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Coordinate Transformation of Vibration Autospectral Density (ASD)

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Abstract

Autospectral Density (ASD), also referred to as Power-Spectral Density (PSD) is the most common way of quantifying vibration for archival of test data and specifications for laboratory tests. The original acceleration vs. time history is no longer available, especially as the ASDs are used in subsequent mathematical processes such as probabilistic assessments leading to test specifications. Later on, it is necessary to determine ASDs in coordinate systems that are oriented at an angle w.r.to the original coordinate system. For instance, tests need to be run in component coordinate axis vs. system coordinate axis in which data was collected and propagated.

This paper summarizes the necessary theoretical background of ASD transformation, as it relates to both correlated and uncorrelated (random) vibration data. Equations are provided to allow the user to calculate relevant ASDs in a transformed coordinate system. Also, the consequence of alternative methods that are often used (such as enveloping the 2 orthogonal specifications or sum of ASDs as upper limit) are explored and compared with more accurate methods and equations. Finally, simulations with various angle of transform and relative phase of vibration in the two axes are used to understand the implications and to quantify the levels of conservatism.

Keywords: Vibration; Transform; ASD; Spectral Density; Simulation

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Background

Auto-spectral Density (ASD) also known as Power-Spectral-Density (PSD) of an acceleration vs. time provides an estimate of the energy content at each frequency. It is derived by multiplying the Fourier transform of the acceleration vs. time with the conjugate of the Fourier transform, and dividing by the bandwidth, thus resulting in real values at each frequency of interest [1]. Underwood and Keller [2] have discussed coordinate transform of acceleration (and rotation) time histories to determine input and outputs for application in 6-Degree of Freedom (6-DOF) shakers. Hale [3] have also described how to convert spectral density matrices of measured accelerations into single-point-input spectral density matrices. This paper discusses the conversion of ASDs from one set of coordinates to another when the underlying acceleration vs. time data is not available.

The next section provides the theoretical background and equations. It will be followed by the section on Numerical simulations and Recommendations.

Theoretical Background

This section provides the theoretical background on transformation of ASDs for each frequency band. The frequency bands are constant (linear band) based on the duration of time-domain signal converted into frequency domain.

A_y and A_z are Fourier domain accelerations in the Y and Z (original coordinates)

A'_y and A'_z are Fourier domain accelerations in the Y' and Z' (new coordinates). New coordinates are oriented at an angle θ clockwise from original coordinates.

a_y & b_y are real and imaginary parts of A_y and a_z & b_z are real and imaginary parts of A_z

$A_y = a_y + ib_y$, $A_z = a_z + ib_z$

Since ASD equals multiplying the Fourier transform of acceleration with its own conjugate,

$$ASD_y = (a_y + ib_y)(a_y - ib_y) = a_y^2 + b_y^2, \ \& \ ASD_z = a_z^2 + b_z^2 \qquad [1]$$

Taking components of accelerations to obtain accelerations A'_y and A'_z in new coordinates,

$$A'_y = A_y\cos\theta - A_z\sin\theta, \ \ A'_z = A_y\sin\theta + A_z\cos\theta \qquad [2a]$$

Combining equations 1 and 2a,

$$A'_y = (a_y + ib_y)\cos\theta - (a_z + ib_z)\sin\theta = [(a_y\cos\theta - a_z\sin\theta) + i\,[(b_y\cos\theta - b_z\sin\theta)] \qquad [2b]$$

$$A'_z = (a_y + ib_y)\sin\theta + (a_z + ib_z)\cos\theta = [(a_y\sin\theta + a_z\cos\theta) + i\,[(b_y\sin\theta - b_z\cos\theta)] \qquad [2c]$$

