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Nonlinear evolution of the interface of the sausage instability

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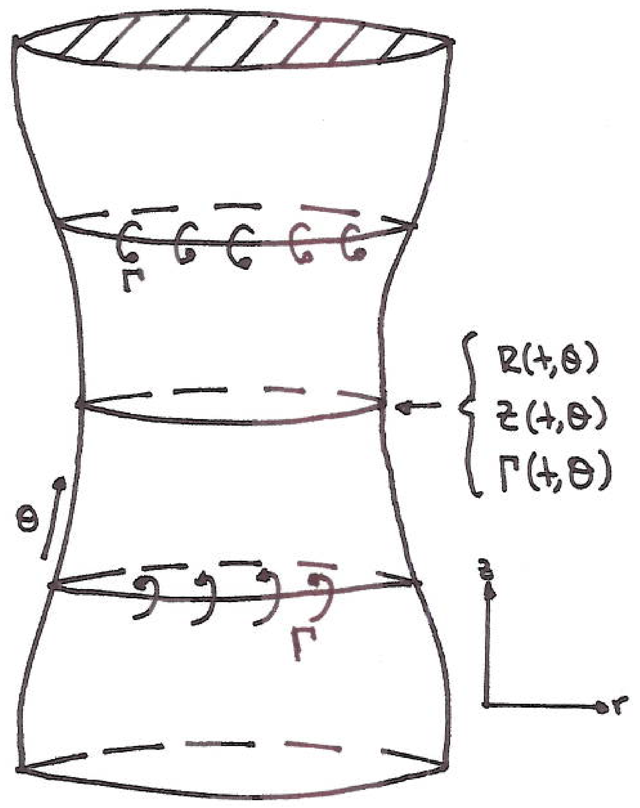
Indroduction

- The $m = 0$ sausage instability is a well-known phenomenon occurring in Bennett-type pinches (z-pinches), that is, axially uniform, axisymmetric, cylindrical plasmas. Nonlinear stages of this instability are known to play a role in dense plasma foci and exploding wire arrays [1]. Given the fundamental nature of the sausage instability, it is important to understand and quantify the nonlinear stages of its evolution [1].
- This work presents a theoretical model to study the nonlinear sausage instability. A **contour-dynamics formulation** [2,3] for the evolution of a plasma column is developed. The interface of the plasma column is described as a series of co-axial vortex rings [4]. The radius, axial location, and vortex strength of each ring is allowed to dynamically evolve, and we derive their corresponding equations of motion.
- The resulting equations are nonlinear and non-local in nature. We numerically solve the derived equations.

Basic problem considered

- We consider a Bennett-type pinch at equilibrium. The plasma column has radius R_0 and density ρ_0.
- We consider the following assumptions for the fluid motion:
  - Axisymmetric: $\mathbf{u} = u_r(t,r,z)\mathbf{e}_r + u_z(t,r,z)\mathbf{e}_z$
  - Incompressible: $\rho = \rho_0$ and $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$,
  - Irrotational: $\boldsymbol{\nabla}\times\mathbf{u} = \mathbf{0}$ (within the fluid),
  - Non-viscous: $\nu = 0$,
  - Perfectly conducting: $\mathbf{B}\cdot\mathbf{t} = 0$.
- The magnetic field is poloidal so that $\mathbf{B} \doteq B_0(R_0/r)\mathbf{e}_\phi$, where $B_0 = \mu_0 I/(2\pi R_0)$ is the unperturbed magnetic-field magnitude at the plasma surface.

