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Connecting functions and nonlinear normal modes for dimension reduction in conservative structural dynamics models

Robert J. Kuether and Andrew Steyer

Sandia National Laboratories

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Motivation

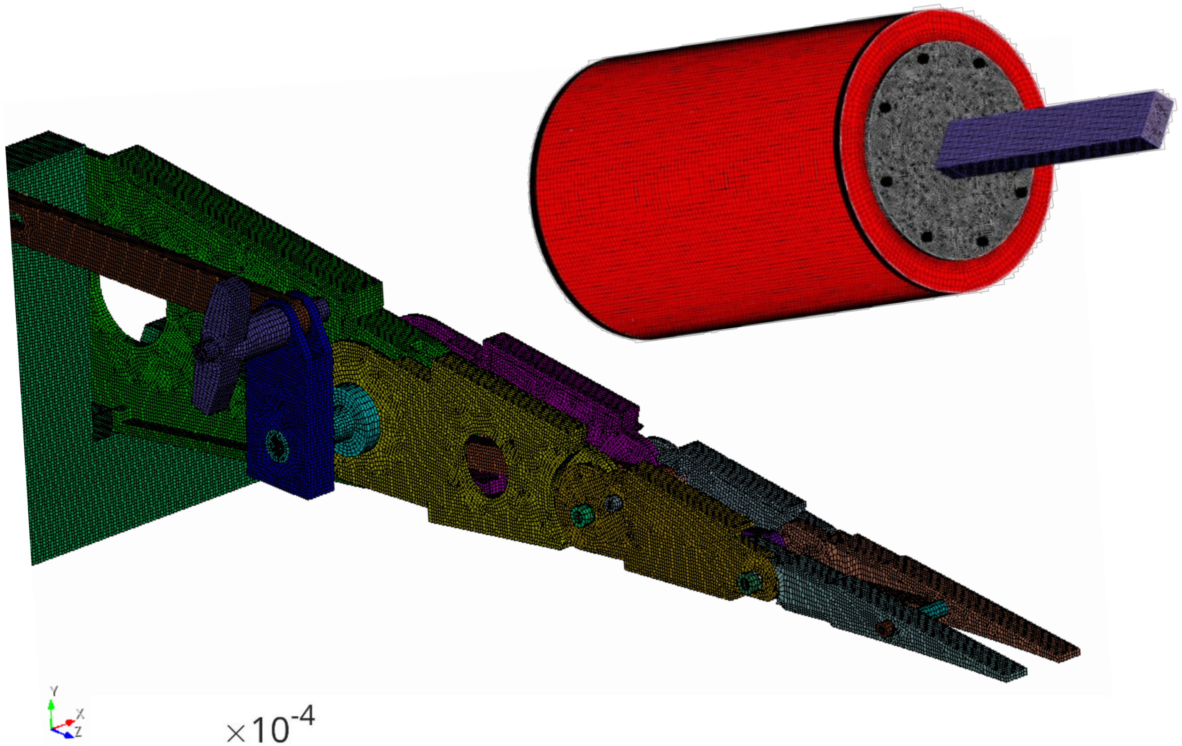
Numerical time integration problematic for solving transient response of nonlinear mechanical systems

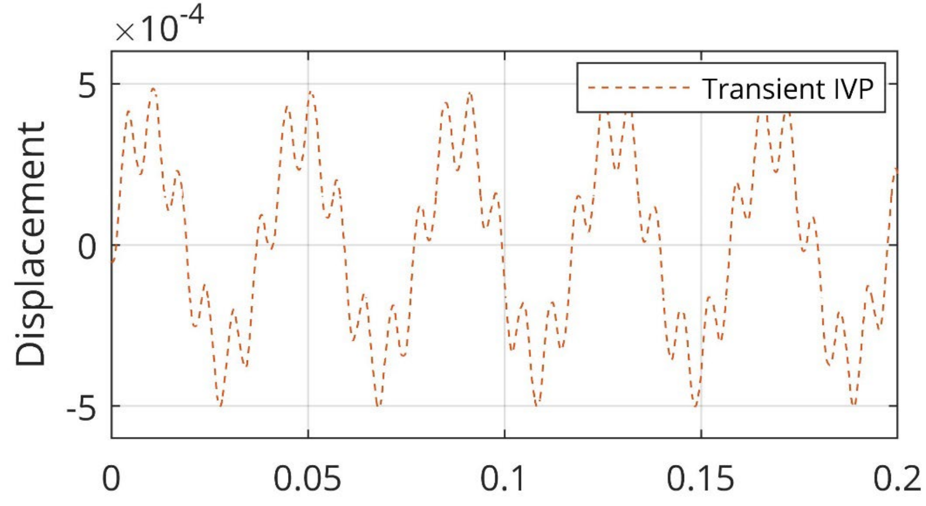
- Computational cost
- Serial solver (next solution depends on the prior)
- Error accumulation over time
- Temporal discretization and convergence

Many options available to address some of these limitations

- Parallel-in-time integration schemes
- Time-spectral methods
- Reduced order modeling
- Etc..

