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Computing Quasiconformal Mappings from Immersed Surfaces

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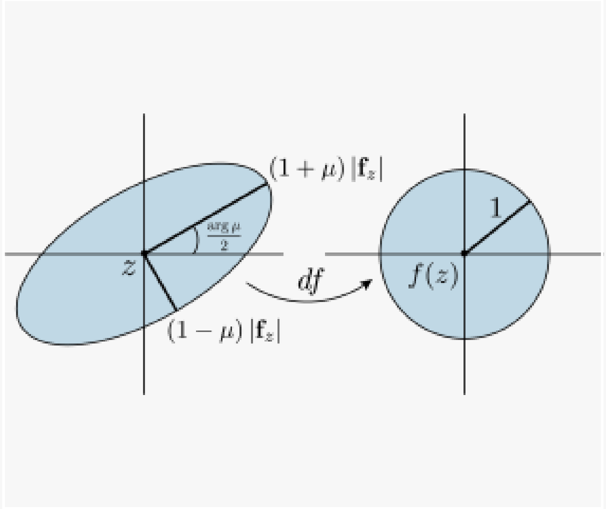
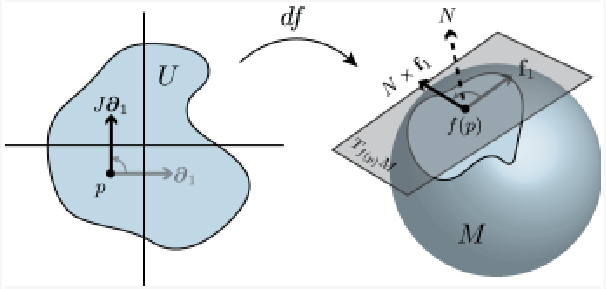
Outline



1

Classical Quasiconformality



2

Quaternionic Quasiconformality

3

Algorithm and Results

* Joint work with **Eugenio Aulisa** at TTU

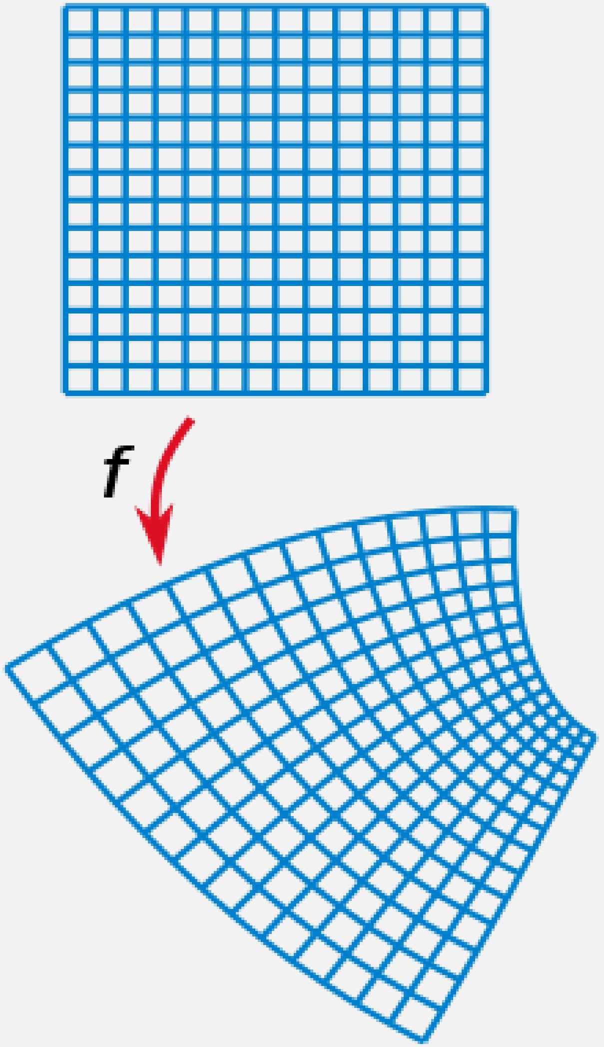
* Preprint at **https://agrubertx.github.io**

Classical Conformality

- Mapping $f : \mathbb{C} \to \mathbb{C}$.
- Write the differential

$$\begin{aligned} df &= f_x\, dx + f_y\, dy \\ &= \tfrac{1}{2}(f_x + i f_y)(dx - i dy) \\ &\quad + \tfrac{1}{2}(f_x - i f_y)(dx + i dy) \\ &:= f_z\, dz + f_{\bar{z}}\, d\bar{z}\ . \end{aligned}$$

- f is conformal if and only if $f_{\bar{z}} \equiv 0$.



