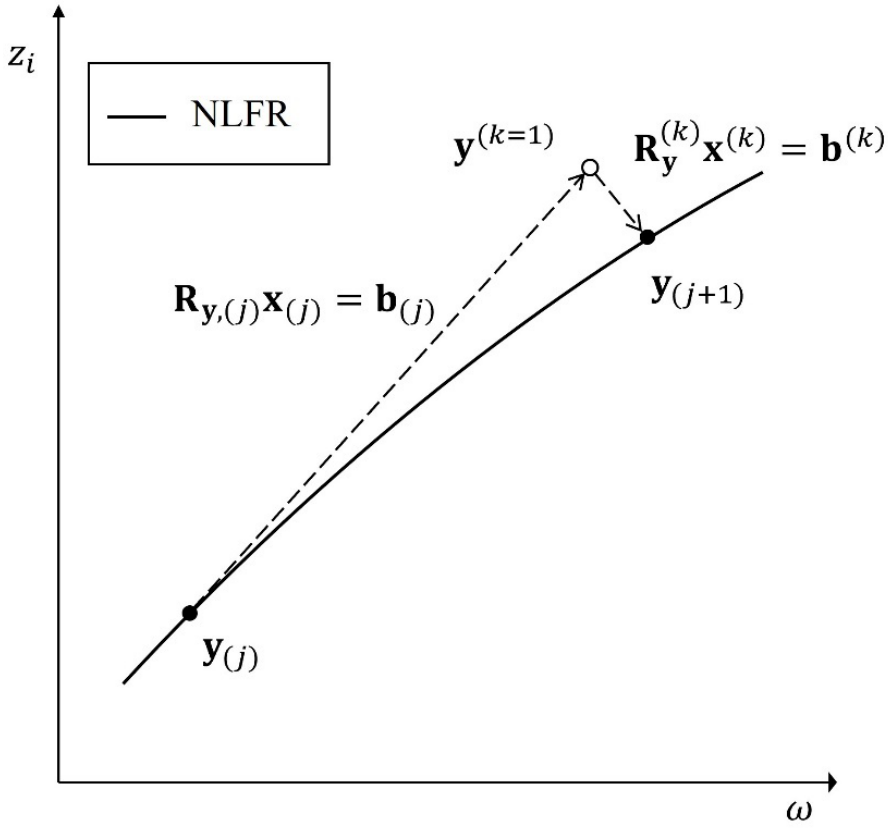
Harmonic balance with numerical continuation is a widely utilized technique to approximate periodic solutions of nonlinear dynamical systems

- Nonlinear normal modes
- **Nonlinear forced response curves**
- Limit cycle oscillations
- Etc..

Costly computations associated with linear solves for each prediction and correction step

- Large system of equations scaled by $2 \times n \times N_h$ where n is the number of DOF and N_h is the number of harmonics in the Fourier approximation

Objective: Utilize Krylov subspace iterative solvers to efficiently solve the large linear system along the predictor and corrector steps



Review of Multi-Harmonic Balance

Starting with the harmonically excited system of equations

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{nl}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}_{pre} + \mathbf{f}_{ext}(t)$$

Assume a truncated Fourier series for the periodic response

$$\mathbf{x}(t) = \frac{\mathbf{c}_0^x}{\sqrt{2}} + \sum_{k=1}^{N_h} [\mathbf{s}_k^x \sin(k\omega t) + \mathbf{c}_k^x \cos(k\omega t)] \qquad \mathbf{f}_{nl}(\mathbf{x}, \dot{\mathbf{x}}) = \frac{\mathbf{c}_0^{nl}}{\sqrt{2}} + \sum_{k=1}^{N_h} \left[\mathbf{s}_k^{nl} \sin(k\omega t) + \mathbf{c}_k^{nl} \cos(k\omega t)\right]$$

$$\mathbf{f}_{ext}(t) = \frac{\mathbf{c}_0^f}{\sqrt{2}} + \sum_{k=1}^{N_h} \left[\mathbf{s}_k^f \sin(k\omega t) + \mathbf{c}_k^f \cos(k\omega t)\right]$$

After substitution and Galerkin projection onto orthogonal periodic functions

$$\mathbf{r}(\mathbf{z}, \omega) = \mathbf{A}(\omega)\mathbf{z} + \mathbf{b}(\mathbf{z}) - \mathbf{b}_{pre} - \mathbf{b}_{ext} = \mathbf{0}$$

Unknowns: vector $\mathbf{z}$ (collection of Fourier coefficients) and scalar ω (fundamental frequency)

See references [1-3] for details and complete derivation of MHB for mechanical systems

[1] T. Detroux, L. Renson, L. Masset, and G. Kerschen, "The harmonic balance method for bifurcation analysis of large-scale nonlinear mechanical systems," *Computer Methods in Applied Mechanics and Engineering,* vol. 296, pp. 18-38, 2015.