Objective: explore efficacy of a nonlinear superposition framework based on nonlinear normal modes to efficiently solve initial value problems for second order ODEs





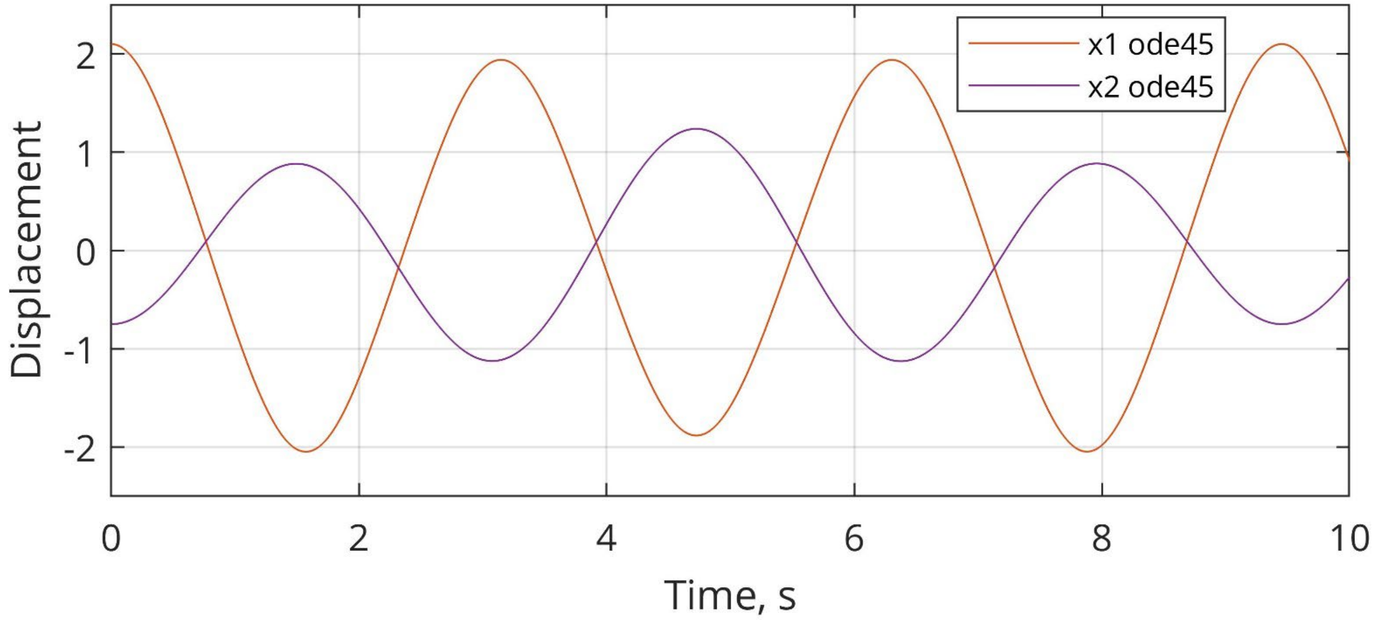
Problem Statement

Solve the initial value problem

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{nl}(\mathbf{x}) = \mathbf{0}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

$$\dot{\mathbf{x}}(t_0) = \mathbf{v}_0$$



Benchmark against the built-in $4^{th}/5^{th}$ order Runge-Kutta explicit solver in Matlab

Review of Linear Superposition

$$\mathbf{M}\ddot{\mathbf{x}} + \widehat{\mathbf{K}}\mathbf{x} = \mathbf{0} \qquad \widehat{\mathbf{K}} = \mathbf{K} + \left.\frac{\partial \mathbf{f}_{nl}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_{eq}} \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \qquad \dot{\mathbf{x}}(t_0) = \mathbf{v}_0$$

The principle of superposition for a linear(ized) second order ODE is defined as

$$\tilde{\mathbf{x}}(t) = a_1\mathbf{u}_1(t) + a_2\mathbf{u}_2(t) + \ldots + a_m\mathbf{u}_m(t)$$ Superposition "function"

$$\mathbf{u}_j(t) = \boldsymbol{\phi}_j \sin(\omega_j t - \theta_j)$$ Linearized modal solution

Components of the superposition solution

a_j, θ_j Determined by the initial conditions $\mathbf{x}_0$ and $\mathbf{v}_0$

$\boldsymbol{\phi}_j, \omega_j$ Real eigenvector and eigenvalue determined by the eigensolution with $\mathbf{M}$ and $\widehat{\mathbf{K}}$

Superposition is a functional relationship between a finite number of modal solutions

Nonlinear Superposition

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{nl}(\mathbf{x}) = \mathbf{0} \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \qquad \dot{\mathbf{x}}(t_0) = \mathbf{v}_0$$

Proposition: a nonlinear superposition principle for a nonlinear second order ODE is defined using a connecting function [1,2]

$$\tilde{\mathbf{x}}(t) = \mathbf{D} + \sum_{k=1}^{m} \mathbf{A}_k \mathbf{u}_k(t) + \mathbf{B}_k \dot{\mathbf{u}}_k(t) + \sum_{j=1,k,l=1}^{d,m} \mathbf{u}_k(t)^{\mathrm{T}} \mathbf{C}_{j,k,l} \dot{\mathbf{u}}_l(t)$$