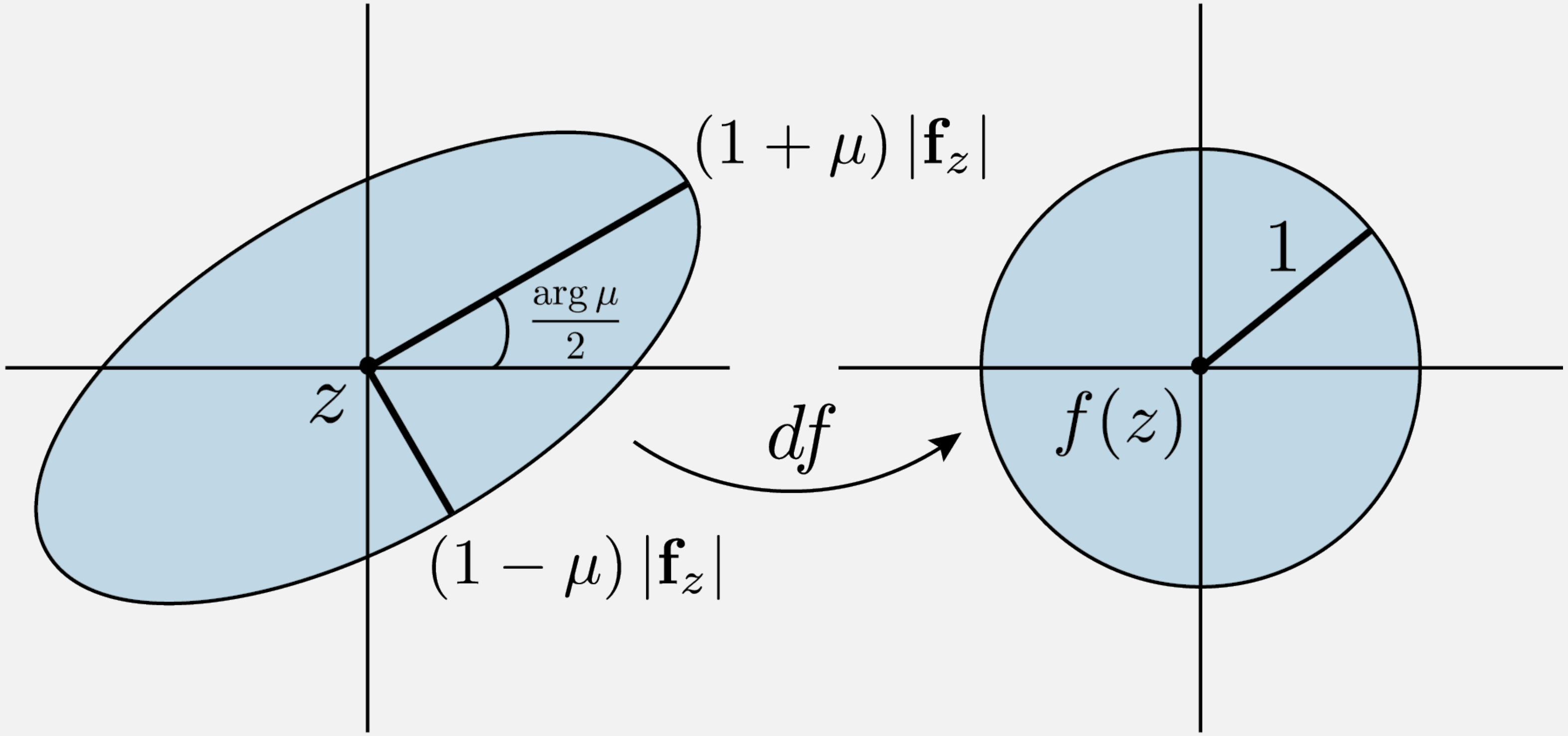
Classical Quasiconformality

- Mapping $f : \mathbb{C} \to \mathbb{C}$.
  - Conformal if $f_{\bar{z}} = 0$.
- *Quasiconformal*:

$$f_{\bar{z}} = \mu\, f_z\,, \quad |\mu| < 1.$$

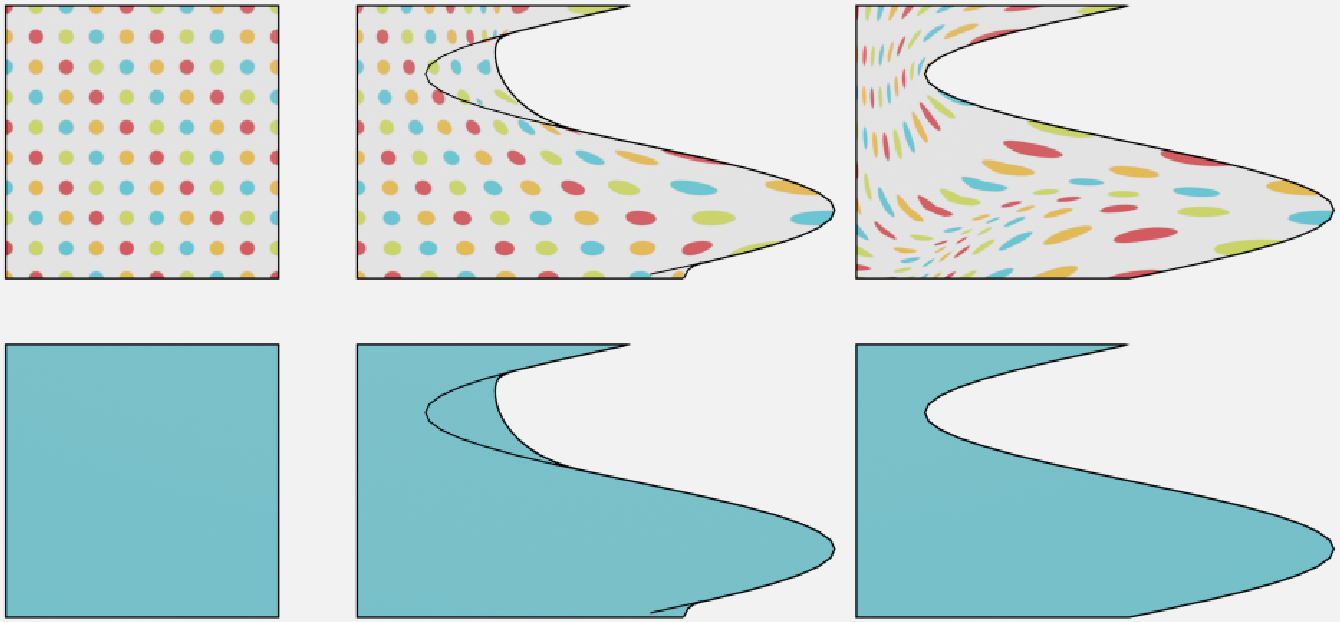
- Allows bounded shearing.
- $\mathrm{Jac}(f) = |f_z|^2 - |f_{\bar{z}}|^2$

$$= |f_z|^2(1 - |\mu|^2)$$



Why quasiconformality?

- Consider least-squares conformal mapping:
    - Minimize $\int_M |f_{\bar{z}}|^2 \, i dz \wedge d\bar{z}$
- *May not interpolate boundary!*
    - Not enough conformal maps*
- Quasiconformal *always* will.
    - Consequence of $\mathrm{Jac}(f) > 0$.



*preserving a discrete boundary correspondence!

What about Riemann surfaces?