[2] Y. Colaïtis and A. Batailly, "The harmonic balance method with arc-length continuation in blade-tip/casing contact problems," *Journal of Sound and Vibration,* vol. 502, 2021.

[3] M. Krack and J. Gross, *Harmonic Balance for Nonlinear Vibration Problems*, 1st ed. Springer International Publishing, 2019.

Predictor-Corrector Method

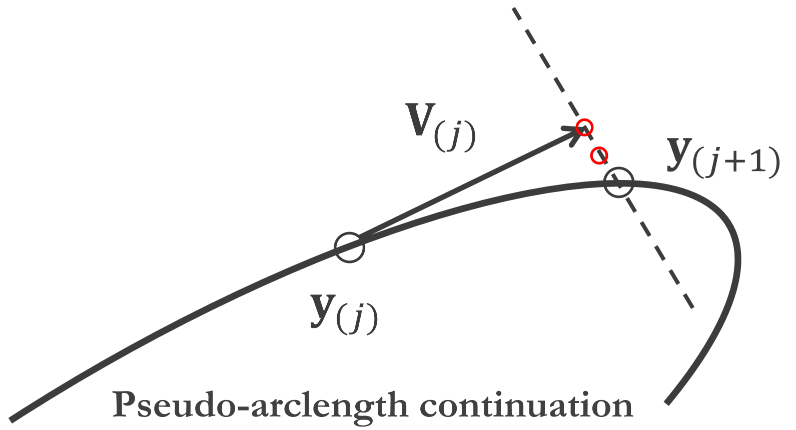
Pseudo-arclength continuation used to trace periodic solution of MHB equations

$$\mathbf{R}(\mathbf{y}) = \begin{bmatrix} \mathbf{A}(\omega)\mathbf{z} + \mathbf{b}(\mathbf{z}) - \mathbf{b}_{pre} - -\mathbf{b}_{ext} \\ \mathbf{V}^{\mathrm{T}}\left(\mathbf{y} - \mathbf{y}^{(k=1)}\right) \end{bmatrix}$$

← MHB Residual

← Tangent hyperplane constraint

$$\mathbf{y} = [\mathbf{z}^{\mathrm{T}} \quad \omega]^{\mathrm{T}}$$



Truncating the Taylor series expansion of above equations results in a system of equations to solve for corrections and predictions

Same form!
Different RHS

$$\begin{bmatrix} \mathbf{r}_{\mathbf{z}}\left(\mathbf{y}^{(k)}\right) & \mathbf{r}_{\omega}\left(\mathbf{y}^{(k)}\right) \\ \mathbf{V}_{\mathrm{z,(j)}}^{\mathrm{T}} & V_{\omega,\mathrm{(j)}} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{z}^{(k)} \\ \Delta\omega^{(k)} \end{bmatrix} = -\mathbf{R}\left(\mathbf{y}^{(k)}\right)$$

Corrector

$$\begin{bmatrix} \mathbf{r}_{\mathbf{z}}\left(\mathbf{y}_{(j)}\right) & \mathbf{r}_{\omega}\left(\mathbf{y}_{(j)}\right) \\ \mathbf{V}_{\mathbf{z},(j-1)}^{\mathrm{T}} & V_{\omega,(j-1)} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{\mathbf{z},(j)} \\ V_{\omega,(j)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

Predictor

Requires solution to large-scale, sparse linear system $\mathbf{Ax} = \mathbf{b}$ that scales as $2 \times n \times N_h$

- Matrix $\mathbf{A}$ has the same form for both correction and prediction solves

Krylov Subspace Iterative Solvers [4]

A Krylov subspace iterative method seeks to solve a high-dimensional linear algebraic system of equations: $\mathbf{Ax} = \mathbf{b}$

The Krylov sequence generated based on the initial residual provides linearly independent vectors in which a solution to the linear system is sought

$$K_{\boldsymbol{r}}(\mathbf{A}, \mathbf{r}_0) = \mathrm{span}(\mathbf{r}_0, \mathbf{A}\mathbf{r}_0, \mathbf{A}^2\mathbf{r}_0, \dots, \mathbf{A}^{n-1}\mathbf{r}_0)$$

$$\mathbf{r}_0 = \mathbf{A}\mathbf{x}_0 - \mathbf{b}$$

An approximate solution is obtained by minimizing the residual over the subspace and iteratively repeated until the solution meets a desired numerical tolerance

Many popular variants available to solve linear systems of equations

- Conjugate Gradient (CG)
- Generalized Minimal Residual Method (GMRES)
- Biconjugate Gradient Stabilized Method (BiCGSTAB)

[4] H. A. Van der Vorst, *Iterative Krylov methods for large linear systems* (no. 13). Cambridge University Press, 2003.