Connecting function

$$\mathbf{u}_j(t) = \frac{\mathbf{c}_0^{\mathbf{u}_j}}{\sqrt{2}} + \sum_{n=1}^{N_h} \mathbf{s}_n^{\mathbf{u}_j} \sin(n\omega_j t) + \mathbf{c}_n^{\mathbf{u}_j} \cos(n\omega_j t)$$

Nonlinear normal mode solution

Components of the connecting function

$\mathbf{A}_k, \mathbf{B}_k, \mathbf{C}_{j,k,l}, \mathbf{D}$ — Determined by the initial conditions $\mathbf{x}_0$ and $\mathbf{v}_0$ and governing equations

$\mathbf{s}_n^{\mathbf{u}_j}, \mathbf{c}_n^{\mathbf{u}_j}, \omega_j$ — Nonlinear normal mode solutions computed from the governing equations

[1] Spijker, M., "Superposition in linear and nonlinear ordinary differential equations," J. Math. Anal. and Appl, 1970.
[2] Ardeh, H, Allen, M.S., "On connecting functions for a class of nonlinear oscillatory systems (a quest for an alternative for superposition)," 17th US National Congress on Theoretical and Applied Mechanics 2014.

Nonlinear Normal Modes

Nonlinear normal mode defined as “a (nonnecessarily synchronous) periodic motion of the conservative system” [3, 4]

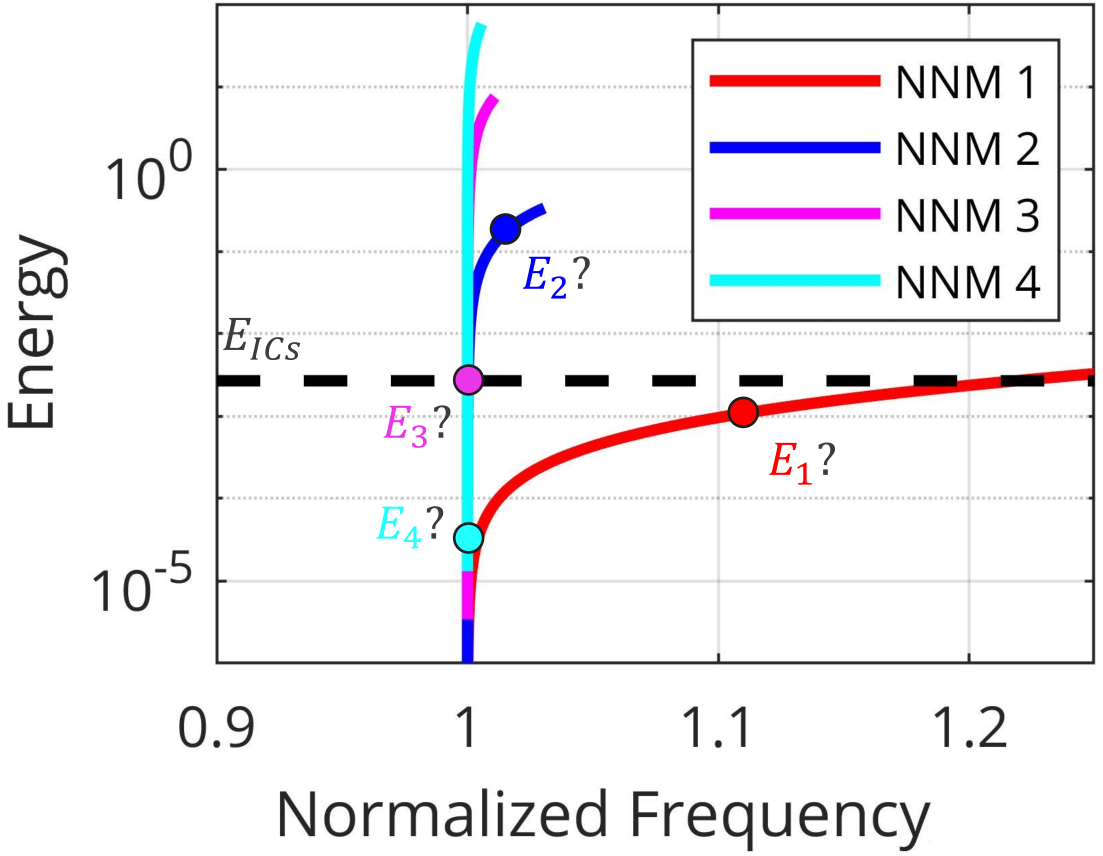
Several numerical and analytical techniques available to obtain the NNM solutions

- **Harmonic balance**, perturbation methods, shooting, etc..

Frequency-energy plots concisely represent the family of periodic solutions initiated at a linearized eigensolution at low energy

- Determine which NNM* solution to utilize within the connecting function

$$\mathbf{s}_n^{\mathbf{u}_j}(E_j), \mathbf{c}_n^{\mathbf{u}_j}(E_j), \omega_j(E_j)$$



*Assume that E_j uniquely defines a periodic NNM solution along the branch (i.e. assume no internal resonances); topic for future investigation

[3] A. F. Vakakis, "Non-linear normal modes (NNMs) and their applications in vibration theory: an overview," Mechanical Systems and Signal Processing, 1997.
[4] G. Kerschen, M. Peeters, J. C. Golinval, and A. F. Vakakis, "Nonlinear normal modes. Part I. A useful framework for the structural dynamicist," Mechanical Systems and Signal Processing, 2009.