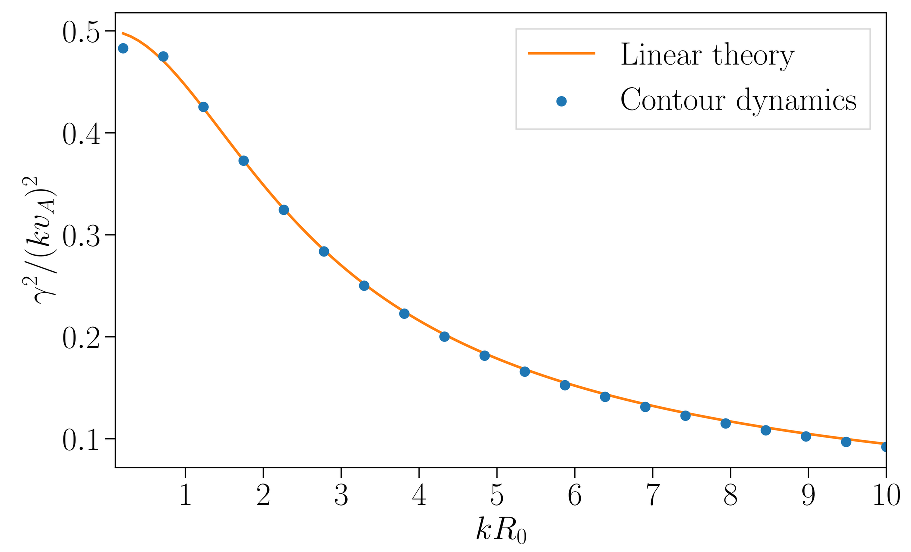


Theoretical model

- The interface of the plasma column is considered as a set of vortex rings and is parameterized as
$$(r,\phi,z) = (R(t,\theta),\phi,Z(t,\theta)),$$
where $R(t,\theta)$ and $Z(t,\theta)$ are the radius and axial location of each vortex ring, respectively. θ is an independent Lagrangian, or labeling, parameter that goes along the axis.
- To satisfy the incompresibility condition, the velocity field can be written as $\mathbf{u} = \boldsymbol{\nabla}\times\mathbf{A}$, where $\mathbf{A} = A_\phi(t,r,z)\mathbf{e}_\phi$ is the vector potential. $\mathbf{A}$ is related to the vorticity $\boldsymbol{\omega} = \omega_\phi(t,r,z)\mathbf{e}_\phi$ in the system by $\boldsymbol{\nabla}^2\mathbf{A} = -\boldsymbol{\omega}$. Since the flow is irrotational, vorticity is only located at the surface of the plasma column. Therefore, we parameterize the vortex sheet as follows:
$$\omega_\phi(t,r,z) = \int \Gamma(t,\theta)\,\delta(r-R(t,\theta))\delta(z-Z(t,\theta))\,\frac{\partial s}{\partial\theta}\,\mathrm{d}\theta,$$
where Γ is the vorticity per-unit-length for each vortex ring and $\partial s/\partial\theta \doteq [(\partial_\theta R)^2+(\partial_\theta Z)^2]^{1/2}$ is the infinitesimal arclength.
- Solving for the vector potential and the velocity field, the dynamical equations of motion for the radii and axial locations of the vortex rings are given by:
$$\frac{\partial}{\partial t}R(t,\theta) = \frac{1}{2}\Gamma\,(\mathbf{t}\cdot\mathbf{e}_r) + U_r(t,R,Z), \quad (1)$$
$$\frac{\partial}{\partial t}Z(t,\theta) = \frac{1}{2}\Gamma\,(\mathbf{t}\cdot\mathbf{e}_z) + U_z(t,R,Z), \quad (2)$$
where is the unitary tangent vector along the surface of the plasma column. The velocities U_r and U_z are
$$U_r(t,r,z) = -\mathrm{P.V.}\int \frac{\partial G}{\partial z}\,\Gamma(t,\theta)\,\frac{\partial s}{\partial\theta}\,\mathrm{d}\theta,$$
$$U_z(t,r,z) = \mathrm{P.V.}\int \frac{1}{r}\frac{\partial(rG)}{\partial r}\,\Gamma(t,\theta)\,\frac{\partial s}{\partial\theta}\,\mathrm{d}\theta,$$
where the kernel $G(r,z;r',z')$ is written as
$$G(r,z;r',z') = \frac{1}{2\pi}\sqrt{\frac{r'}{r}}\left[\left(\frac{2}{m}-m\right)K(m^2) - \frac{2}{m}E(m^2)\right].$$
Here $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively. The parameter $m = m(r,z;r',z')$ is given by
$$m = \sqrt{\frac{4rr'}{(z-z')^2+(r+r')^2}}.$$
- The governing equation for Γ is obtained from the momentum-conservation equation [5]. The resulting equation of motion is
$$\frac{\partial}{\partial t}\Gamma(t,\theta) = -2\left(\frac{\partial}{\partial t}\mathbf{U}(t,\theta)\right)\cdot\mathbf{t} + v_A^2\frac{R_0^2}{R^3}(\mathbf{t}\cdot\mathbf{e}_r), \quad (3)$$
where v_A is the Alfven velocity corresponding to the magnetic field evaluated at the unperturbed plasma surface.