Since ASD equals multiplying the Fourier transform of acceleration with its own conjugate,

$$ASD'_y = [(a_y \cos\theta - a_z \sin\theta) + i\,[(b_y \cos\theta - b_z \sin\theta)]\ x\ [(a_y \cos\theta - a_z \sin\theta) - i\,[(b_y \cos\theta - b_z \sin\theta)]$$

$$= a_y^2 \cos^2\theta + a_z^2 \sin^2\theta - 2a_y a_z \sin\theta\cos\theta + b_y^2 \cos^2\theta + b_z^2 \sin^2\theta - 2b_y b_z \sin\theta\cos\theta \qquad [3a]$$

$$ASD'_z = [(a_y \sin\theta + a_z \cos\theta) + i\,[(b_y \sin\theta + b_z \cos\theta)]\ x\ [(a_y \sin\theta + a_z \cos\theta) - i\,[(b_y \sin\theta + b_z \cos\theta)]$$

$$= a_y^2 \sin^2\theta + a_z^2 \cos^2\theta + 2a_y a_z \sin\theta\cos\theta + b_y^2 \sin^2\theta + b_z^2 \cos^2\theta + 2b_y b_z \sin\theta\cos\theta \qquad [3b]$$

Summing equations 3a and 3b, and comparing with equation 1,

$$ASD'_y + ASD'_z = a_y^2(\cos^2\theta + \sin^2\theta) + a_z^2(\sin^2\theta + \cos^2\theta) +$$
$$b_y^2(\cos^2\theta + \sin^2\theta) + b_z^2(\sin^2\theta + \cos^2\theta) = a_y^2 + a_z^2 + b_y^2 + b_z^2 = ASD_y + ASD_z \qquad [3c]$$

The equation above shows that the sum of the two ASDs is the same in both the original and the new coordinate systems.

Reorganizing equations 3a and 3b,

$$ASD'_y = (a_y^2 + b_y^2)\cos^2\theta + (a_z^2 + b_z^2)\sin^2\theta - 2\sin\theta\cos\theta\,(a_y a_z + b_y b_z) =$$

$$ASD_y \cos^2\theta + ASD_z \sin^2\theta - 2\sin\theta\cos\theta\,(a_y a_z + b_y b_z) \qquad [4a]$$

$$ASD'_z = (a_z^2 + b_z^2)\cos^2\theta + (a_y^2 + b_y^2)\sin^2\theta + 2\sin\theta\cos\theta\,(a_y a_z + b_y b_z) =$$

$$ASD_z \cos^2\theta + ASD_y \sin^2\theta + 2\sin\theta\cos\theta\,(a_y a_z + b_y b_z) \qquad [4b]$$

If $\underline{A}_y$ & $\underline{A}_z$ are amplitudes and ϕ_y & ϕ_z phase of accelerations in Y and Z

$$\underline{A}_y = \sqrt{ASD_y}\ \&\ \underline{A}_z = \sqrt{ASD_z},\ a_y = \underline{A}_y \cos\phi_y,\ b_y = \underline{A}_y \sin\phi_y,\ a_z = \underline{A}_z \cos\phi_z,\ b_z = \underline{A}_z \sin\phi_z \qquad [5]$$

Combining equations 4a, and 4b with 5

$$ASD'_y = ASD_y \cos^2\theta + ASD_z \sin^2\theta - 2\sin\theta\cos\theta\,(\underline{A}_y \cos\phi_y\, \underline{A}_z \cos\phi_z + \underline{A}_y \sin\phi_y\, \underline{A}_z \sin\phi_z) =$$

$$ASD_y \cos^2\theta + ASD_z \sin^2\theta - 2\sin\theta\cos\theta\sqrt{(ASD_y ASD_z)}(\cos\phi_y \cos\phi_z + \sin\phi_y \sin\phi_z) =$$

$$ASD_y \cos^2\theta + ASD_z \sin^2\theta - \sin 2\theta\sqrt{(ASD_y ASD_z)}\cos(\phi_y - \phi_z) \qquad [6a]$$