Subspace Recycling with GCRO-DR

GCRO method with deflated restarting (GCRO-DR) [5] used to successively solve sequence of linear systems that arise from the prediction and correction steps

$$\mathbf{A}^i \mathbf{x}^i = \mathbf{b}^i \qquad i = 1,2, \dots,$$

Utilize a selected subspace to retain between linear systems to improve convergence, assuming the linear system changes slowly between iterations, i, $i+1$, etc..

- No assumptions about the vector $\mathbf{b}^i$

GCRO-DR($\tilde{m}$,$\tilde{k}$) utilizes $\tilde{k}$ harmonic Ritz vectors corresponding with smallest magnitude values to initialize the subspace for the iterative solver

- $\tilde{m}$ is the maximum size of the subspace within the solver
- Details and pseudo-code provided in [1]

[5] M. L. Parks, E. De Sturler, G. Mackey, D. D. Johnson, and S. Maiti, "Recycling Krylov subspaces for sequences of linear systems," *SIAM Journal on Scientific Computing,* vol. 28, no. 5, pp. 1651-1674, 2006.

Delayed Frequency Preconditioner

Reuse a preconditioner computed at a point along the solution branch – delayed frequency preconditioner [6]

- Avoid costly refactorization at each solution

Monitor performance of preconditioner based on the number of iterations in the iterative solver

- **If** # iterations exceeds a threshold (based on % increase in iterations from last update), **Then** update the preconditioner (details in [7])

Zero fill-in Incomplete LU (iLU) factorization used as preconditioner iLU(0)

- Supported in MATLAB

[6] G. Jenovencio, A. Sivasankar, Z. Saeed, and D. Rixen, "A Delayed Frequency Preconditioner Approach for Speeding-Up Frequency Response Computations of Structural Components," in *XI International Conference on Structural Dynamics*, 2020.
[7] R.J. Kuether and A. Steyer, “Large-Scale Harmonic Balance Simulations with Krylov Subspace and Preconditioner Recycling”, (in preparation).

Numerical Example I

Coupled shaker-structure model of mock pylon subcomponent mounted to a stiff fixture

Frictionless contact between hanging strip and mounting blocks of pylon

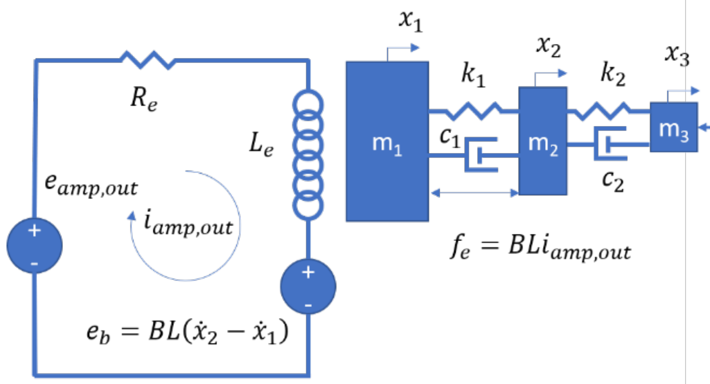
- Hardening type nonlinearity at sufficient vibration amplitude

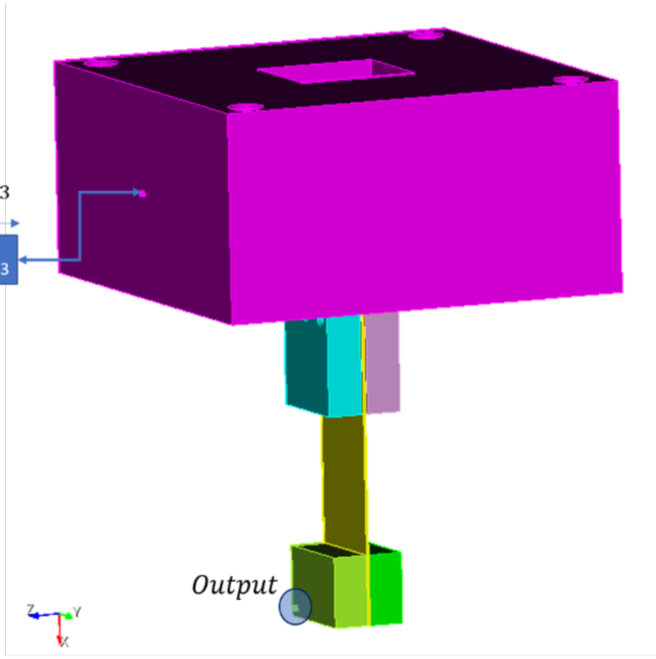
Hurty/Craig-Bampton model developed to reduce structural model degrees-of-freedom