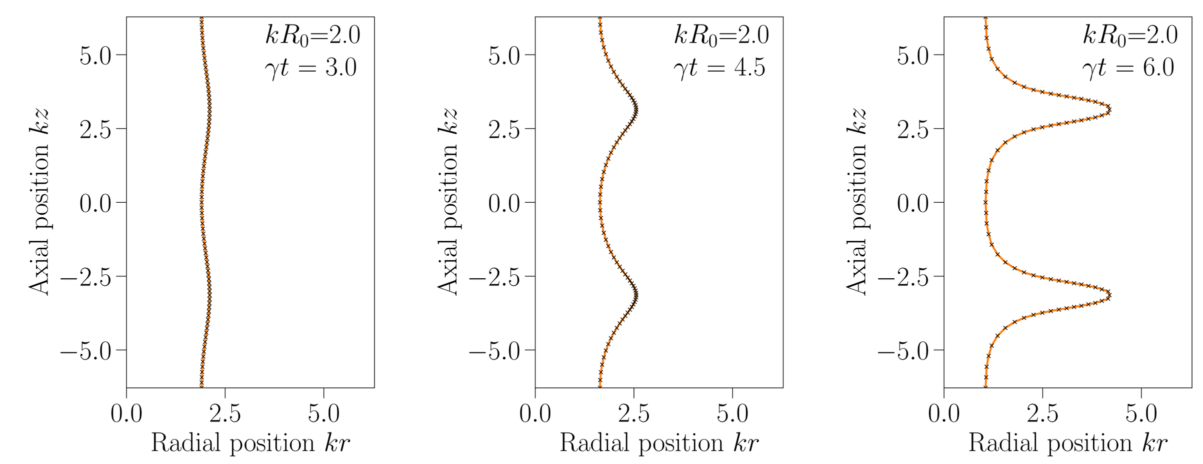
Linear growth

- Beyond the assumptions initially stated, Eqs. (1)–(3) are exact and no approximations have been made. The dimensionality of the problem has been greatly reduced using the contour-dynamics formulation. However, two main difficulties arise:
  1. The governing equations are nonlocal.
  2. Equation (3) is, in fact, a Fredholm-integral equation of the second kind since $\partial\Gamma/\partial t$ explicitly appears on the left-hand side of Eq. (3) and implicitly within the $\partial\mathbf{U}/\partial t$ term.
- To verify the numerical implementation of Eqs. (1)–(3), we calculated the linear growth rate γ and compared it to the well-known growth rate obtained using the Eulerian framework. For a sinusoidal perturbation with wavenumber k, one has
$$\gamma^2 = \frac{(kv_A)^2}{kR_0}\frac{I_1(kR_0)}{I_0(kR_0)},$$
where $I_\nu(x)$ is the modified Bessel function of the first kind.