$$ASD'_z = ASD_z \cos^2\theta + ASD_y \sin^2\theta + 2\sin\theta\cos\theta\,(\underline{A}_y \cos\phi_y\, \underline{A}_z \cos\phi_z + \underline{A}_y \sin\phi_y\, \underline{A}_z \sin\phi_z) =$$

$$ASD_z \cos^2\theta + ASD_y \sin^2\theta + 2\sin\theta\cos\theta\sqrt{(ASD_y ASD_z)}(\cos\phi_y \cos\phi_z + \sin\phi_y \sin\phi_z) =$$

$$ASD_z \cos^2\theta + ASD_y \sin^2\theta + \sin 2\theta\sqrt{(ASD_y ASD_z)}\cos(\phi_y - \phi_z) \qquad [6b]$$

Only the last part of equations 6a and 6b capture the effect of relative phase (hence correlation) between the Y and Z accelerations.

If the original accelerations A_y and A_z are uncorrelated the expected value of $\cos(\phi_y - \phi_z)$ is 0,

$$ASD'_y = ASD_y \cos^2\theta + ASD_z \sin^2\theta \text{ and } ASD'_z = ASD_z \cos^2\theta + ASD_y \sin^2\theta \qquad [7]$$

If the original accelerations A_y and A_z are correlated the extreme values of $cos(\phi_y - \phi_z)$ is ±1

Then the Maximum value of $ASD'_y = ASD_y cos^2\theta + ASD_z sin^2\theta + sin2\theta\sqrt{(ASD_y ASD_z)}$ and

Maximum value of $ASD'_z = ASD_z cos^2\theta + ASD_y sin^2\theta + sin2\theta\sqrt{(ASD_y ASD_z)}$ [8]

The following equations pertain to comparison of actual transformed ASDs vs. envelope of original ASDs.

for instance, for the example shown below, If $ASD_y = n$ x ASD_z , ($n>1$, i.e. ASD_y is higher)

from [Eq 8] maximum value of $ASD'_y = (n\ cos^2\theta + sin^2\theta + \sqrt{n}$ x $sin2\theta)\ ASD_z$ [9a]

and maximum value of $ASD'_z = (cos^2\theta + n\ sin^2\theta + \sqrt{n}$ x $sin2\theta)\ ASD_z$ [9b]

Therefore, maximum value of ASD'_y relative to the maximum of ASD_y and ASD_z $(=ASD_y$ in the example below where $ASD_y>ASD_z)$

$= (n\ cos^2\theta + sin^2\theta + \sqrt{n}$ x $sin2\theta))/n$ [10]

The ratio of the Maximum value of ASD'_y relative to the sum of the 2 ASDs is

$= (n\ cos^2\theta + sin^2\theta + \sqrt{n}$ x $sin2\theta))/(n+1)$ [11]

Equations 7 & 8 can be used to determine ASDs in new coordinate system depending on expectations of relative phase. Equations 9-11 can be used to determine overestimation if sum of ASDs or max of the 2 original ASDs is used instead of Equations 7 & 8.

If however, the original accelerations A_y and A_z are uncorrelated (random) the expected value of $\cos(\phi_y - \phi_z)$ is 0, then

Maximum value of ASD'_y relative to the max of ASD_y and $ASD_z = (n \cos^2\theta + \sin^2\theta)/n$ [12]

This represents how much the envelope ($=ASD_y$ for n>1) overestimates the actual ASD

When the two ASDs are equal (n=1), the transformed ASDs will off-course also be the same and this ratio =1. The ratio decreases with increasing n, approaching $\cos^2\theta$ for $n=\propto$ and a lowest value of 0.5 at θ=45◦.

