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Equivalencing of Sine-Sweep and Random Vibration Specification with Considerations of Nonlinear Statistics

Arup Maji, Org 1557

Abstract

Comparison of pure sinusoidal vibration to random vibration or combinations of the two is an important and useful subject for dynamic testing. The objective of this paper is to succinctly document the technical background for converting a sine-sweep test specification into an equivalent random vibration test specification. The information can also be used in reverse, i.e. to compare a random vibe spec with a sine-sweep, although that is less common in practice. Because of inherent assumptions involved in such conversions, it is always preferable to test to original specifications and conduct this conversion when other options are impractical.

The paper outlines the theoretical premise and relevant equations. An example of implementation with hypothetical but realistic data is provided that captures the conversion of a sinusoid to an equivalent ASD. The example also demonstrates how to account for the rate of sine-sweep to the duration of the random vibration.

A significant content of this paper is the discussion on the statistical distribution of peaks in a narrow-band random signal and the consequences of that on the damage imparted to a structure. Numerical simulations were carried out to capture the effect for various combinations of narrow-band random and pure sinusoid superimposed on each other. The consequences of this are captured to provide guidance on accuracy and conservatism.

Keywords: Vibration; Random; Sinusoid; Testing; Simulation

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Theoretical Background

Random vibe can be due to various sources of vibration typically propagated through structural connections while the sine sweep could be due to engine humming at different rpm over time. The random vibe environment is specified as ASD (Autospectral Density) as g^2 vs. Frequency (typically log scale), while the sine-sweep is specified as g vs. Frequency (typically linear scale). The overall premise for equivalencing the two is to consider how these inputs lead to resonance in the structure being tested. For simplicity, structural dynamic equations pertaining to a single degree of freedom (SDOF) structure will be used, but the results are universally applicable.

Amplification at resonance $Q=1/(2\xi)$ where ξ is the viscous damping factor. The half-power (3db) bandwidth (Δf) captures the frequency band around the resonant frequency fn within which most of the energy associated with resonance is concentrated: $\Delta f = 2\xi f_n = f_n/Q$. The time constant $t_n=T/(2\pi\xi)$ represents the duration for decay to 1/e of initial amplitude for free vibration (T is the period of vibration). It is also an indicator for the time needed for resonance to develop; in this application it is small enough for resonance to develop during the sine-sweep. The basic premise of equivalencing is to look at each frequency, consider the energy within the half power (3db) bandwidth (Δf), and evaluate the equivalent imparted damage (note: $\Delta f= f_n /Q$).

Grms of a pure sinusoid of amplitude G_s is $G_s/\sqrt{2}$, Grms of response is therefore $QG_s/\sqrt{2}$. Grms of a ASD (where G_{xx} is the ASD at frequency f_n) within the half-power (3db) bandwidth Δf is $\sqrt{(G_{xx} \Delta f)} = \sqrt{(G_{xx} f_n/Q)}$.

Grms response of a SDOF oscillator to a narrow band (Δf) random vibe is captured by the Miles' equation: $Grms=\sqrt{(\pi Q G_{xx} f_n/2)}$. Note that this is similar to amplification factor x the Grms of input, $Q \times \sqrt{(G_{xx} f_n/Q)}$, but differs by the factor $\sqrt{(\pi/2)}$. This is due to the fact that the peaks of the response to a narrow-band random vibe follows a Raleigh distribution due to the superposition of various frequencies and random phase.

The implication of this Raleigh distribution is the greater possibility of peak amplitudes that are farther from the mean. Since fatigue damage follows a power law, these larger peaks have a far more significant effect. Cap [1] did a detailed analysis with various fatigue damage parameters to determine the appropriate factor N_σ to capture this effect. N_σ is defined as the ratio between the peak of a pure sinusoid and the grms of the corresponding narrow-band random that provides the same damage (note: ratio of grms of pure sinusoid to grms of equivalent narrow-band = $N_\sigma /\sqrt{2}$).