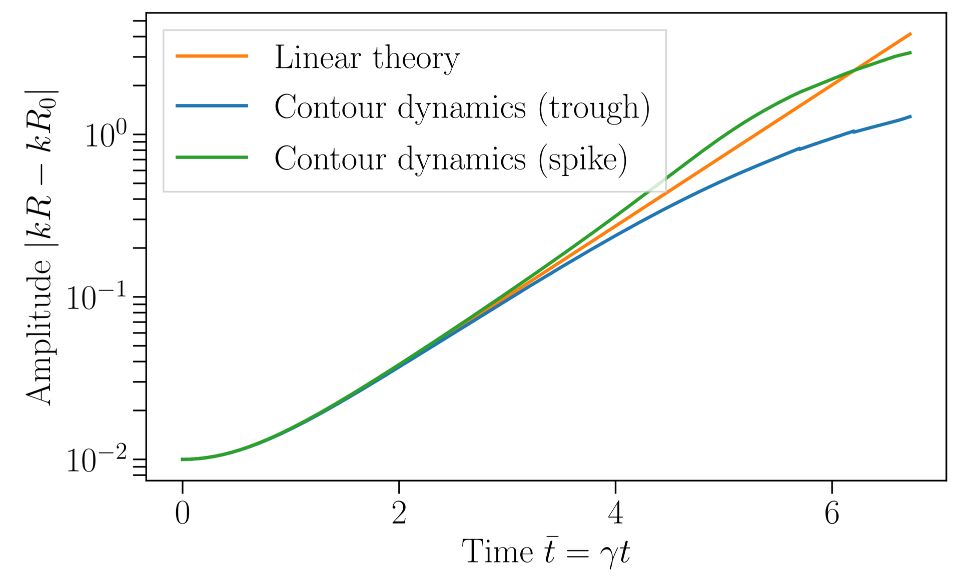


Nonlinear evolution of the interface

- Equations (1)–(3) were written in dimensionless form and solved for the intermediate case of $kR_0 = 2$.
- The images below show the nonlinear evolution of the sausage instability. In regions where the plasma radius is smaller (referred to as "troughs"), the magnetic force is greater and causes an inwards radial motion. This increases the amplitude of the troughs, and a runaway effect occurs. In the nonlinear stage, the sausage instability evolves into a "spindle"-like structure with broad troughs and sharp spikes.



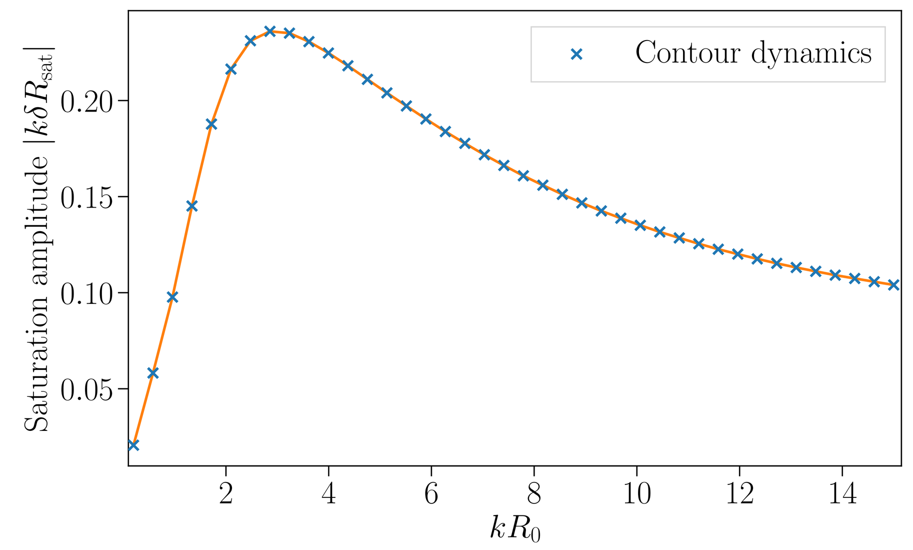
- The amplitude of the sausage-mode troughs and spikes are shown in the figure below.



- The trough amplitude is always smaller than the amplitude calculated from linear theory. In contrast, the spike amplitude initially grows faster than the linear amplitude. However, in the nonlinear stage, the spike velocity saturates, and the spike amplitude only grows linearly in time.

Saturation amplitude of the linear growth

- The strength of nonlinear effects can be measured by the difference between the linear and nonlinear solutions.
- Let the saturation amplitude be defined as the trough amplitude δR_{sat} of the linear solution calculated at the time when the nonlinear solution of Eqs. (1)–(3) differs by 10%, i.e., when
$$|\delta R_{\mathrm{lin}} - \delta R|/\delta R_{\mathrm{lin}} = 0.1.$$
- Our calculations suggest that the saturation amplitude depends quite significantly on the dimensionless quantity kR_0. Ongoing work is focused towards analytically computing [6] and physically understanding the observed trend.



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Future research

1. Additional work must be done to improve the developed algorithm. For example, the Lagrangian trajectories of the vortex rings lead to bunching along the spike structures. This was overcome by a relaxation algorithm. However, introducing an artificial velocity field tangent to the surface may offer a cleaner, more natural solution to this problem.
2. Additional physics may be incorporated to the model, e.g., adding an axial magnetic field and allowing for 3D perturbations.
3. Results from this semi-analytical study may serve as a benchmark suite for more complex, magnetohydrodynamics codes.

Contact info and acknowledgments

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