Fitting Connecting Functions

$$\tilde{\mathbf{x}}(t) = \mathbf{D} + \sum_{k=1}^{m} \mathbf{A}_k \mathbf{u}_k(t) + \mathbf{B}_k \dot{\mathbf{u}}_k(t) + \sum_{j=1,k,l=1}^{d,m} \mathbf{u}_k(t)^{\mathrm{T}} \mathbf{C}_{j,k,l} \dot{\mathbf{u}}_l(t)$$

Following the approach by [2], the CF matrices are reduced to

$$\mathbf{A}_k = a_k \mathbf{I} \qquad \mathbf{B}_k = b_k \mathbf{I} \qquad \mathbf{C}_{j,k,l} = c_{jkl} \mathbf{I}$$

Define residual function that satisfies both the initial conditions and governing equations over a defined period between t_0 and t_p

$$\mathbf{R}(\mathbf{a}, \mathbf{b}, \mathbf{C}, \mathbf{D}, \mathbf{E}) = \begin{cases} \{\tilde{\mathbf{x}}(t=0) - \mathbf{u}_0\}/|\mathbf{u}_0| \\ \{\dot{\tilde{\mathbf{x}}}(t=0) - \mathbf{v}_0\}/|\mathbf{v}_0| \\ \mathbf{r}(t_p) \end{cases} \qquad \mathbf{r}(t_p) = \frac{\int_{t_0}^{t_p} \{\mathbf{M}\ddot{\tilde{\mathbf{x}}} + \mathbf{K}\tilde{\mathbf{x}} + \mathbf{f}_{nl}(\tilde{\mathbf{x}})\}^{\mathrm{T}} \{\mathbf{M}\ddot{\tilde{\mathbf{x}}} + \mathbf{K}\tilde{\mathbf{x}} + \mathbf{f}_{nl}(\tilde{\mathbf{x}})\}}{\int_{t_0}^{t_p} \{\mathbf{K}\tilde{\mathbf{x}} + \mathbf{f}_{nl}(\tilde{\mathbf{x}})\}^{\mathrm{T}} \{\mathbf{K}\tilde{\mathbf{x}} + \mathbf{f}_{nl}(\tilde{\mathbf{x}})\}}$$

Key Question: Can we fit a CF where m << N to achieve dimension reduction for the IVP?

[2] Ardeh, H, Allen, M.S., "On connecting functions for a class of nonlinear oscillatory systems (a quest for an alternative for superposition)," 17th US National Congress on Theoretical and Applied Mechanics 2014.

Computational Approach

Many methods available to solve the system of equations (see [2])

- Newton-Raphson
- Continuation methods
- Etc..

Three step approach based on nonlinear solver (fsolve in Matlab [5,6])

1. For the prescribed initial conditions, calculate the total kinetic and potential energy in the system. Fix NNMs solutions for each branch at $E_j = E_{ICs}$. Fit linear CF (i.e. fix $\mathbf{C}_{j,k,l} = 0$).
2. Using initial guess from Step 1, repeat linear CF solve by allowing NNM energy, E_j, to be a variable.
3. Using initial guess from Step 2, fit the nonlinear CF and still allowing NNM energy, E_j, to be a variable.

[2] Ardeh, H, Allen, M.S., On connecting functions for a class of nonlinear oscillatory systems (a quest for an alternative for superposition), 17th US National Congress on Theoretical and Applied Mechanics 2014.
[5] Levenberg, K., "A Method for the Solution of Certain Problems in Least Squares," *Quart. Appl. Math*., 1944.
[6] Marquardt, D., "An Algorithm for Least-Squares Estimation of Nonlinear Parameters," *SIAM J. Appl. Math*., 1963.

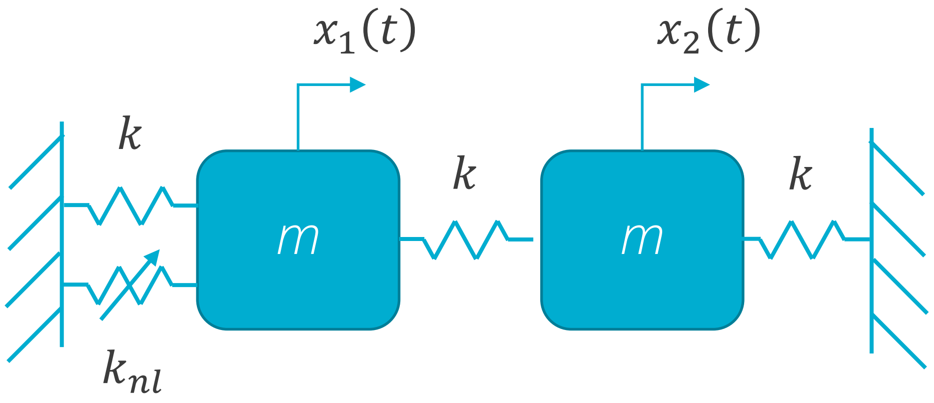
2DOF Example

Two degree-of-freedom (DOF) example with a cubic spring nonlinearity

No dimension reduction ($m = N$)