Our specifications are based on low-cycle-fatigue coefficient of 6.67, which corresponds to a 3dB increase for time compression of factor of 10. Cap's numerical analyses [1] provided a N_σ value ≈ 2 for a fatigue coefficient of 6.67. This N_σ is based on the following equivalencing that considers both the rms response and the consequence to fatigue damage between a sine tone of peak amplitude G_s and an ASD of G_{xx}:

From [1] $G_s = N_\sigma \sqrt{(\pi Q G_{xx} f_n/2)}/Q = N_\sigma \sqrt{(\pi G_{xx} f_n/2Q)}$

Therefore $G_{xx} = 2QG_s^2/(\pi f_n N_\sigma^2) \approx QG_s^2/(2\pi f_n)$ (assumed $N_\sigma \approx 2$) [Eq.1]

Note that ignoring N_σ (assuming $=\sqrt{2}$) leads to a more conservative (higher) value of G_{xx}.

Equation 3-19 in document by Fackler (page 55) provides an equation for equivalent stress imparted $=C_r\sqrt{(\pi Q G_{xx} f_n/2)}$. This is identical in form to the equations used here. This reference [2] also provides detailed discussion on numerous technical aspects relevant to this memo.

Equivalencing of Duration of Sine-Sweep vs. Random vibe

The time duration for which sine-sweep is within each subsequent half-power bandwidth is $\Delta f/R = f_n/QR = 2\xi f_n/R$, where R is the rate of sine-sweep (example: if sine sweep is between 1-4Khz for a 30min test, R=100Hz/min). If the equivalent random vibe test has a duration of t minutes, G_{xx} needs to be multiplied by $(2\xi f_n/Rt)^{0.3}$ to account for the relative durations. The time-compression factor 0.3 in the last equation depends on the relevant low-cycle fatigue parameter (0.3 for ASDs corresponds to 0.15 for G, and is inverse of the fatigue coefficient of 6.67 used).

It is now important to recognize that the Stage5 vibe occurs simultaneously with the sine-tone and therefore the ASDs need to be added. However, the individual sine-tones occur in sequence (at different points in time that add up to 30 min).

Implementation Process with Example

The Specifications shown in Table 1 were used to illustrate the process. The moving sine-tone is to occur simultaneously with the Stage 5 random vibe ASD for a duration of 30 min. Hence the sine-sweep rate is to go from 1 to 4 kHz over 30 min (=100Hz/min). Damping factor is 0.05 (Q=10). The ASD has a g-rms = 2.45 in the 1-4kHZ regime, comparable to the sine-tone g-levels.

Step1: Determine desired octave spacing based on half-power bandwidth (Δf).

Step2: Convert the straight-line specs into octave-spaced data.

Step3: Convert each sine-tone into equivalent narrow-band ASD using [Eq1].

This is shown in Figure 1; dashed blue and green are original sine-tone and Stage5 data, solid green is octave-spaced Stage5 data and solid blue is sine-tone converted to equivalent ASD for each octave-spaced data point. It is now important to recognize that the Stage5 vibe occurs simultaneously with the sine-tone and therefore the ASDs need to be added. However, the individual sine-tones occur in sequence (at different points in time that add up to 30 min).

Step 4: Each equivalent sine-tone ASD is added to the Stage5 ASD (solid blue & green lines in Figure 1) since they occur simultaneously. Result is shown in Figure 2 (original Stage5 ASD is shown in).

Step 5: Since the different ASDs (examples shown in Figures 3a-3c) occur in sequence, they are combined using fatigue-time-compression algorithm (based on Miner's rule and the time-compression factor mentioned earlier).

Figure 4 shows the final combined specification (red) along with the original Stage5 spec and the sine-tone equivalent ASD separated from the Stage5 spec (same as solid blue line in Figure 1).

Table 1. Data (Specifications) used in Example

Original Sine-Tone Spec		Original Stage5 Spec	
Freq (Hz)	(g)	Freq (Hz)	ASD (g^2/Hz)
1000	1.5	1000	0.002
1600	2	4000	0.002
2100	1		
2500	1		
3200	1.9		
4000	2.2		