As an example, for n=2, for θ = 30◦, this ratio = 0.875, vs. $\cos^2 30°$ =0.867. *So, when the phase is uncorrelated (random), the maximum of the two original ASDs can significantly overestimate the maximum of the two transformed ASDs (by a factor =1/* $\cos^2\theta$ *, use the smaller angle, i.e.* θ<45*). It is best to use Equation 7 to calculate the ASDs in the transformed coordinate axes.*

Whether or not the original accelerations in Y and Z were correlated depends on the underlying phenomena. For instance, vibration from an engine could be correlated whereas that from road noise could be uncorrelated. In the absence of the original accelerations, a test may be conducted to reveal the correlation. Else, assuming they are corelated will lead to a conservative but overestimated answer.

Numerical Simulations

The objectives of numerical simulations are listed below:

i. To validate the equations provided above to transform ASDs from one coordinate system to another.
ii. To evaluate the extent of overestimation or underestimation if one were to use the sum of ASDs or envelope of original ASDs instead of using the equations provided here.
iii. To further illustrate the value of Equations [10-12] in evaluating how the coordinate transformation angle θ and the ratio of original ASDs n influences the results.

Numerical simulations were conducted with various values of ASDs and relative phase to examine the result depicted in Equations [7] and [8] using the following steps:

- Start with some value of ASD_y and ASD_z in the original coordinate system
- Generate corresponding acceleration time history with random phase A_y and A_z
- Compute the acceleration time history in the transformed coordinates to obtain A'_y & A'_z
- Using A'_y and A'_z compute ASD'_y and ASD'_z in the transformed coordinate system
- Compare the results from those in Equations [8-10].
- Monte-carlo simulations were done by generating various acceleration vs. time histories for the same ASDs with various relative phase. This was done to validate the accuracy of the ASD transformation equations [Eq.7&8] for the uncorrelated and correlated cases.

Example 1: $ASD_y > ASD_z$

ASD_z was obtained by multiplying ASD_y with a random number <1.0 to ensure that $ASD_y > ASD_z$ for all frequencies. Figure 1a shows the ASDs in the original coordinates Y & Z; Figure 1b shows a segment of the corresponding acceleration vs. time. Figure 2 shows that the sum of the ASDs is the same in both coordinate systems (θ=30◦ in this simulation) as proven by [Eq 3c].

Figure 3a shows the maximum possible value (upper bound) of ASD'_y calculated using [Eq.8] (when phase is correlated) along with the original ASD_y and ASD_z. It can be seen that the possible upper bound of ASD_y is significantly higher than the envelope of the original ASDs.

Note that from [Eq. 10], for θ = 30◦, this ratio (*upper bound of* ASD'_y/ASD_y) = 1.87 for n=1 (two ASDs are equal), which is the case at 120Hz. The ratio = 1.49 for n=2 (at 220 Hz). *So, when the phase is correlated, the maximum (envelope) of the two original ASDs is significantly lower than the maximum possible transformed ASD (by up to a factor of 2 for θ = 45◦ when n=1).*