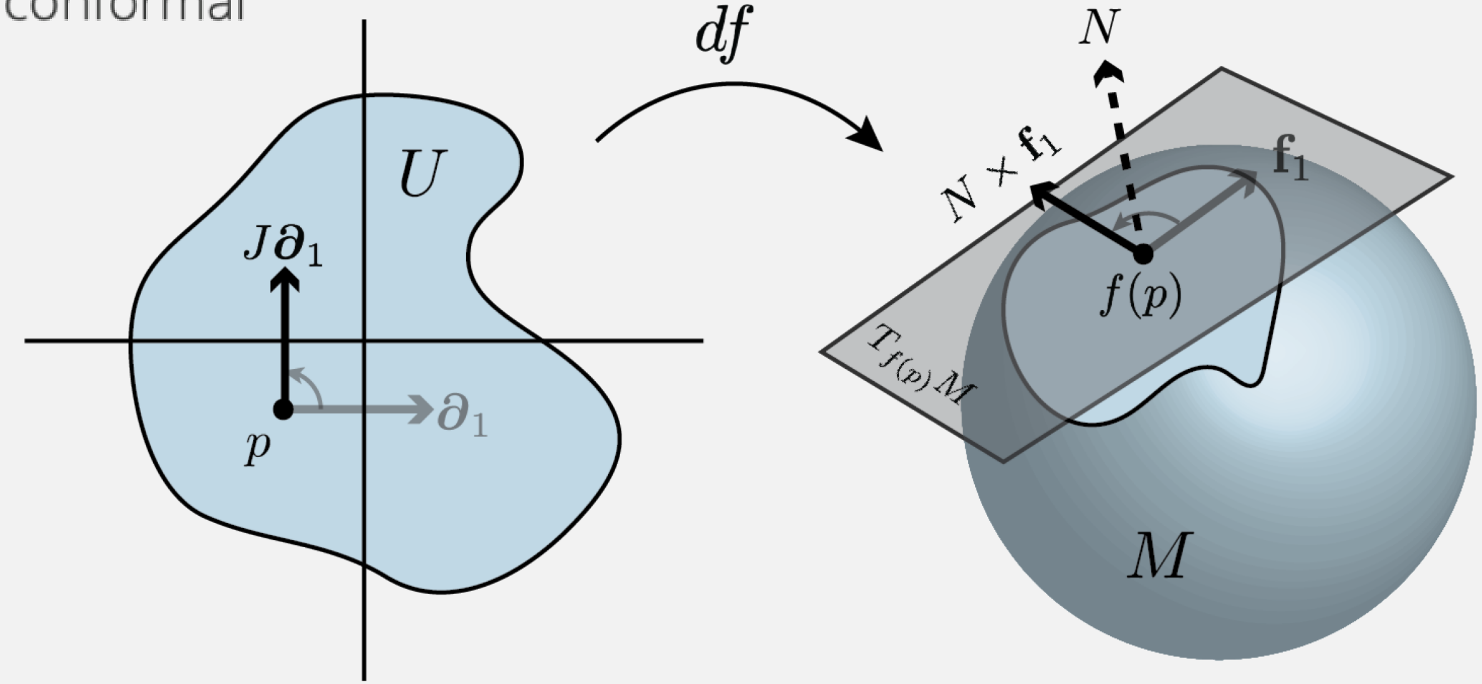
- How to remove coordinate dependence?
- Consider $f\colon M \to P,\;\; df = df^+ + df^-$.
- $df^- = \mu\, df^+$ *won't work* without modification.
  - Left side $\mathbb{C}$-antilinear, right side $\mathbb{C}$-linear!
- $df^- = df^+ \circ \mu$ makes sense. ("Classical version")
  - $\mu\colon TM \to TM$ is now $\mathbb{C}$-antilinear.
  - But we compose instead of multiply!

Quaternionic Conformality

- Suppose $M, P \subset \mathbb{R}^3$ immersed.
- (Kamberov et al. 1992) $f : M \to \mathbb{R}^3$ is conformal

  iff $\quad * \, df = N \, df.$
- $J^2 = I_{TM}$ is complex structure

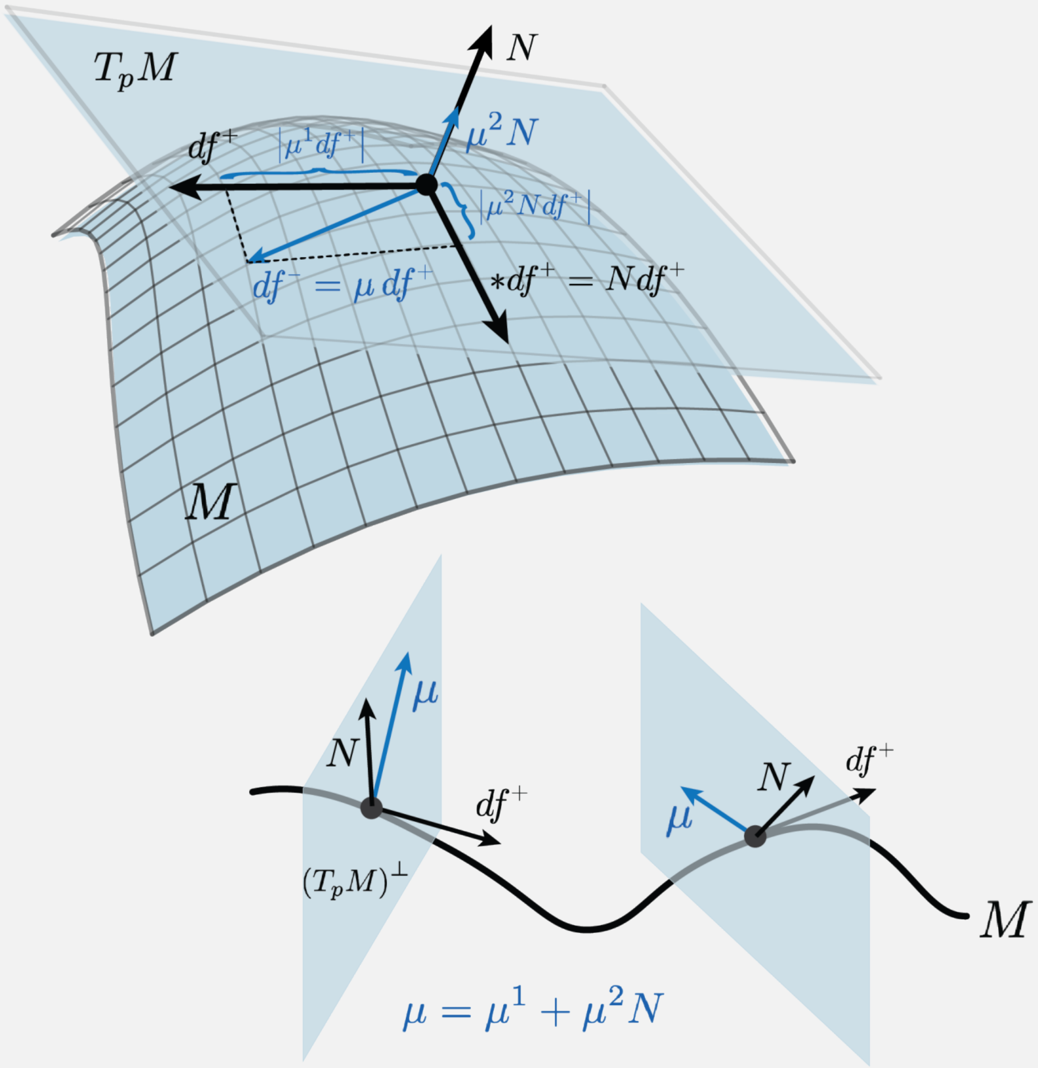
  and $\quad * \, df = df \circ J.$
- Connects *intrinsic* notion with *extrinsic* representation!