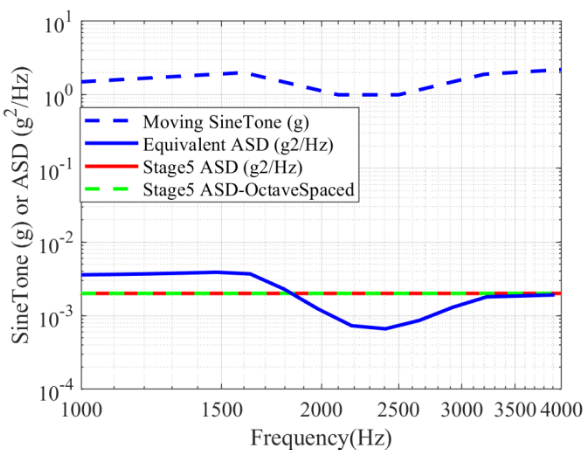

Figure 1. SineTone converted to ASD

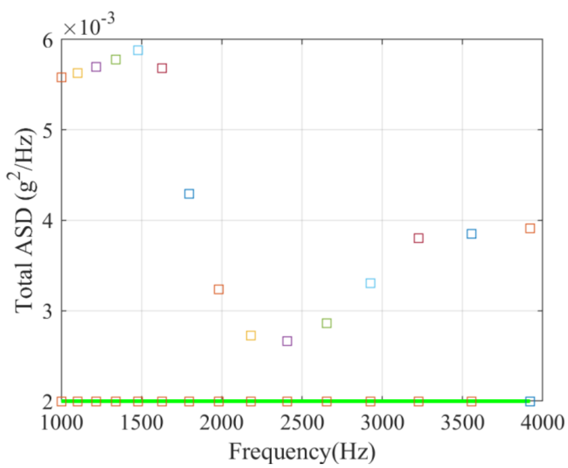

Figure 2. Total ASDs vs. Stage5 ASD

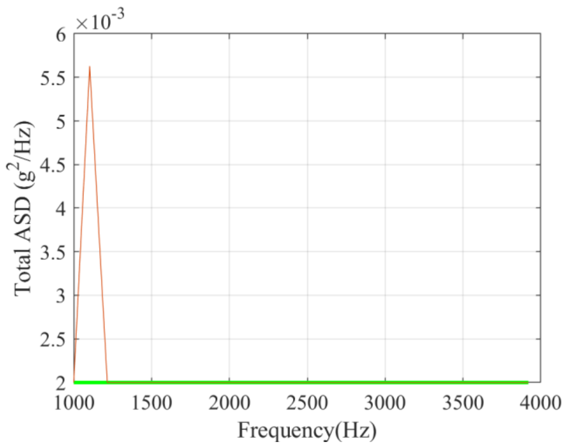

Figure 3a. Total ASD at 2nd Time Interval

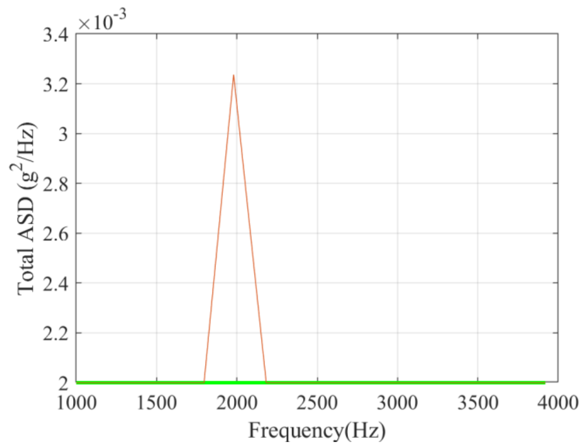

Figure 3b. Total ASD at 8th Time Interval

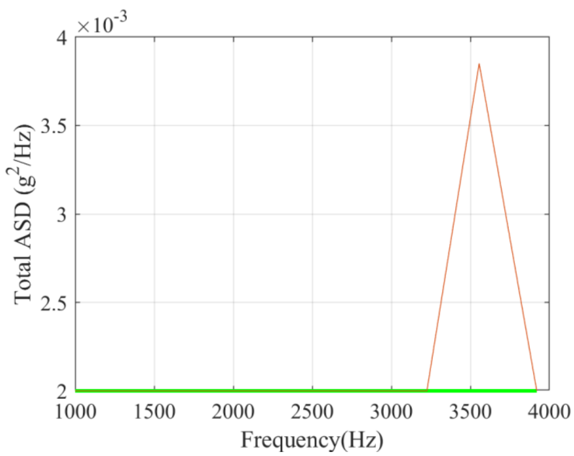

Figure 3c. Total ASD at 14th Time Interval

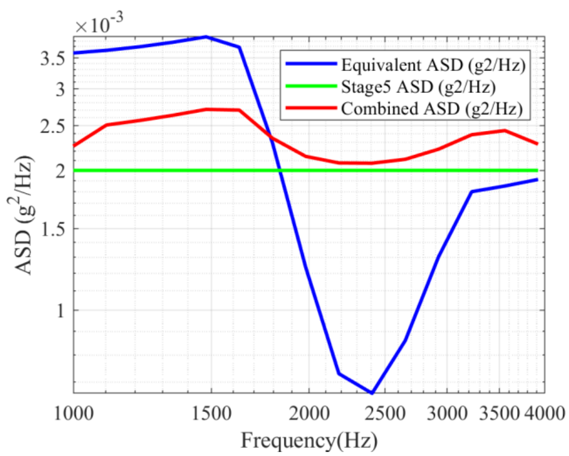

Figure 4. Equivalent Test Specification (red)