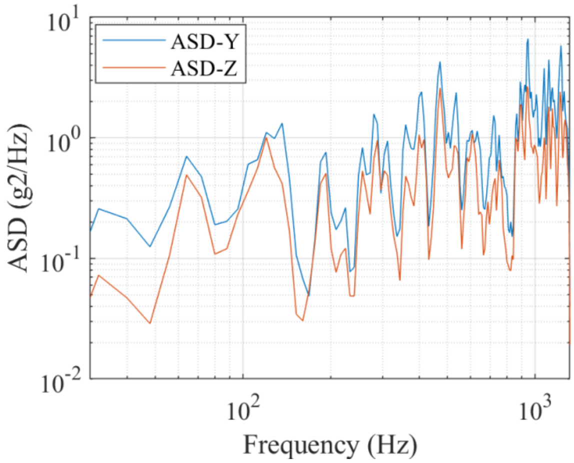

Figure 1a ASDs in Original Coordinates Y&Z

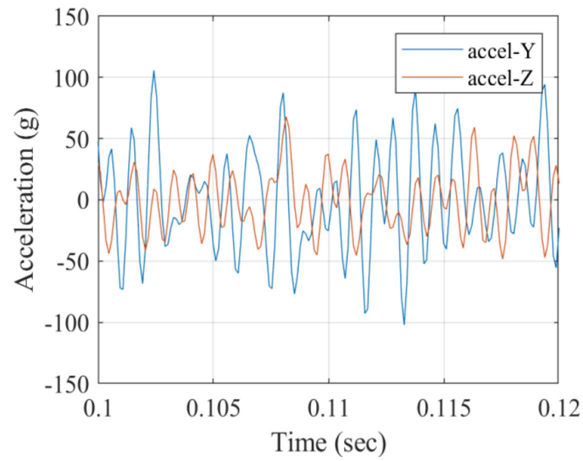

Figure 1b. Corresponding Acceleration vs. time

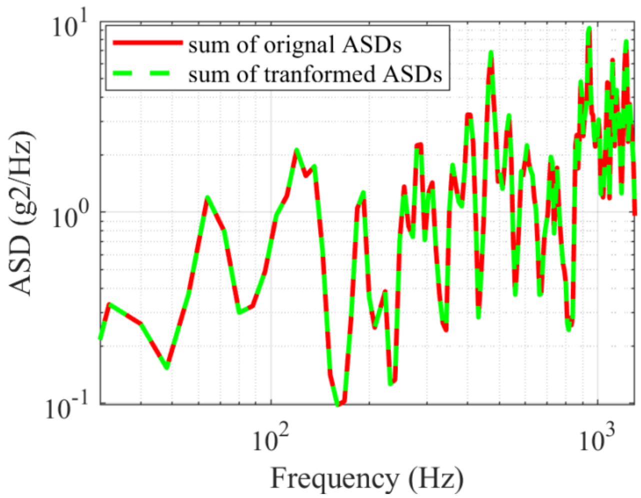

Figure 2. Sum of ASDs in Original and Transformed Coordinates

Figure 3b shows the maximum possible value (upper bound) of ASD_y calculated using [Eq.8] (for the situation when phase is correlated) along with the sum of ASD_y & ASD_z. It can be seen that the possible upper bound of ASD_y is almost the same as the sum of the two ASDs. Note that per [Eq. 11] for θ = 30◦, this ratio = 0.93 for n=1 (two ASDs are equal), and = 0.99 for n=2. *So, when the phase is correlated, the sum of the two original ASDs is always greater than and close to the maximum possible transformed ASD. However, the envelope of original ASDs is unconservative.*

Figure 4a and 4b show the maximum possible value (upper bound) of ASD_y and ASD_z calculated using [Eq.8] (for the situation when phase is correlated) along with the transformed ASD'_y and ASD'_z. It can be seen that the possible upper bound of ASD_y is significantly greater since the actual accelerations in Y and Z were uncorrelated.

