Taking PDE Solutions from Low-Fidelity to High-Fidelity Using Function-on-Function Regression

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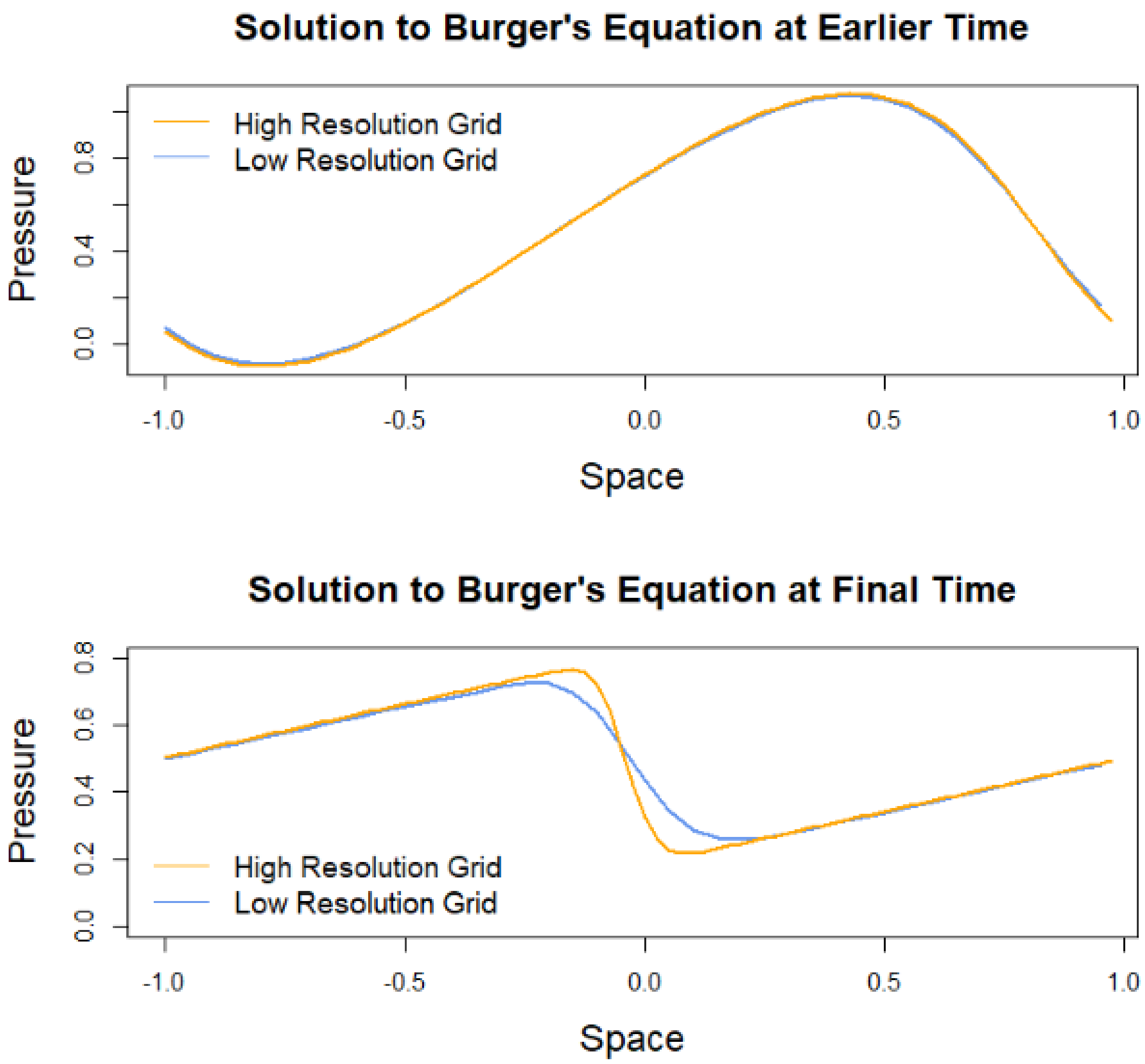
Introduction

- Partial differential equations (PDEs) are used to model complex physical processes
- Burger's equation models phenomena such as wave propagation and shock flows.