Appendix A Nonlinear Statistics and Equivalent Fatigue Damage

This section examines through numerical simulations how to account for the parameter N_σ defined earlier [Eq. 1]. N_σ captures the nonlinear effect of statistical distribution of peaks in a narrow-band random signal and the consequent damage imparted. The issues and the consequent effects are captured for both pure sine-tone and that superimposed on a random vibration. Effect of various fatigue damage coefficients is also demonstrated.

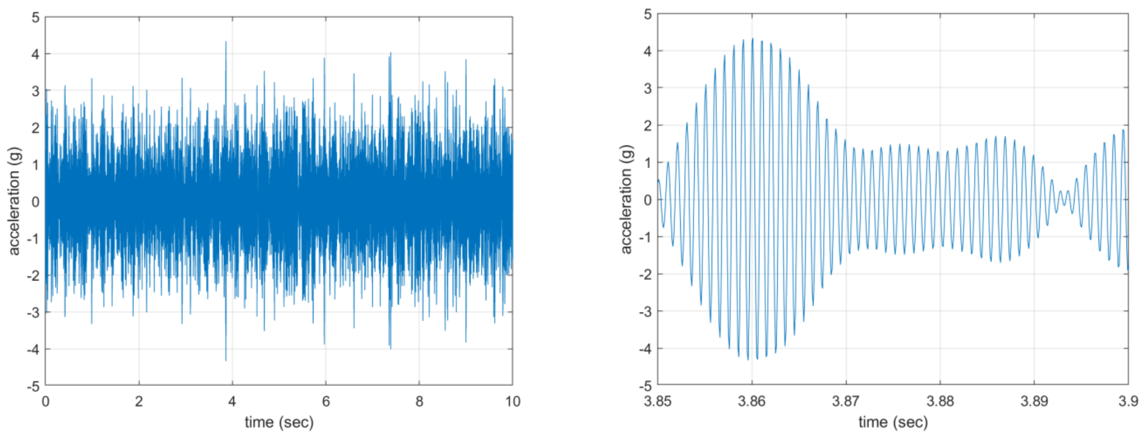

Figure A1. Time-domain Realization of a Narrowband ASD of grms=1.0

Narrowband signal was created with ASD = 0.01g^2/Hz between 950 and 1050 Hz (bandwidth = $\Delta f = 2\ \xi\ f_n$ = 100 Hz for fn = 1000Hz). Therefore grms = (100x0.01)^0.5=1.0. A realization of this ASD for a 10 second duration with sampling rate of 10000 is shown in Figures A1. A zoomed-in version of the time series is also shown to illustrate the modulated sinusoid.

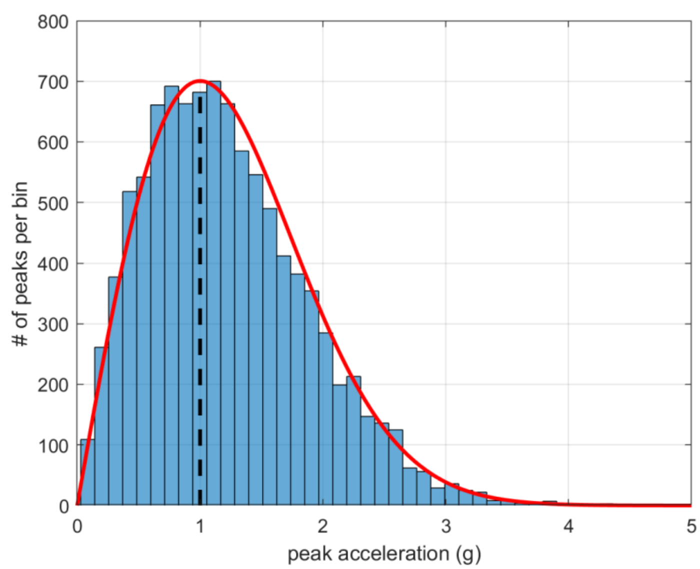

Figure A2. Distribution of Peaks of the Narrow-band Random Signal (grms=1.0)