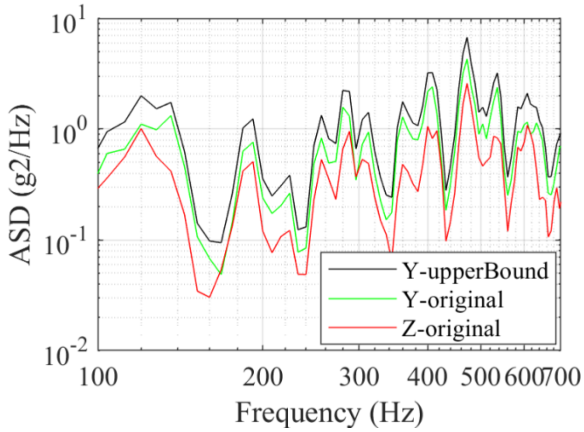

Figure 3a. *ASD'y* Maximum vs. Original ASDs

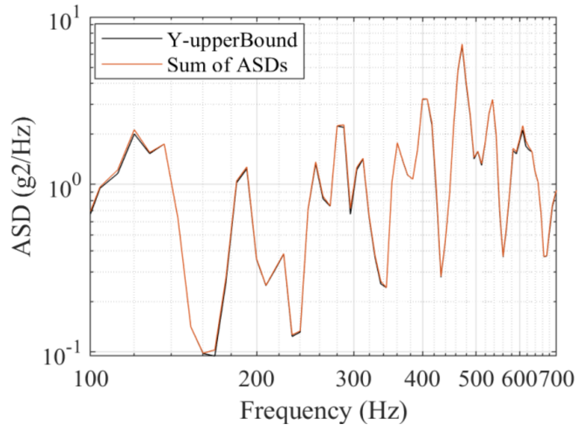

Figure 3b. *ASD'y* Maximum vs. Sum of ASDs

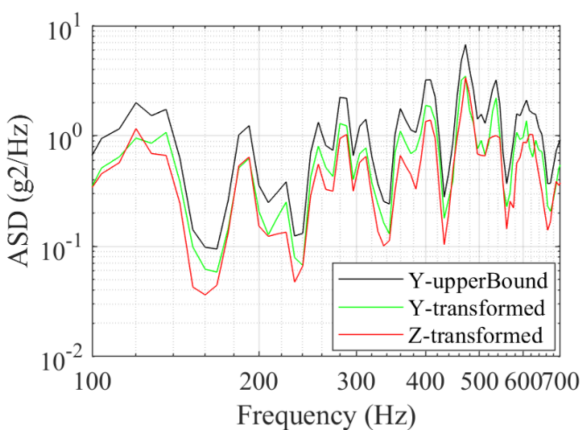

Figure 4a. *ASD'y* Max & Transformed ASDs

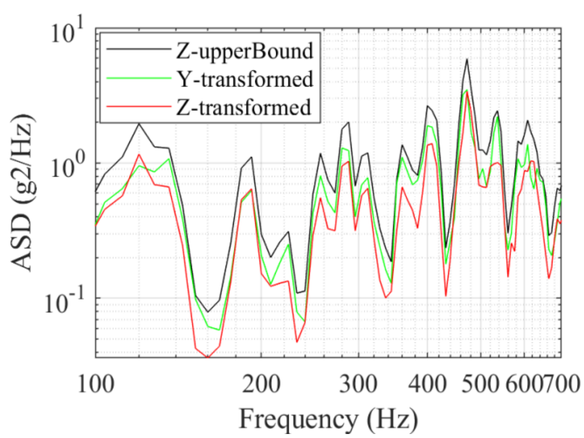

Figure 4b. *ASD'z* Max & Transformed ASDs

<u>Example 2: Monte-Carlo Simulation</u>

The process described above was repeated with 100 acceleration vs. time data generated from the same ASDs. The phase at each frequency is randomly generated, so ϕ_y-ϕ_z remains constant for the entire duration for each run. Sampling rate was 8192 for a total duration of 8.0 sec. The results are shown in Figures 5a and 5b. The black lines are the upper-bound ASDs in the transformed coordinates *Y' & Z'* [Eq. 8], the red lines are a single simulation and the cyan depicts the 100 simulations with random phases. This confirms the upper bound as the upper limit of possible values of the ASDs. Any one of the 100 simulations may touch the black line (upper limit) at a particular frequency when coincidentally the phase in Y and Z are equal (ϕ_y=ϕ_z).