Initial conditions:

$$\mathbf{x}_0 = \begin{Bmatrix} 2.1 \\ -0.75 \end{Bmatrix} \quad \mathbf{v}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



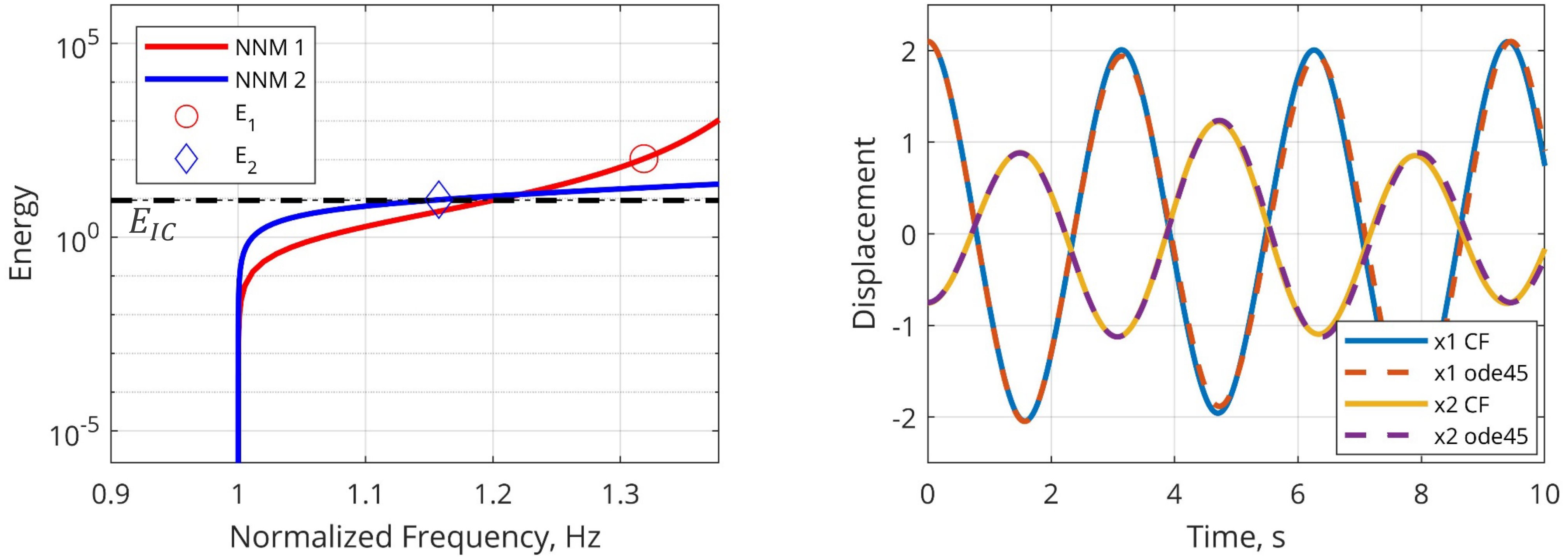
Parameter	Value
m	1
k	1
k_{nl}	0.5

2DOF Results

Identified NNM solutions do not necessarily occur at the energy of the initial conditions

Relative error provides an accuracy metric of connecting function (CF)

- $\mathbf{r}(t_p = 5) = 0.0159$ used for identification of CF coefficients
- $\mathbf{r}(t_p = 500) = 0.0201$ used for validation over long time history



Nonlinear beam example

Cantilever beam example with a cubic spring nonlinearity

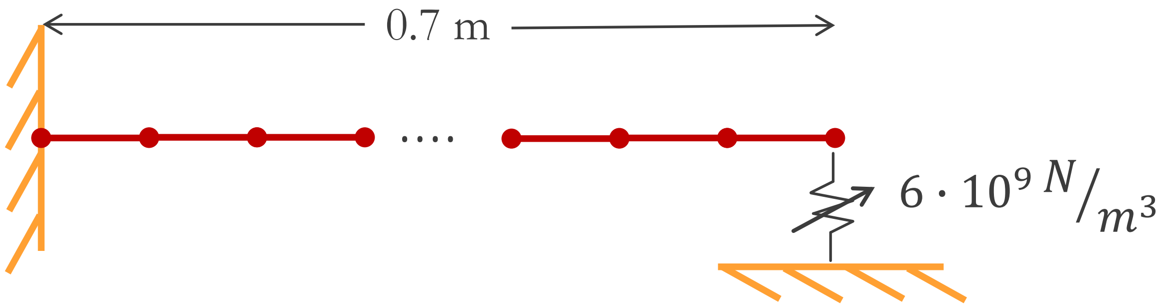
- Structural steel material properties
- 19 Euler-Bernoulli beam elements