* Burstall, Ferus, Kamberov, Leschke, Pedit, Pinkall

Quaternionic Quasiconformality

- With quaternionic algebra:
  - $f: M \to P \subset \mathbb{R}^3, \;\; df = df^+ + df^-$.
- $df^\pm = \frac{1}{2}(df \mp N * df)$
  - Conformal/anticonformal parts.
- $df^- = \mu \, df^+$ *is now possible*!
  - If $\mu: TM \to (TM)^\perp$ is normal-valued!



Quaternionic Quasiconformality

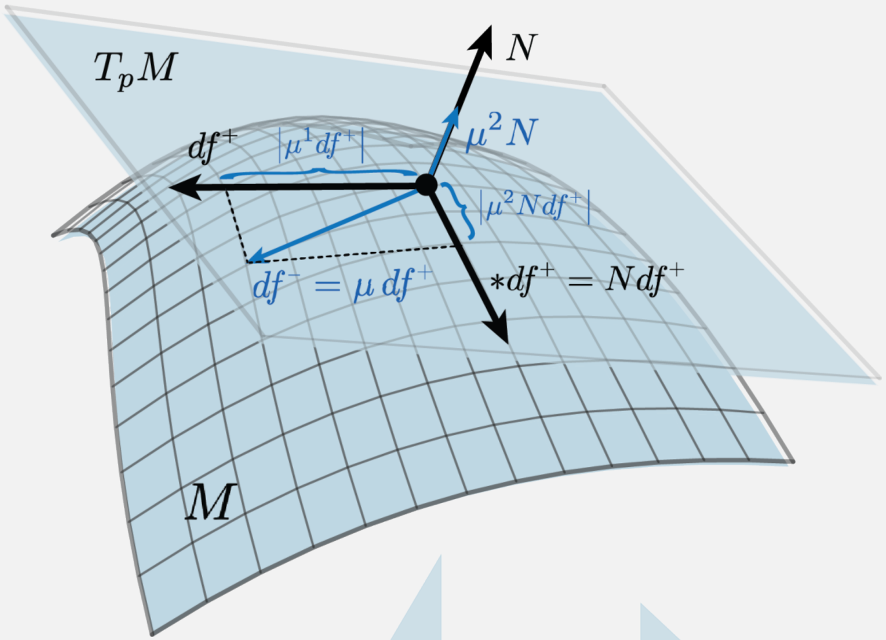
- $f: M \to \mathbb{R}^3$ is quasiconformal iff

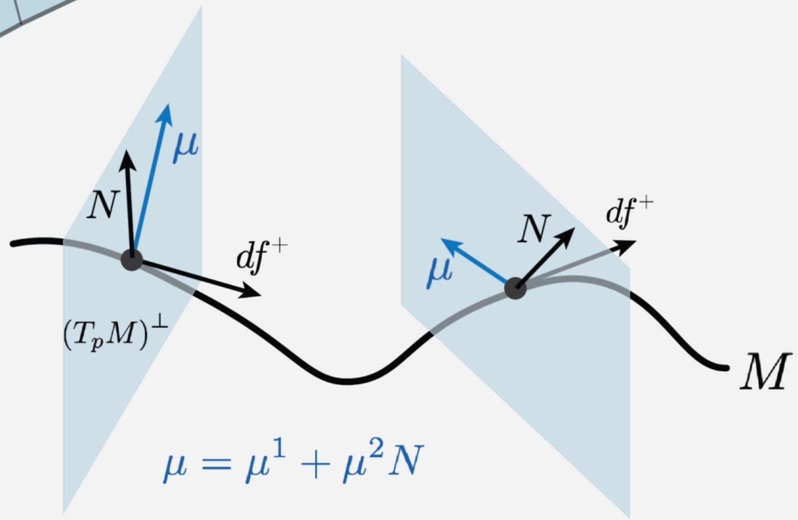
$$df^- = \mu \, df^+, \quad |\mu| < 1.$$

- $Proof$: write $\mu(v) = \mu^1(v)v + \mu^2(v)Jv$,

$$\begin{aligned} df^+ \circ \mu(v) &= df^+(\mu^1(v)v + \mu^2(v)Jv) \\ &= (\mu^1(v) + \mu^2(v)N)df^+(v) \\ &= \mu(v)df^+(v). \end{aligned}$$

- Quaternionic algebra converts *composition* to *multiplication*!