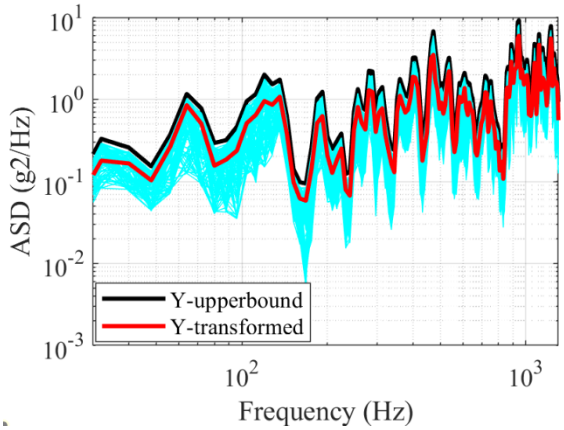

Figure 5a. *ASD'y* Max & 100 Simulations

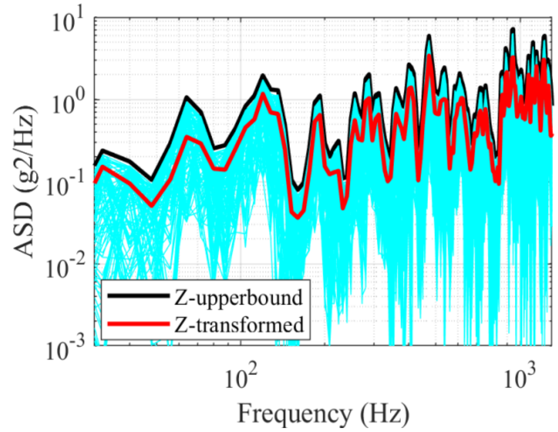

Figure 5b. *ASD'z* Max & 100 Simulations

<u>Example 3: $ASD_y \approx ASD_z$</u>

The goal of this simulation was to examine the overestimation caused when the sum of the ASDs is used instead of the upper limits provided in [Eq.8]. The discussion pertaining to Figure 3b will be further illustrated based on [Eq. 11] with varying values of $n = ASD_y/ASD_z$ at different frequencies and for different values of θ. Figure 6a shows the results of [Eq. 11] (ASD'_y -max)/ sum of ASDs) for various n & θ. As to be expected, the result is symmetric about θ=45∘ and for reciprocal values of n (=0.5 vs. 2.0). So, only the left half of the plot (θ<45∘) is sufficient.

ASD_z was changed by multiplying the previous ASD_z with 2 to make ASD_y more similar in magnitude to the new ASD_z (called *z2*). Figure 6b shows the maximum possible value (upper bound) of ASD_y calculated using [Eq.8] (for the situation when phase is correlated) along with the sum of ASD_y & ASD_z. As expected, based on Figure 6a, at 104Hz, n=1.0 and ratio=0.93, at 120Hz, n=0.68 and ratio=0.88, at 130Hz, n=2.0 and ratio=0.99, at 136Hz, n=0.61 and ratio=0.98. *Therefore, [Eq. 11] or Figure 6a can be used to determine by how much the sum of the two original ASDs exceeds the maximum possible transformed ASDs.*