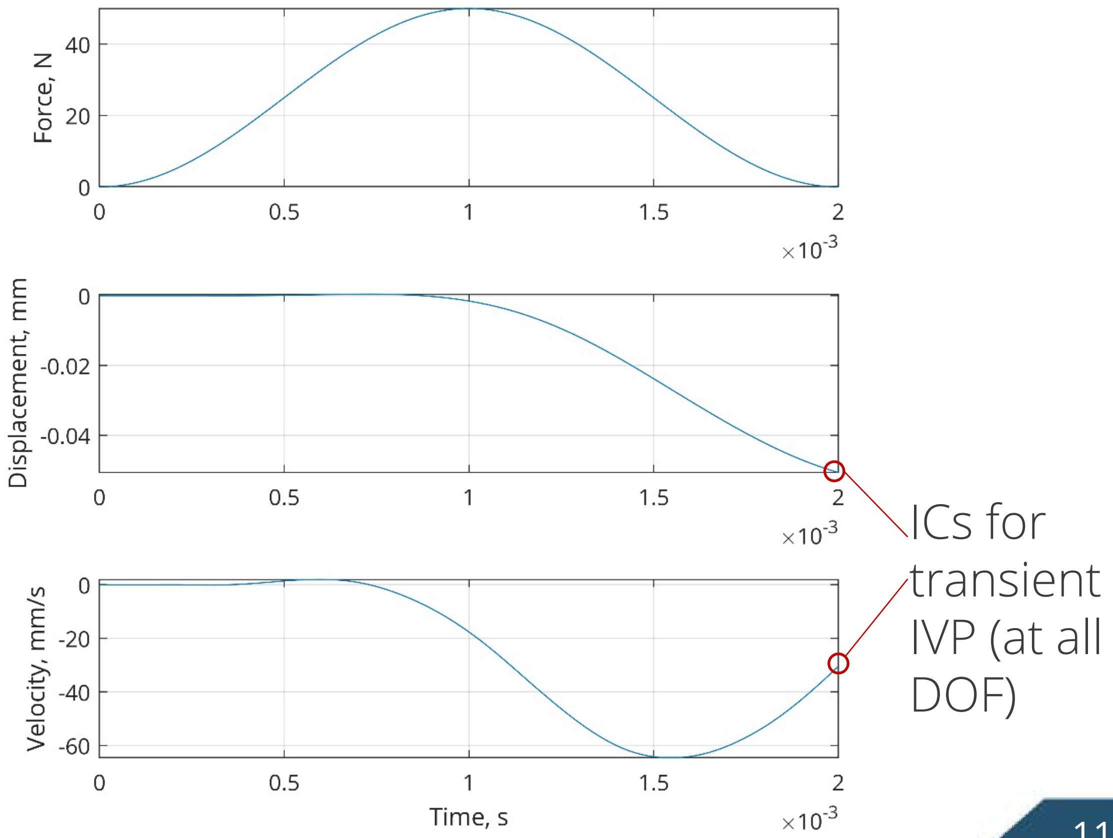
With dimension reduction ($m < N$)

- $N = 57$
- $m = 5$

Initial conditions based on the response after the application of a haversine impulse with a pulse width of 2 ms near the midpoint

- Presenting results from a 50N impulse, which excites system in the nonlinear regime