Figure A2 shows the histogram of 10033 peaks into 40 bins demonstrating Raleigh distribution with a peak value at 1.0 (equal to the grms of the signal) and max beyond 4.0. A normalized Raleigh distribution [Eq. 2] is superimposed on the Histogram (red line). The formula for Raleigh distribution of peaks (x) of a narrow-band random signal formula is shown below for a waveform of grms = σ.

$$P(x/\sigma) = \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} \qquad \text{[Eq.2]}$$

Figure A2 shows that the narrow-band random has many peaks above √2≈1.4 (which corresponds to the peak of a pure sinusoid of the same grms =1.0. This leads to a higher imparted damage since the low-cycle-fatigue damage law (*damage* = *Σstress^b*) exaggerates the effect of higher amplitudes. The exponent *b* depends on the type of failure; *b*=6.65 (ductile failure) is typically used in our analyses. Based on this Raleigh distribution of the sinusoid peaks, the corresponding damage can be calculated. An equivalent sinusoid (single frequency) that imparts the same damage can be determined by adjusting the peak of that pure sinetone. This equivalency leads to various values of the factor N_σ, discussed earlier [Eq. 1] for corresponding values of b. These are shown in Figure A.2 (b=2, 4 & 16.65 corresponds to energy, fretting corrosion and quasi-brittle failure respectively, b = 6.65 leads to $N_\sigma \approx 2$ shown by the red line).

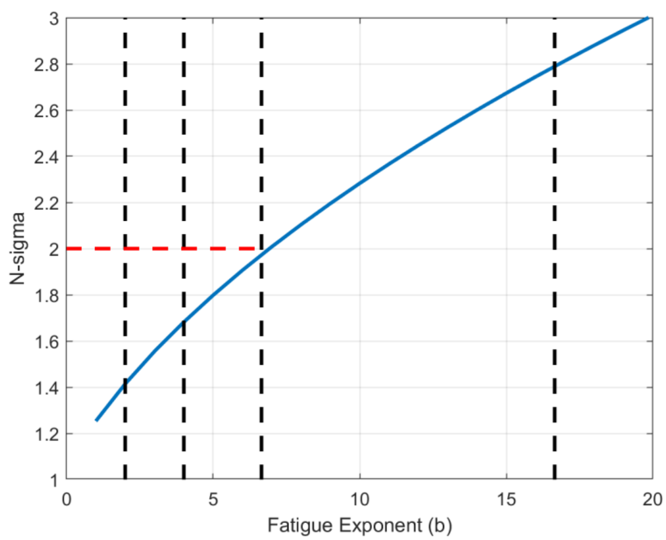

Figure A3. Factor N_σ for Various Values of b

Figure A4 shows the time-domain realization of pure Sinusoid superimposed on a Narrowband ASD, each with grms=1.0. The narrow-band random had an Autospectral Density (ASD) of 0.01 between 950 and 1050 Hz.m and the sinusoid had a peak amplitude of √2 at 1000Hz. The resulting signal has a rms value of ≈√2. The same zoomed-in portion is also shown to compare with the realization of only the narrow-band signal. Note that in contrast with Figure A1 (narrow-band random only) the new signal has many exceedences above 4g and a few above 5g.

A ‘rainflow’ analysis was done on several combined (sinusoid + narrow-band random) time-domain signals each with correspondingly greater ratio of the g-rms values of the sinusoid to the narrow-band random signals. The signals with this ratio of 0 (random-only) and 1.0 (same g-rms) have been shown earlier in Figures A1 and A4. Signals with this ratio = 2.0 and 5.0 (g-rms of pure sine-tone = 5.0) were also generated and added to the narrow-band random.