Aside: what is μ ?

- Write the induced metric

$$f^*\delta = |df|^2 = |df^+|^2 + |df^-|^2 + 2\,\mathrm{Re}\left(df^+\overline{df^-}\right)$$

- The (2,0)-part is the *Hopf differential,* $Q = df^+\overline{df^-}$.
- Expanding yields $4Q = |df|^2 - |*\,df|^2 - \langle df, *\,df\rangle N$.
  - Q is normal-valued!
- For quasiconformal f, can show $\mu = *\,\bar{Q}$.

Least-squares QC Map

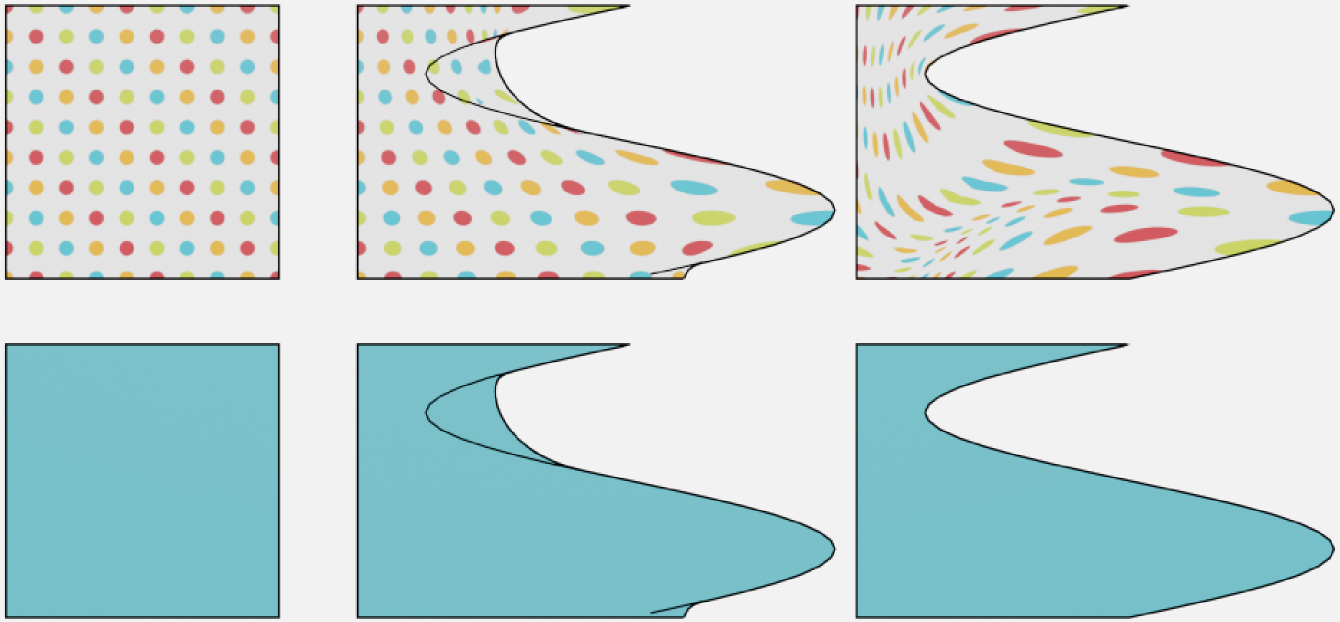
- Consider the *quasiconformal distortion*

$$QC_\mu(f) = \frac{1}{2}\int_M |df^- - \mu\, df^+|^2 \; dS_g$$

- Want to solve:

$$\arg\min_{f\in[f_0]} QC_\mu(f), \quad f|_{\partial M} = f_0|_{\partial M} .$$

- $f_0 : M \to \mathbb{R}^3$ given, defines homotopy class and boundary data.



Least-squares QC Map

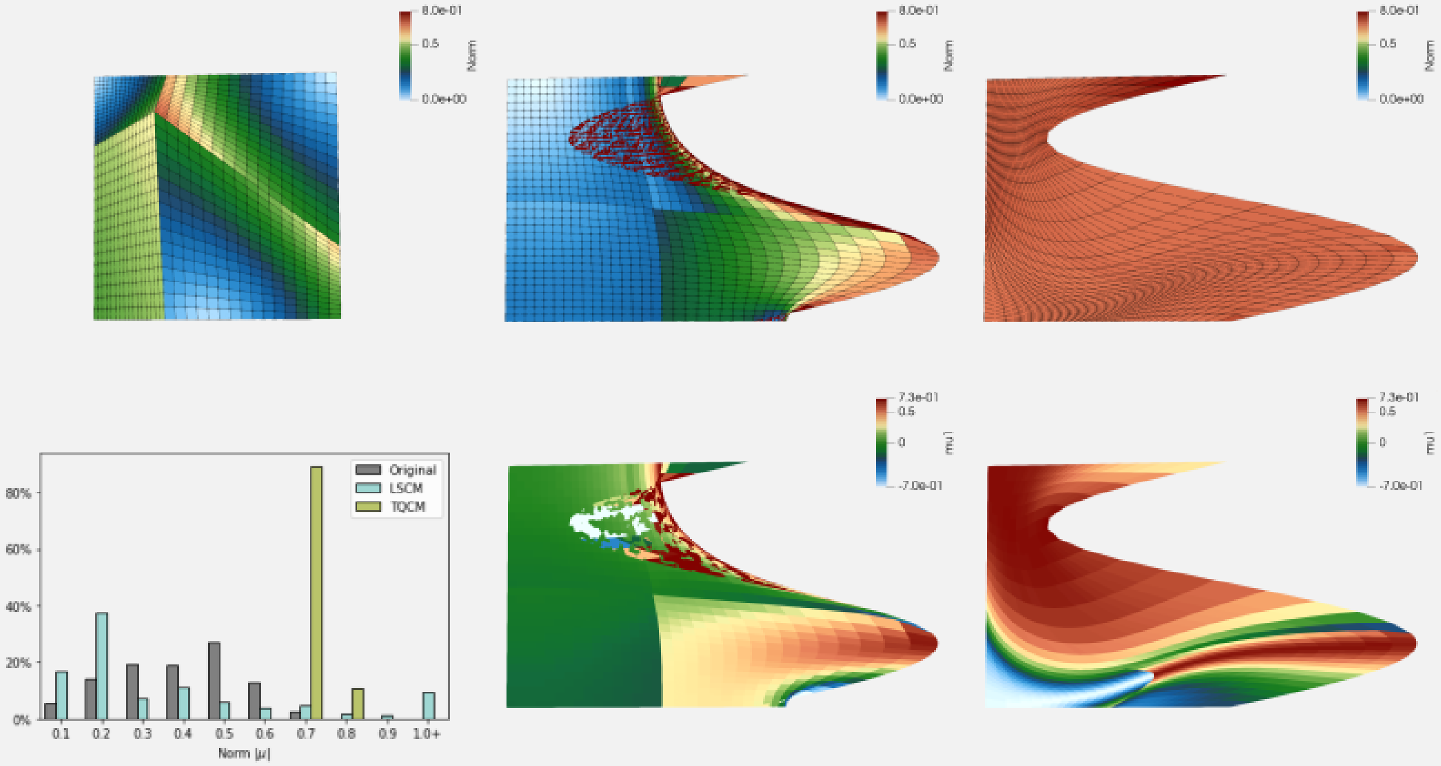
- Fix g, N, and write $f = f_0 + t\varphi$ for some $\varphi: M \to \mathbb{R}^3$, then