Linearized natural frequencies of beam

Mode #	1	2	3	4	5
f_n (Hz)	23.66	148.3	415.2	813.7	1345

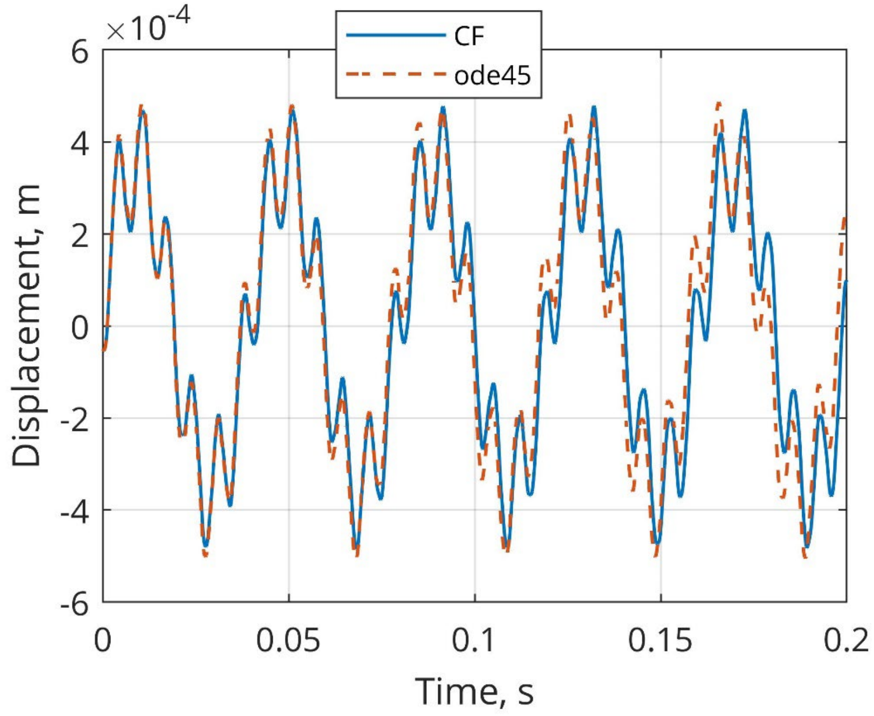
Nonlinear beam results

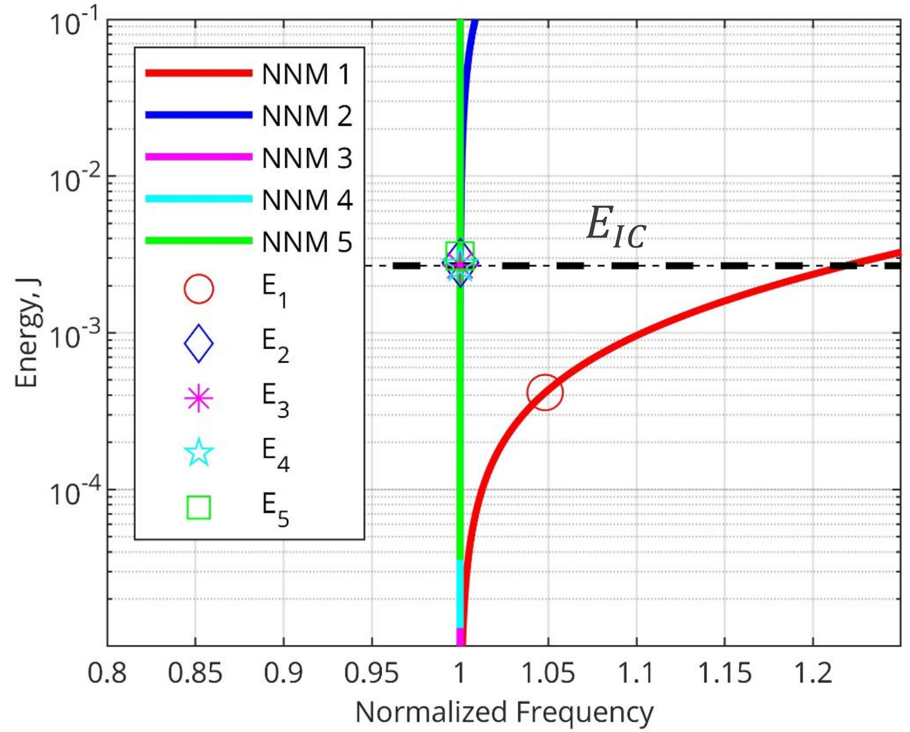
Comparison of transverse transient response (displacement and velocity) at beam tip location

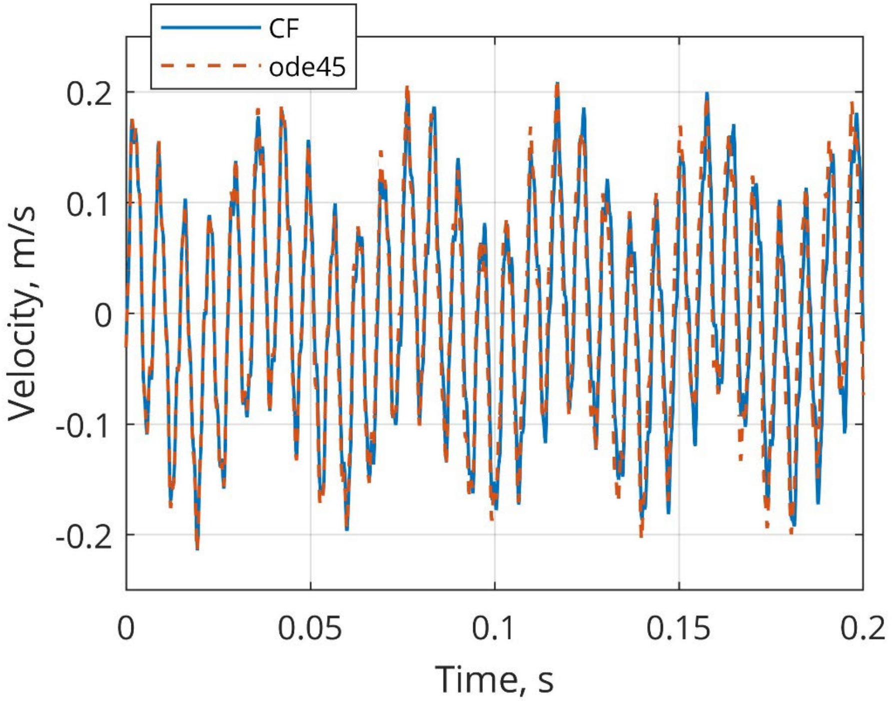
E_1 is below E_{IC}, all other NNMs near E_{IC}

Relative error of connecting function

- $\mathbf{r}(t_p = 0.1) = 0.0184$ used for identification of CF coefficients
- $\mathbf{r}(t_p = 1.0) = 0.0188$ used for validation over long time history







