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Data-Driven Whitney Forms for Structure-Preserving Control Volume Analysis

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Given:
physics **data** that obeys physics and conservation

- high-fidelity model
- expensive
- compute-intensive
- observational

Goal:
recover surrogate model that preserves underlying physics laws

Challenges:
1. How to **enforce conservation exactly** from sampled point data?
2. How to extract a generalizable **parameterized model** from small amounts of data?
3. How to guarantee **stability** and **convergence** properties?

1. Build Chain Complex

- Guarantee Exact Sequence property
- Enforce Stokes' Theorem

2. Discover Control Volumes

- Use Partitions of Unity to define Whitney *k*-forms
- Assemble variational problem with compatible spaces

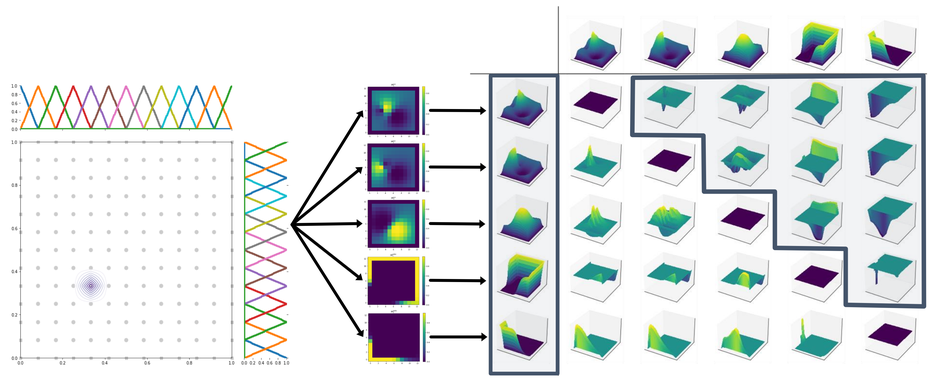
3. Parameterize Model

- Induce learnable metrics for boundary, coboundary operators
- Use free-knot splines with trainable knots for exact integration

$$\begin{array}{ccccccccccc} C^0 & \xrightarrow{\mathrm{d}_0} & C^1 & \xrightarrow{\mathrm{d}_1} & C^2 & \xrightarrow{\mathrm{d}_2} & C^3 & \xrightarrow{\mathrm{d}_3} & \cdots & \xrightarrow{\mathrm{d}_{d-1}} & C^d \\ \uparrow{\scriptstyle \mathbf{B}_0} & & \uparrow{\scriptstyle \mathbf{B}_1} & & \uparrow{\scriptstyle \mathbf{B}_2} & & \uparrow{\scriptstyle \mathbf{B}_3} & & & & \uparrow{\scriptstyle \mathbf{B}_d} \\ C^0 & \underset{\delta_1^*}{\xrightarrow{\delta_0}} & C^1 & \underset{\delta_2^*}{\xrightarrow{\delta_1}} & C^2 & \underset{\delta_3^*}{\xrightarrow{\delta_2}} & C^3 & \underset{\delta_{d-1}^*}{\xrightarrow{\delta_3}} & \cdots & \underset{\delta_d^*}{\xrightarrow{\delta_{d-1}}} & C^d \\ \uparrow{\scriptstyle \mathbf{D}_0} & & \uparrow{\scriptstyle \mathbf{D}_1} & & \uparrow{\scriptstyle \mathbf{D}_2} & & \uparrow{\scriptstyle \mathbf{D}_3} & & & & \uparrow{\scriptstyle \mathbf{D}_d} \\ C^0 & \xleftarrow{\mathrm{d}_0^*} & C^1 & \xleftarrow{\mathrm{d}_1^*} & C^2 & \xleftarrow{\mathrm{d}_2^*} & C^3 & \xleftarrow{\mathrm{d}_3^*} & \cdots & \xleftarrow{\mathrm{d}_{d-1}^*} & C^d \end{array}$$

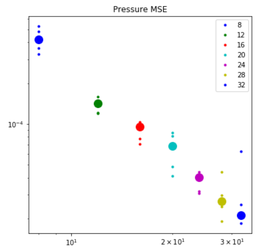
$$\min_{\boldsymbol{\xi}} \left\| p_{\text{data}} - \left(p_D + \sum_i \hat{p}_i \psi_i \right) \right\|_2^2 + \alpha^2 \left\| F_{\text{data}} - \sum_{ij} \hat{F}_{ij} \psi_{ij} \right\|_2^2$$

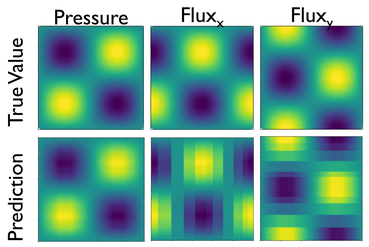
$$\text{such that} \begin{bmatrix} \mathbf{M}_1 & -\mathbf{M}_1 \mathbf{D}_1^{-1} \delta_0 \mathbf{D}_1 \\ -\mathbf{B}_0^{-1} \delta_0^T \mathbf{B}_1 \mathbf{M}_1 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{F}} \\ \hat{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} b_D \\ -b_f - b_N \end{bmatrix}$$

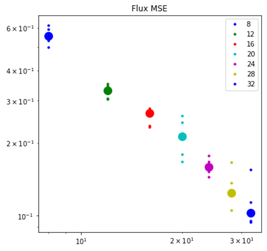


Poisson Problem

Maintains stability, *hp*-convergence of FEM methods

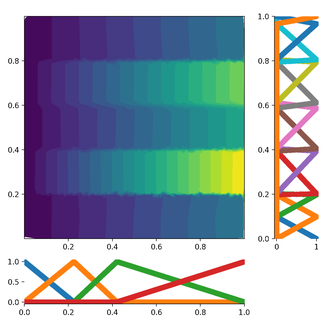






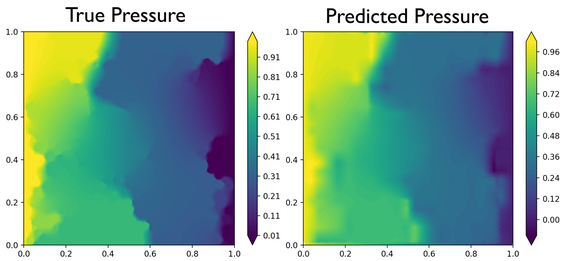
Five Strip Problem

Free spline knots allow POUs to adaptively fit regions and interfaces



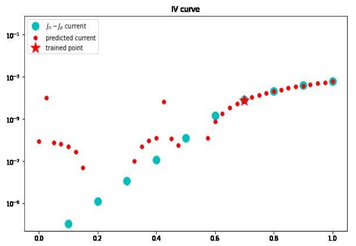
Battery Problem

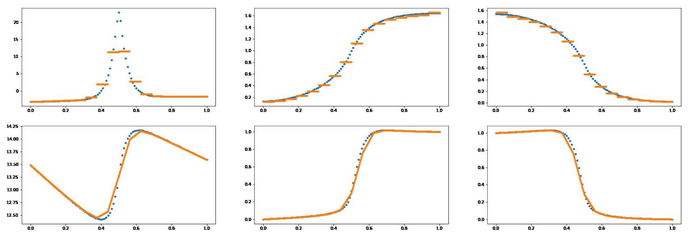
Learned model reduces system from 5.9M to 136 DoF while preserving quantity of interest



PN Diode

Method extends to nonlinear and advection problems





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