$$\delta QC_\mu(f_0)\varphi = \int_M \langle df_0^- - \mu\, df_0^+, d\varphi^- - \mu\, d\varphi^+ \rangle\, dS_g$$

- Solving $\delta QC_\mu(f_0)\varphi = 0$ yields LSQC map matching initial data.
- In practice, we discretize with p.w. linear finite elements.
  - Can also enforce constraints through Lagrange multipliers.

How do we get μ?

- Developed quaternionic version of QC Iteration *(Lui et. al. 2014).*
  - Computes optimal Teichmuller (TM) map between planar domains.
  - TM maps *minimize* the *maximal dilatation* $K = \frac{1+|\mu|_\infty}{1-|\mu|_\infty}$.



QC Iteration

- Idea is alternating minimization.
- (3) involves heat flow on norm/argument separately.

Algorithm 2 Overview of the quaternionic QC Iteration

Require: Surface $M \subset \mathbb{R}^3$ and homotopy class $[f']$. Beltrami coefficient $\mu_0 = 0$. Stopping tolerance $\varepsilon > 0$ and maximum iteration number $n_t > 0$.

while $0 \leq k \leq n_t$ and $QC_{\mu_k}(f_k) > \varepsilon$ **do**

    (1) Given μ_k, Minimize QC_{μ_k} for $f_k : M \to \mathbb{R}^3$.

    (2) Compute μ_{k+1} algebraically given f_k.

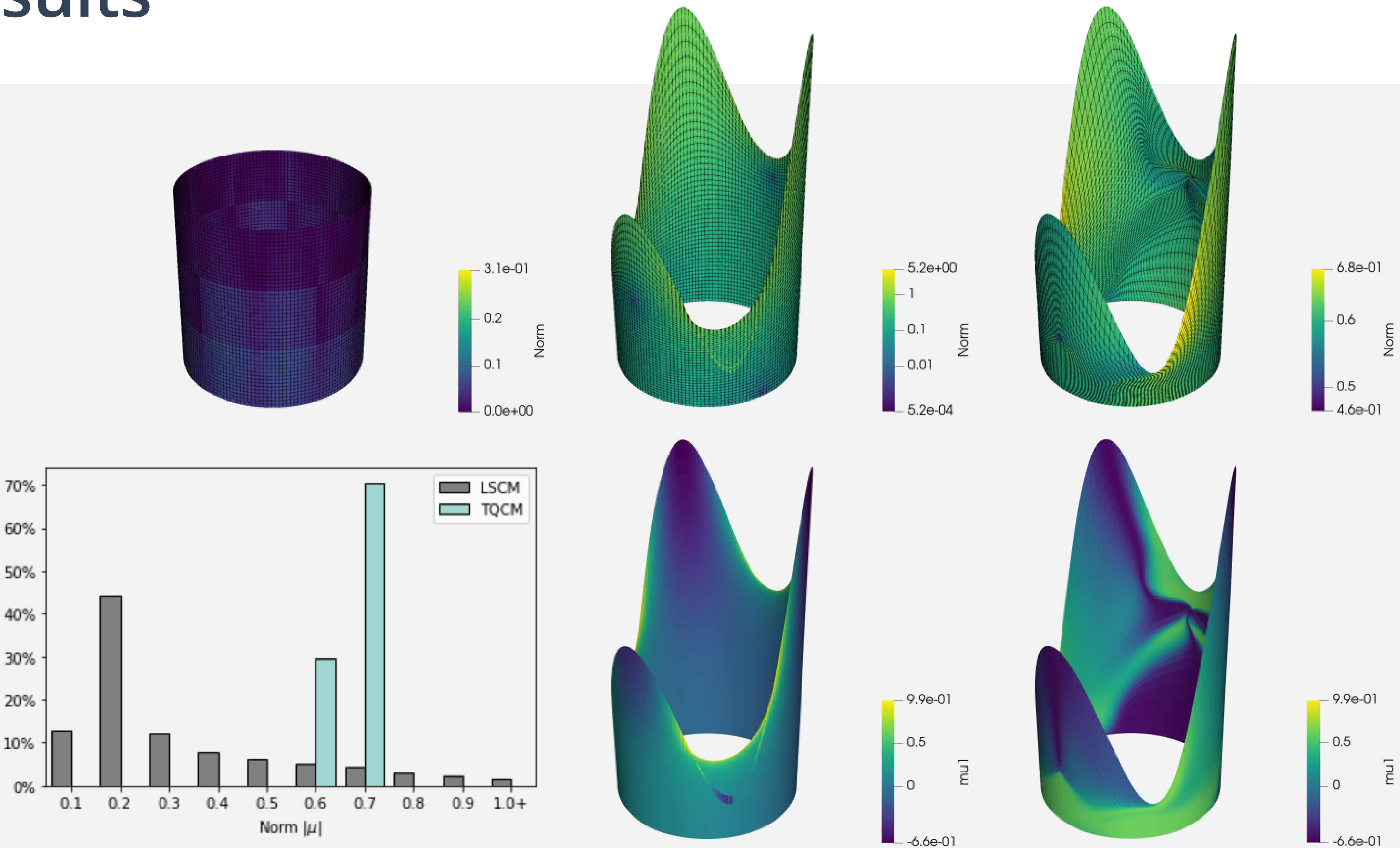
    (3) Post-process μ_{k+1} to bring it closer to Teichmüller form.