Nonlinear beam results

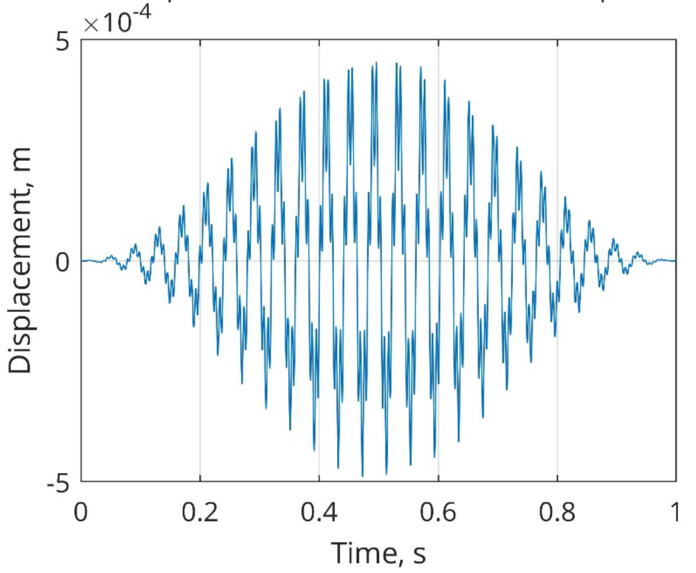
The spectrum of the transient response plotted by applying a Hanning window to the signal integrated to 1.0 sec, followed by FFT

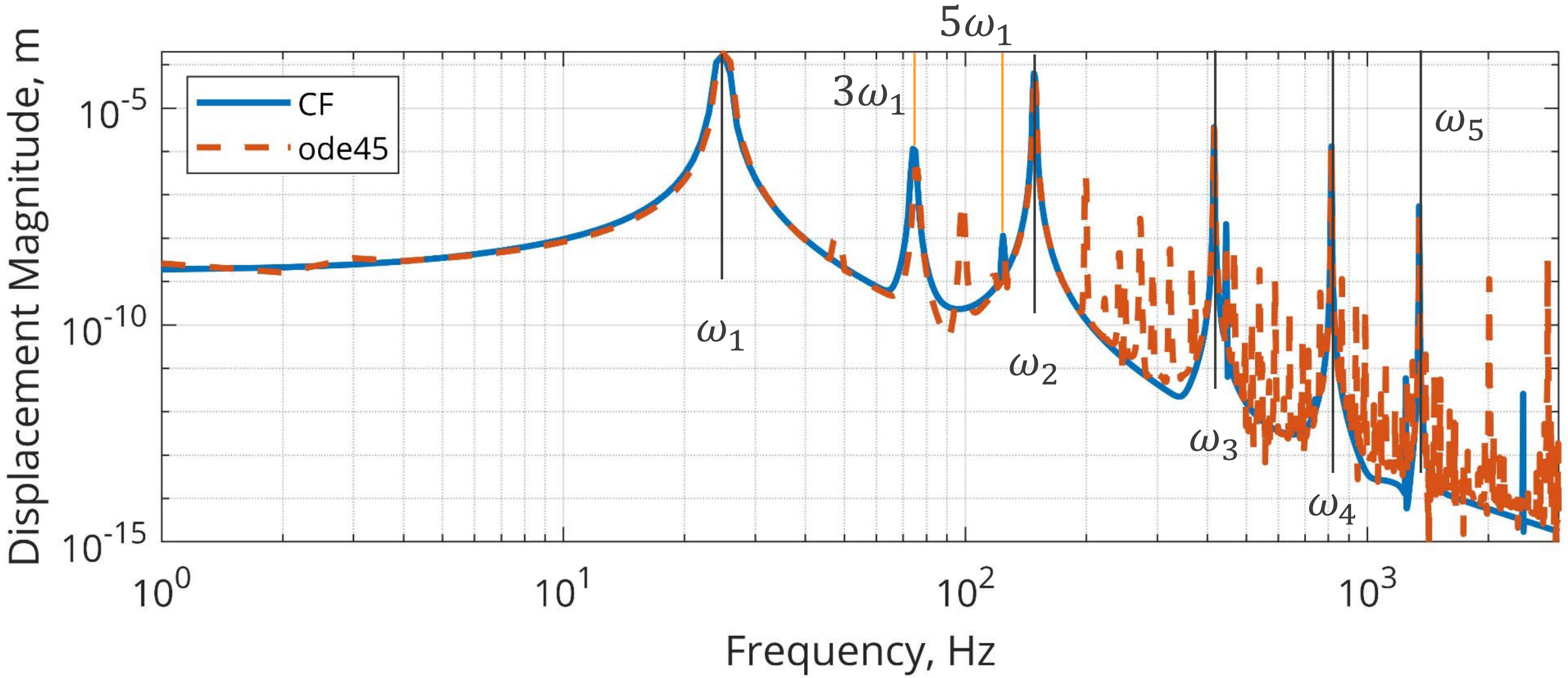
Reveals dominant frequency content at NNM fundamental frequencies and 3rd harmonic of NNM 1

Other peaks not captured with CF due to fit error of $\mathbf{C}_{j,k,l}$ terms

- $\mathbf{C}_{j,k,l} = 0$ for the resultant nonlinear CF fit, topic of future investigation

Example of windowed response





Conclusion

Developed a numerical framework to perform nonlinear superposition with nonlinear normal modes using connecting function theory

- Solution to the initial value problem

Applied the methodology to a nonlinear cantilever beam with a cubic spring nonlinearity

Explored the connecting function framework when using a subset of nonlinear normal modes for the modal solution basis

- Form of dimension reduction analogous to modal superposition

Observed good accuracy of the connecting function for transient response when compared to traditional time integrated solutions

Acknowledgement

Contact information

- Robert Kuether, rjkueth@sandia.gov
- Andrew Steyer, asteyer@sandia.gov

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