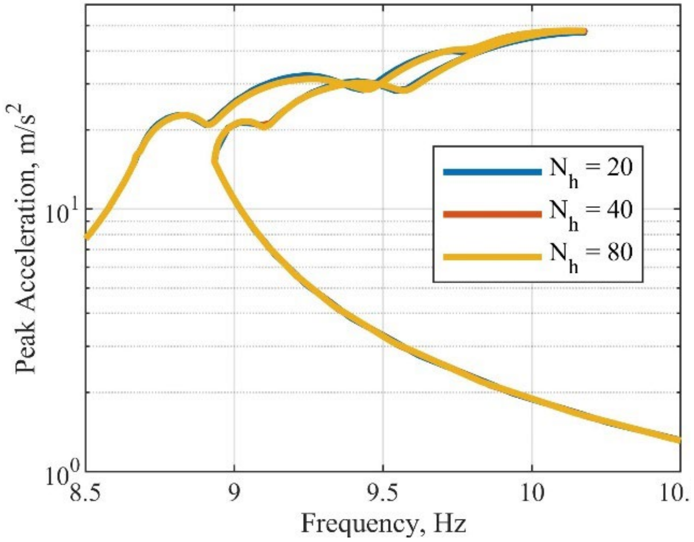
- 50 fixed-interface modes
- 189 boundary DOF (input locations + nonlinear DOF)

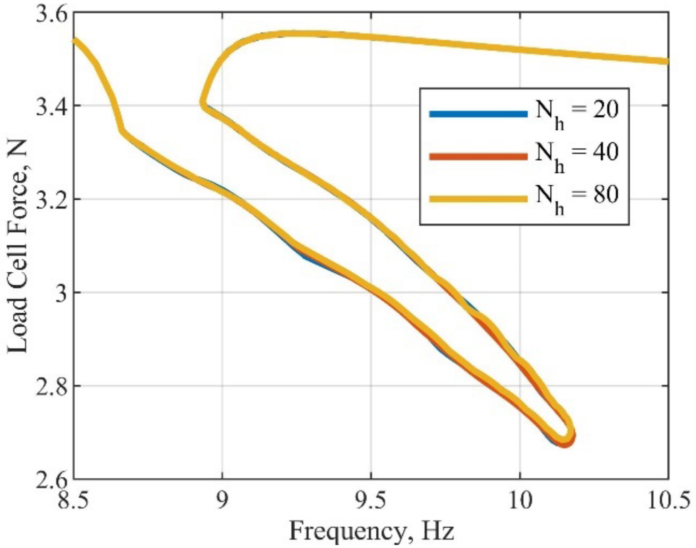
Driven with a harmonic excitation at a constant DAQ voltage and continuing along frequency

- Excitation around 1st elastic mode (pylon bending)









Numerical Example I Results

Examined solver performance for iterative solver with subspace recycling (i.e. GCRO-DR) and without recycling (i.e. GMRES)

- Total Cost = Iterative Solver + Preconditioner Evaluation

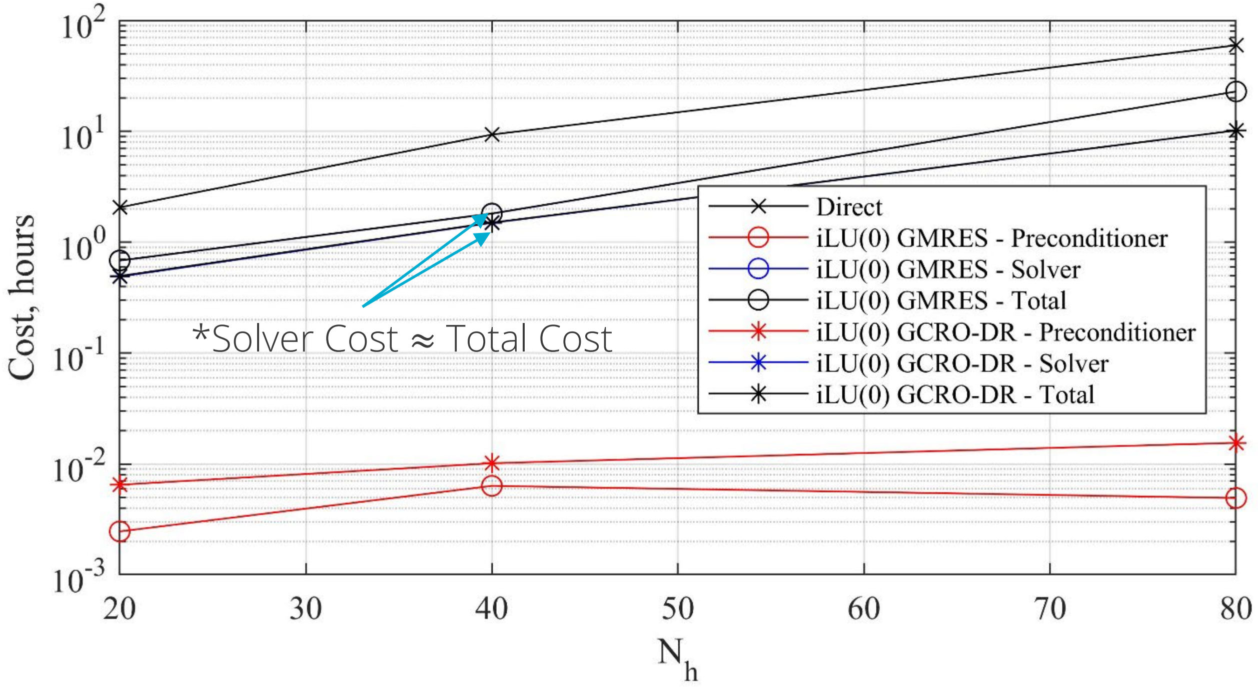
Subspace recycling method less sensitive to maximum size of subspace and number of eigenvectors to recycle between linear systems