    (4) Minimize $QC_\mu(f_k)$ for μ on the line between μ_k and μ_{k+1}, generating $\mu_{k+1} \leftarrow \mu$.

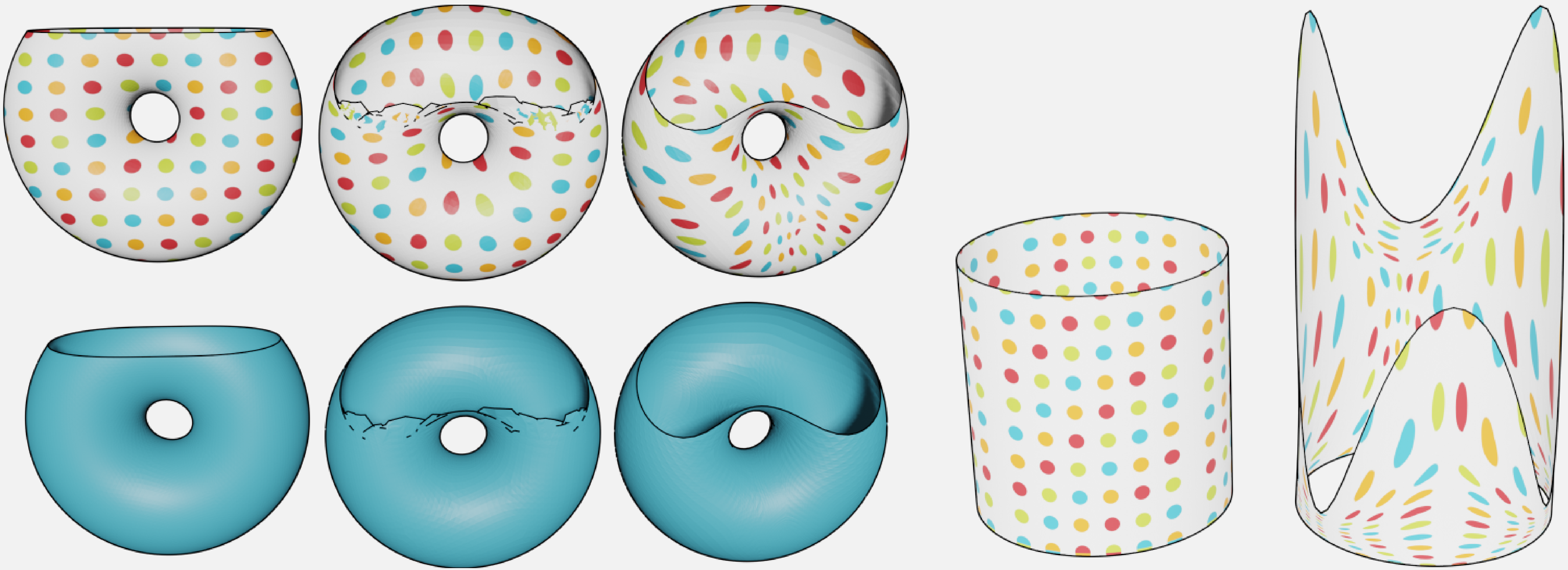
end while

return (f, μ)

Results


3.1e-01
0.2
0.1
0.0e+00
Norm
5.2e+00
1
0.1
0.01
5.2e-04
Norm
6.8e-01
0.6
0.5
4.6e-01
Norm
70%
60%
50%
40%
30%
20%
10%
0%
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0+
Norm |μ|
LSCM
TQCM
9.9e-01
0.5
0
-6.6e-01
mu1
9.9e-01
0.5
0
-6.6e-01
mu1