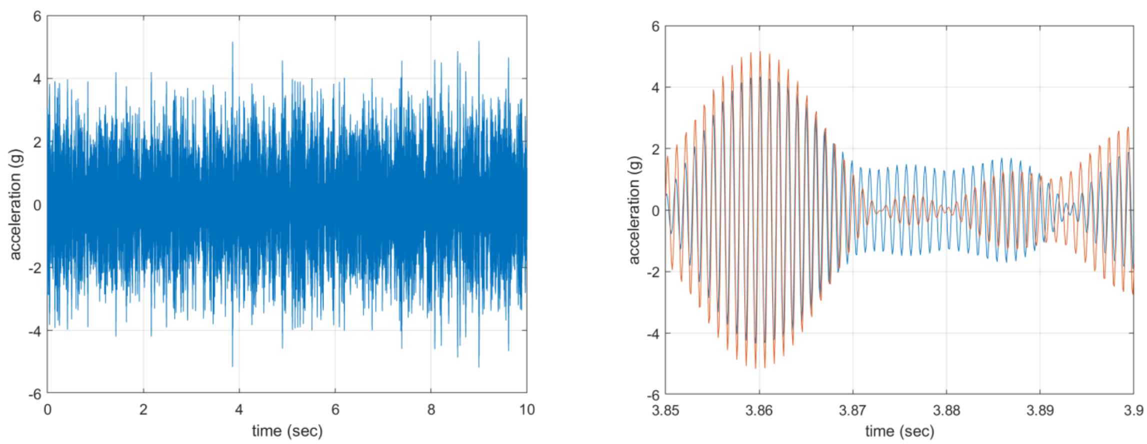

Figure A4. Realization of pure Sinusoid superimposed on a Narrowband ASD

Figure A5 shows the histograms of the cycles from the ‘Rainflow’ analyses on the 4 signals described earlier (progressively increasing ratios). For comparison, the histograms were all normalized for the grms of the total signal being 1.0. It can be seen that the narrow-band random had a Raleigh distribution discussed earlier with a most probably occurrence near 1.0, when the pure sinusoid dominates (ratio=5) the distribution is more ‘normal’ with peak close to 1.4 (peak of a pure sinusoid of grms=1.0 is √2). On the other hand when the Sinusoid gets smaller compared to the narrow-band random, the distribution is progressively closer to that of the narrow-band random (blue line depicting Raleigh distribution). Since with decreasing ratio of sinusoid/narrow-band-random the resulting distributions progressively deviates from that of the pure sinetone (single peak at √2), it can be presumed that Nσ will decrease progressively from ≈2.0 to ≈1.0.

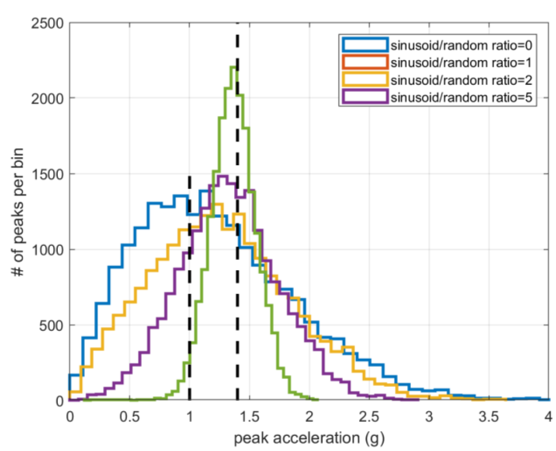

Figure A5. Distribution of Cycles for Signals with Different Sinusoid/Narrow-band Random

Figure A.6 shows this change of N_σ for various ratio of sinusoid to narrow-band random using simulations described in the steps below. The total simulation time was extended to 100 seconds to get more statistically representative results.

- Signals were created by adding (in time domain) the narrow-band random signal to the sinusoid multiplied by various factors (0.25, 0.5, 1.0, 2.0, 3.0, 4.0, 5.0).
- 'Rainflow' analysis was done to determine the number of fatigue cycles of various magnitude; some of them are shown in Figure A5.
- Total damage for each combined signal was determined using the equation for damage above ($D_{total}=\Sigma$(fatigue cycles^6.65))
- A total equivalent grms of a narrow-band random that would have the same damage was determined $grms_{total}= D_{total}^{1/6.65}$
- This $grms_{total}$ is due to both the original narrow-band random $grms_{random}$ plus the sinusoid (of grms=1.0) converted to an equivalent narrow-band random with $grms_{sine}$. Hence
- $grms_{sine}^2= grms_{total}^2 - grms_{random}^2$
- by original definition N_σ = sinusoid-peak / $grms_{sine}$ = sinusoid-peak / $\sqrt{(grms_{total}^2 - grms_{random}^2)}$ [Eq.3]

Figure A6 shows that N_σ approaches the values corresponding that of the pure sinusoid for ratios approaching 5. The effect of N_σ is insignificant ($\approx\sqrt{2}$) for small values of sinusoid (based on relative grms values) and approaches 2.0 when the sinusoid is dominant.