- Lower values of $\tilde{m}$ and $\tilde{k}$ can cause the solver to stall

Favorable speedups achieved for GCRO-DR relative to direct solver cost (4.2x, 6.2x, 5.9x)

- GMRES speedups (3.0x, 5.1x, 2.6x)

All solvers achieve the same relative residual for the solution along the NLFR branch ($\varepsilon_R = 10^{-6}$)



GCRO-DR solver settings

N_h	$\tilde{m}$	$\tilde{k}$
20	150	75
40	150	75
80	200	50

Numerical Example I Results

Subspace recycling reduced the number of iterations within the iterative solver, leading to improvements in the solver cost

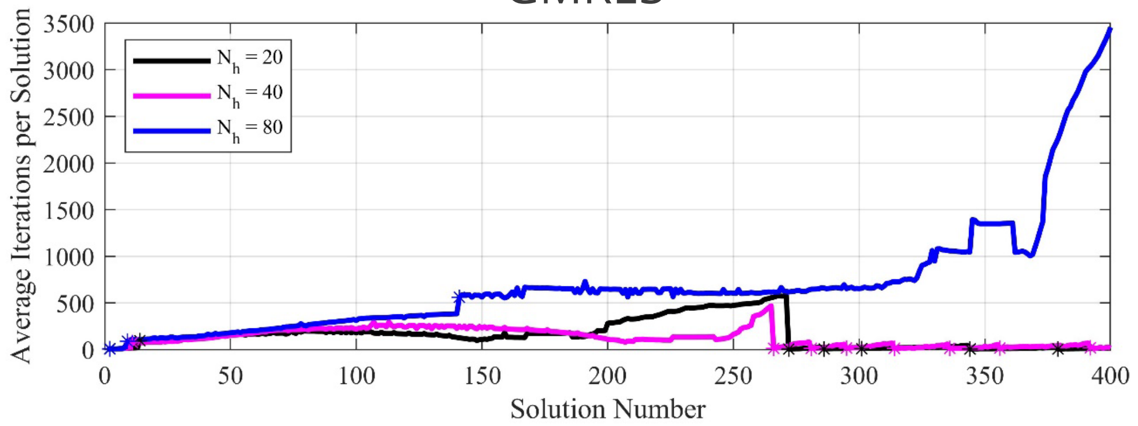
Overhead costs (not reported here) associated with generation of harmonic Ritz vectors used for subspace recycling

GCRO-DR works favorably with iLU(0) preconditioner, which is inexpensive to compute