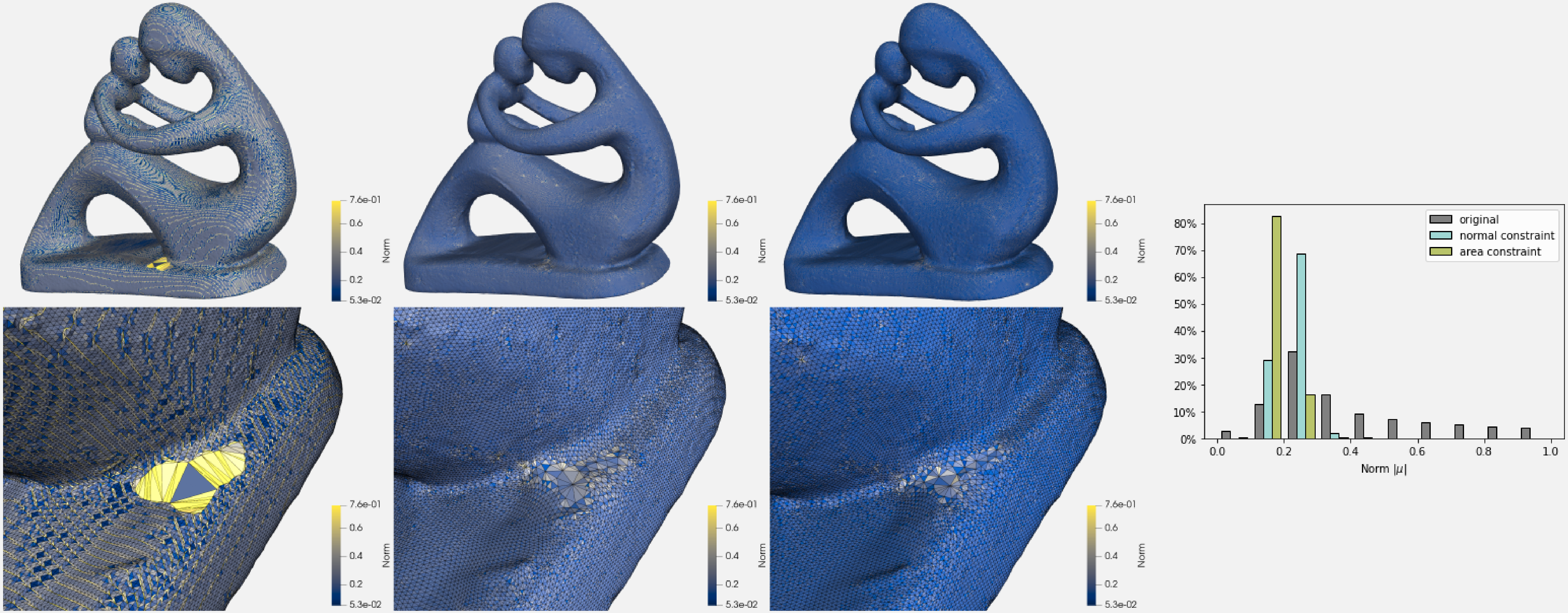
Results



Application: mesh editing

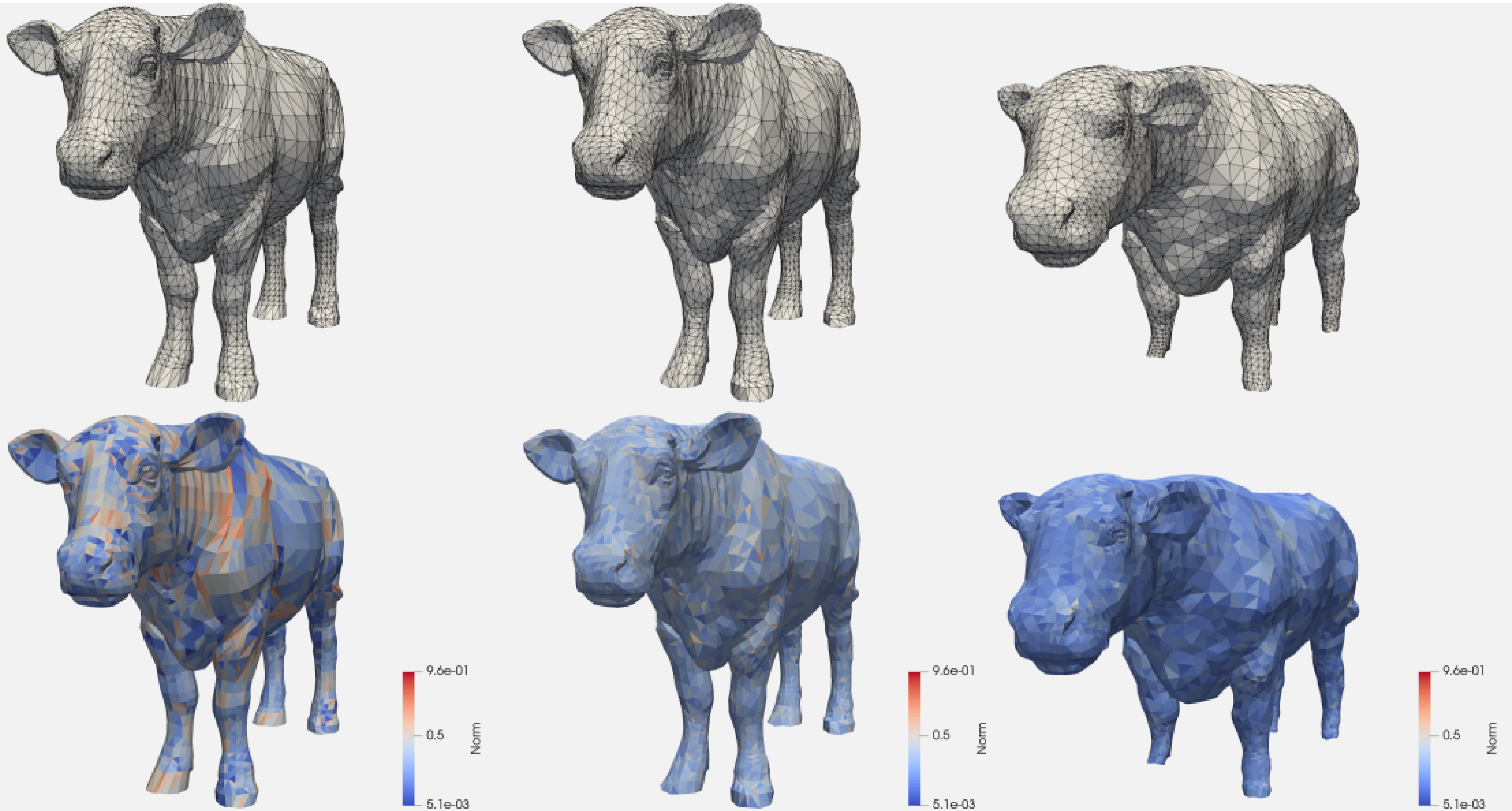
- Can fix "bad" triangulations of M (determined by g).
- Compute optimal TM map $(M, g_{\mathrm{ref}}) \to (M, g)$.
  - Metric g_{ref} is *not necessary*, only need $[g_{\mathrm{ref}}]$.
  - Conformal class specified by target interior angles.
- Minimize QC_{μ} w.r.t. g_{ref} .
  - Include constraint on extrinsic geometry.

Results



7.6e-01
0.6
0.4
0.2
5.3e-02
Norm
original
normal constraint
area constraint
80%
70%
60%
50%
40%
30%
20%
10%
0%
0.0
0.2
0.4
0.6
0.8
1.0
Norm |μ|

Results



Thank you!

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