Figure A7 shows the variation of N_σ for various recreations of time-domain signals from the same ASD (due to randomly generated relative phase). The variation is greater for higher values of the fatigue coefficient (b) since that amplifies the effect of the few instances the signal has high amplitude. The variation is also greater for small relative values of the sinusoid since the denominator of Eq. 3 is small and hence subject to greater fluctuations for each generation of random signal from the same ASD.

For values of $b<10$, the variation is <5% for higher ratio of sinusoid to random, which is negligible relative to most other sources of errors in real-life data. For small ratio of sinusoid to random, if the goal is to convert a sinusoid to an equivalent narrow-band ASD (G_{xx}), it is conservative to assume the lower limiting value of $\sqrt{2}$ (i.e. simply ignore this issue), since per Eq. 1 a lower value of N_σ would lead to a greater value of ASD.

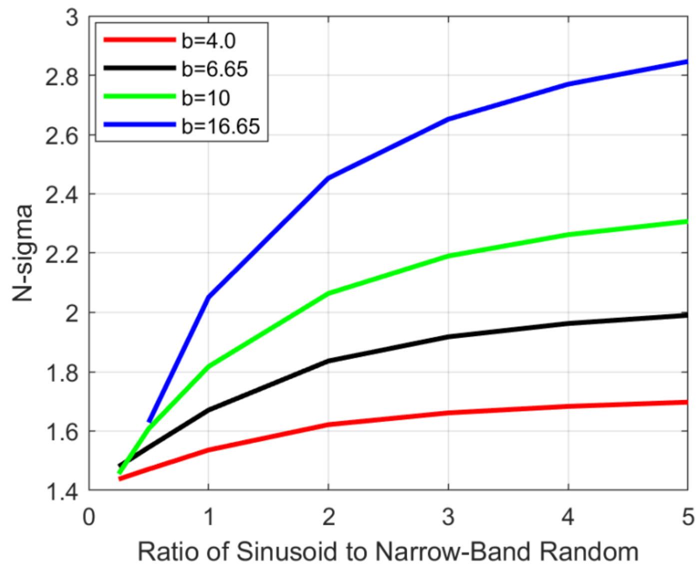

Figure A6. Value of N_{σ} for Various Ratio of Sinusoid to Narrow-band-random

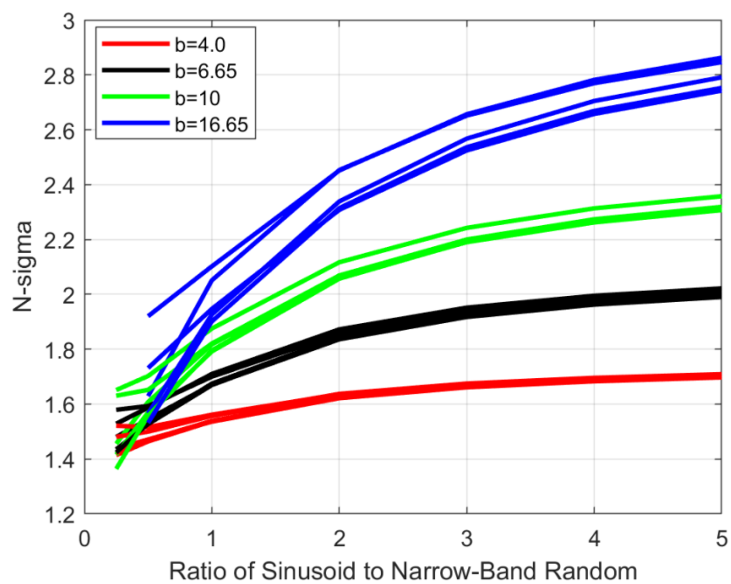

Figure A7. Variation of N_{σ} for Various Recreations of Narrow-band-random

References

[1] Jerome Cap, "Environments Specifications Short Course, Sinusoidal Vibration", es2p3_sinevibration_103120.pptx.

[2] Fackler, Warren C., 1972, "Equivalence Techniques for Vibration Testing", SVM-9, Shock and Vibration Monograph Series (DTIC, 1987).