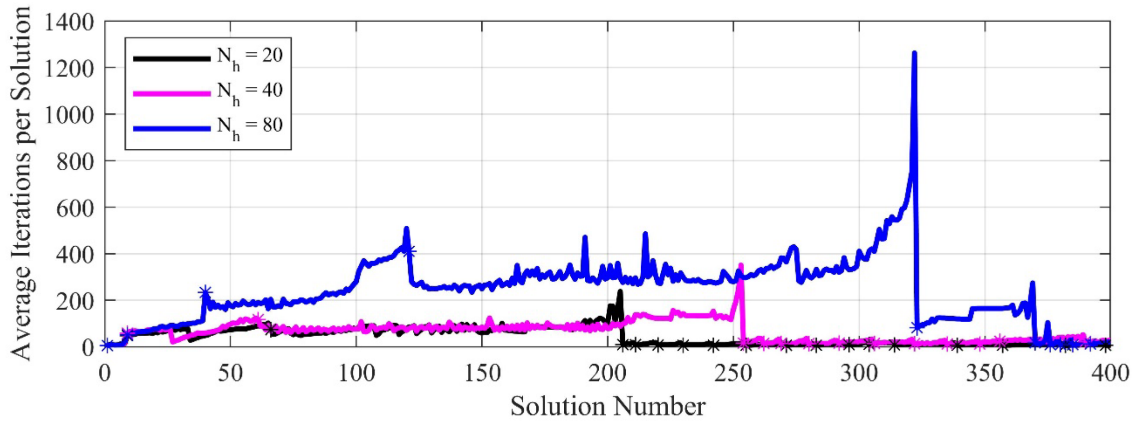
Total number of iterations

N_h	GMRES	GCRO-DR
20	126,261	47,840
40	163,278	76,313
80	750,248	362,368

GMRES



GCRO-DR



(*) represent solution at which iLU(0) factorization occurs

Numerical Example II

Bolted C-Beam assembly with node-to-node frictional contact elements at bolted interface

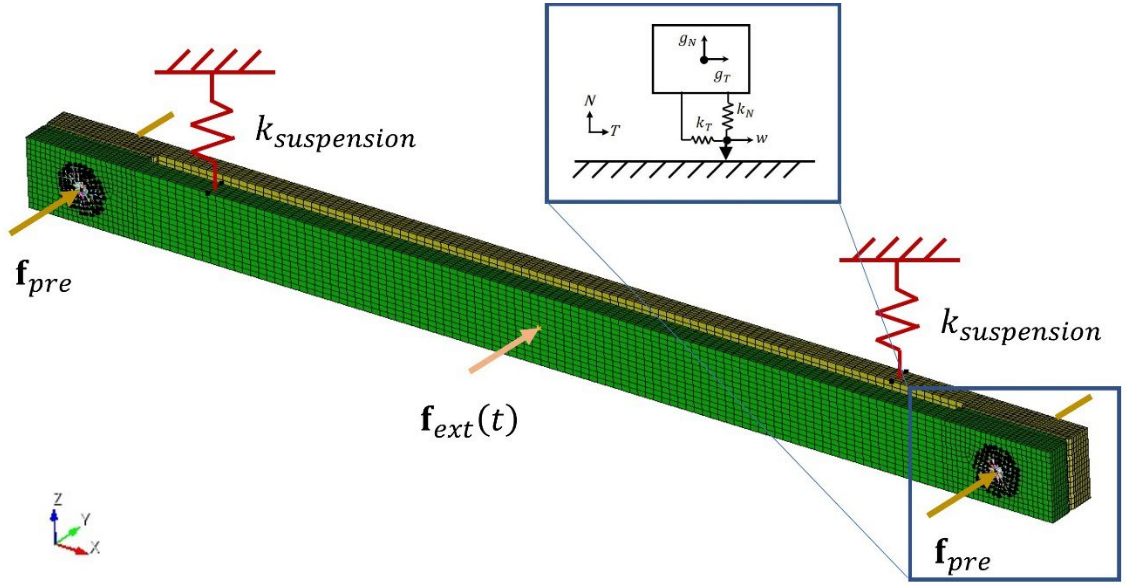
- 1,220 Jenkins elements

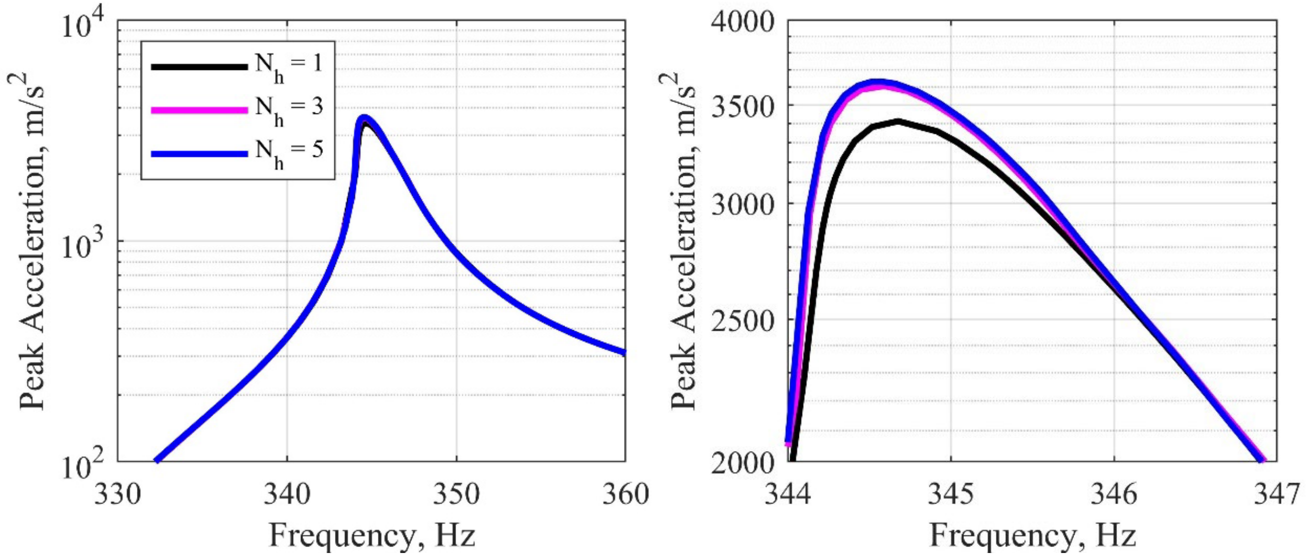
Hurty/Craig-Bampton model developed to reduce structural model degrees-of-freedom