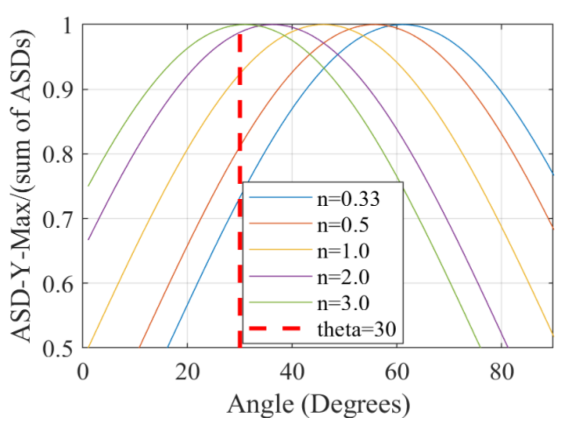

Figure 6a. Ratio of ASD'_y Max to sum of ASDs

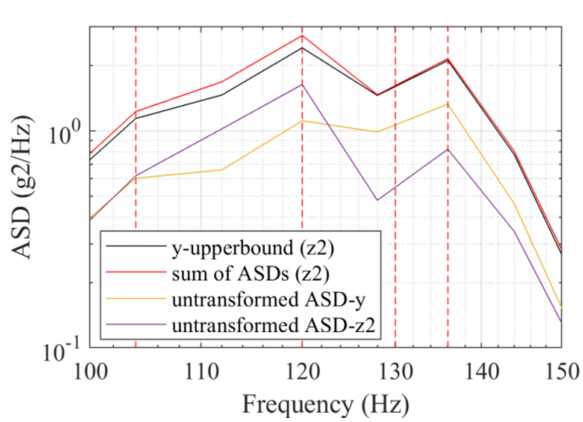

Figure 6b. ASD'_y Max vs. sum of ASDs

Figure 7a and 7b compares an additional 100 simulations with the upper bound of ASD'_y for the uncorrelated case [Eq.7]. The duration each acceleration was increased to 80 seconds to obtain enough data segments for the expected value of $cos\ (\phi_y - \phi_z)$ to reach 0. As a result, the 100 simulations were very similar (the cyan lines are in a very narrow band vs. those in Figures 5a and 5b). *This confirms the accuracy of [Eq.7] for the uncorrelated case for sufficiently long duration. Using the correlated upper bound would result in a factor of 2 (≈3dB) overestimation.*

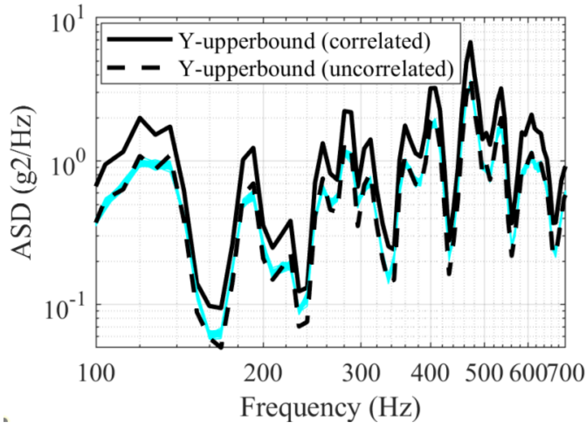

Figure 7a. ASD'_y Max Uncorrelated (<700Hz)

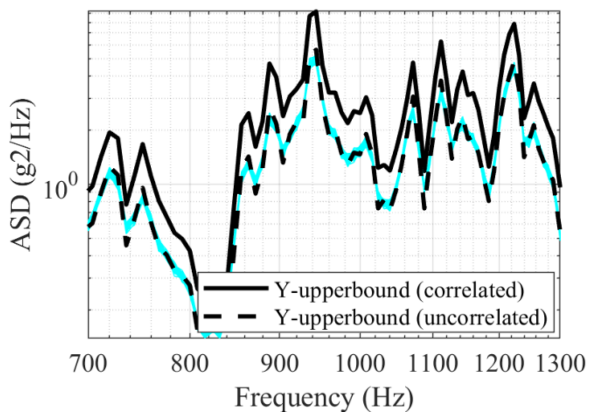

Figure 7b. ASD'_y Max Uncorrelated (>700Hz)

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Conclusions

i. Equations were provided to rotate a pair of orthogonal ASDs from one coordinate system to another if phases of the ASDs are known. [Eq. 6a & 6b].

ii. Equations were provided to calculate maximum envelopes for the ASDs in a rotated coordinate system if phases of the ASDs are assumed to be correlated or uncorrelated. [Eq. 7 & 8].

iii. Equations were provided [Eq. 10-12] to evaluate the extent of overestimation or underestimation if one were to use the sum of ASDs or envelope of original ASDs instead of using the equations provided here. This allows one to evaluate how the coordinate transformation angle θ and the ratio of original ASDs n influences the results.

iv. Numerical simulations were conducted with various values of ASDs and relative phase to validate the various Equations towards their intended applications.