- 25 fixed-interface modes
- 3,675 boundary DOF (input locations + nonlinear DOF)

Driven with a harmonic excitation at a constant force level (20 N) and continuing along frequency

- Excitation around 2nd elastic mode (in-phase bending)





Numerical Example II Results

Examined solver performance for iterative solver with subspace recycling (i.e. GCRO-DR) and without recycling (i.e. GMRES)

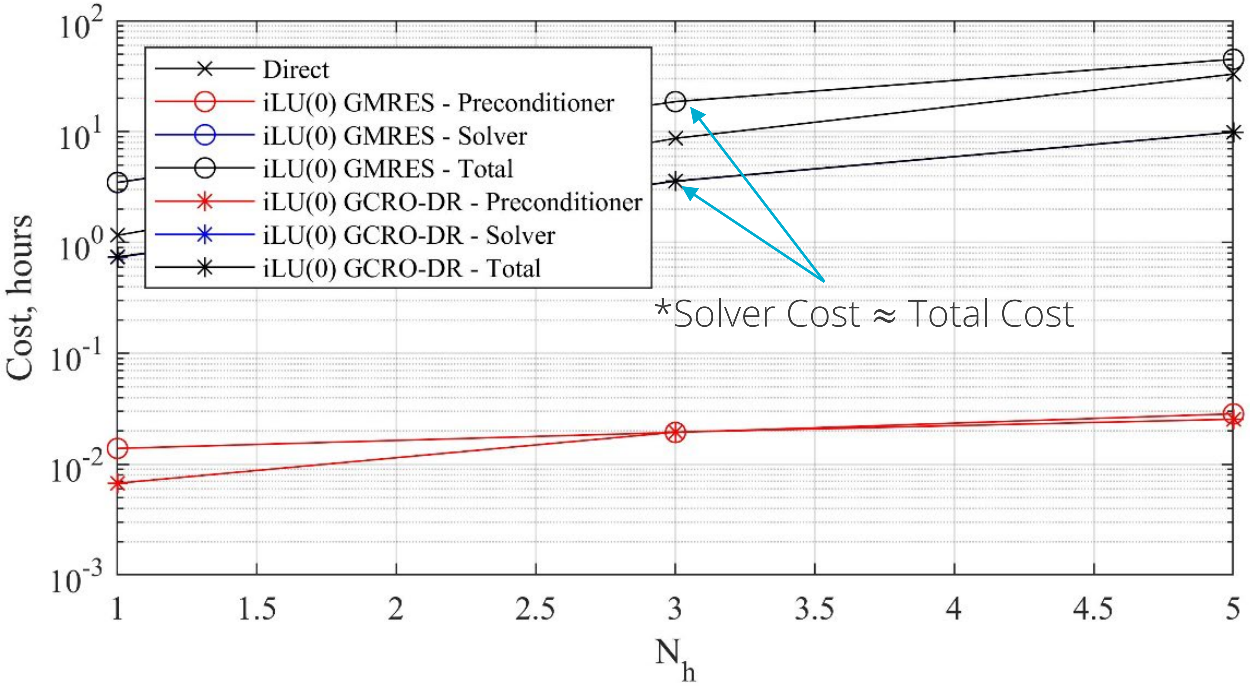
Subspace recycling method more sensitive to maximum size of subspace and number of eigenvectors to recycle between linear systems

- Needed to increase $\tilde{m}$ and $\tilde{k}$ to avoid stalling of the solver

Favorable speedups achieved for GCRO-DR relative to direct solver cost (1.6x, 2.4x, 3.3x)

- "Slowdown" observed for the GMRES solver relative to direct solver (0.33x, 0.47x, 0.73x)

All solvers achieve the same relative residual for the solution along the NLFR branch ($\varepsilon_R = 10^{-6}$)



GCRO-DR solver settings

N_h	$\tilde{m}$	$\tilde{k}$
1	100	50
3	200	100
5	400	200

Numerical Example II Results

Subspace recycling significantly reduced the number of iterations within the iterative solver, leading to improvements in the solver cost

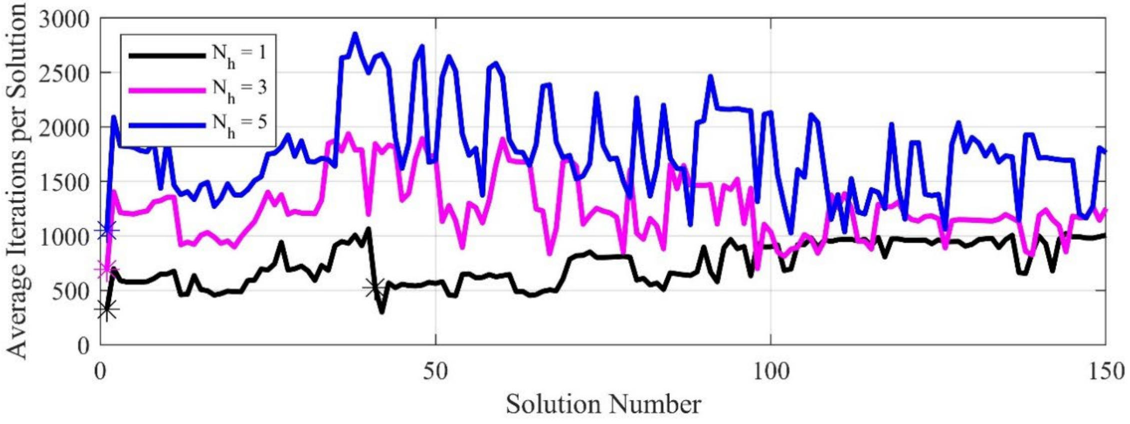
- GCRO-DR performed better than direct solver, where GMRES did not

Subspace recycling works favorably with iLU(0) preconditioner, which is relatively inexpensive to compute

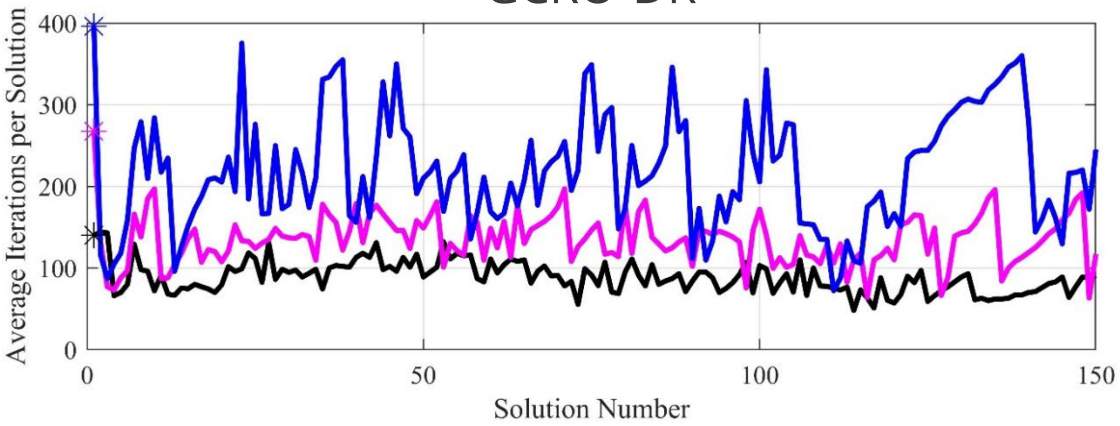
Total number of iterations

N_h	GMRES	GCRO-DR
1	151,535	19,634
3	305,963	33,360
5	464,212	59,850

GMRES



GCRO-DR



(*) represent solution at which iLU(0) factorization occurs

Conclusion

Implemented an iterative solver (GCRO-DR) that recycles Krylov subspace between solution of linear systems

Combined subspace recycling method with delayed frequency preconditioner to avoid costly refactorization of the preconditioner

Demonstrated solver on two numerical examples with frictionless and frictional contact nonlinearities

Zero fill-in iLU(0) preconditioner inexpensive to compute; Krylov subspace recycling reduce the number of solver iterations leading to faster speedups

Acknowledgement

Contact information

- Robert Kuether, rjkueth@sandia.gov
- Andrew Steyer, asteyer@sandia.gov

Supported by the Laboratory Directed Research and Development program at Sandia National Laboratories, a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia LLC, a wholly owned subsidiary of Honeywell International Inc. for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525.