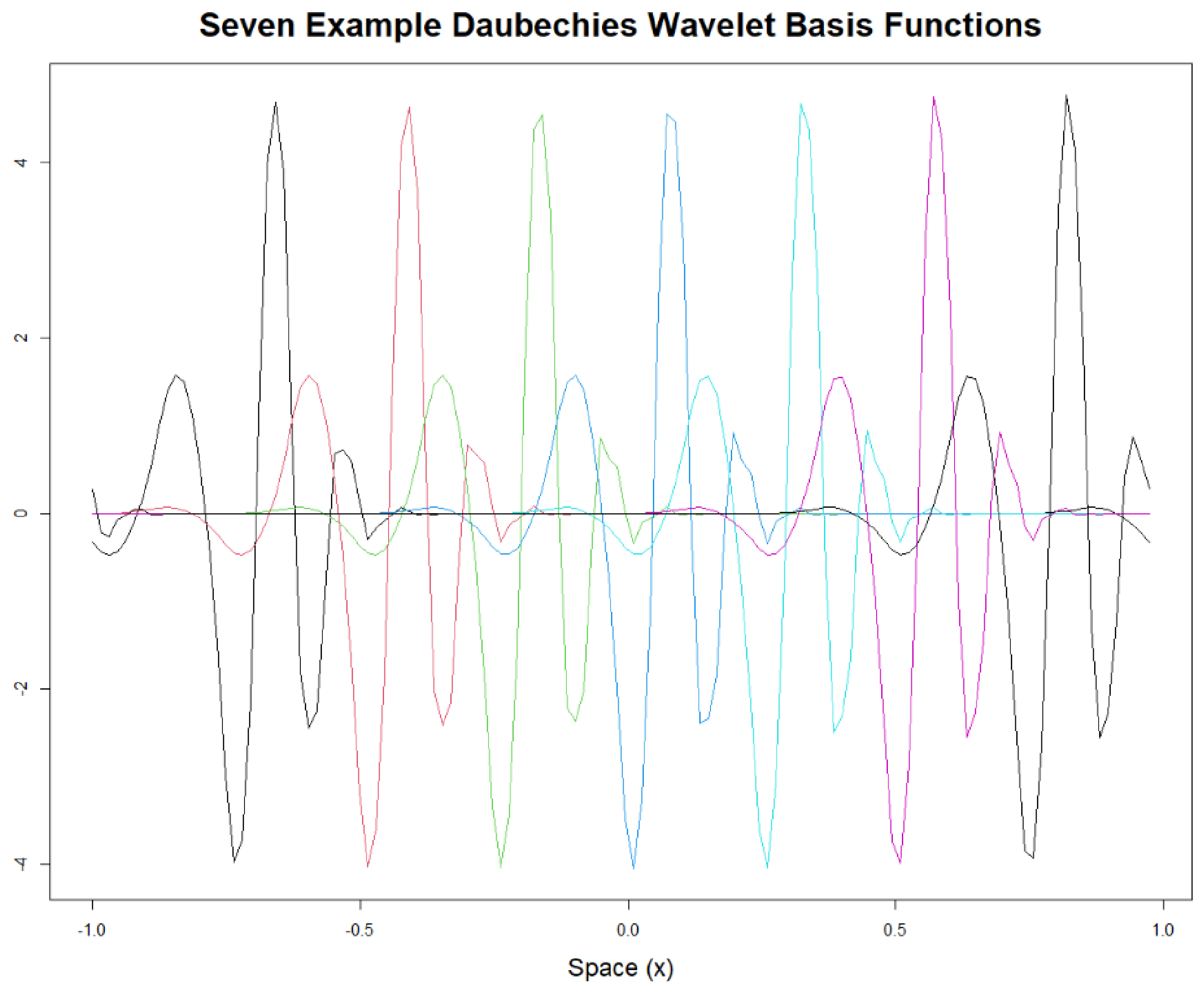
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

- Low fidelity solutions are more computationally efficient, but often misrepresent key features



Methods

$u_{h,t}(x)$ and $u_{l,t}(x)$ represented using wavelet basis functions



Methods

$u_{h,t}(x)$ and $u_{l,t}(x)$ represented using wavelet basis functions

→ Data split into testing and training sets.

→ **Bootstrap**

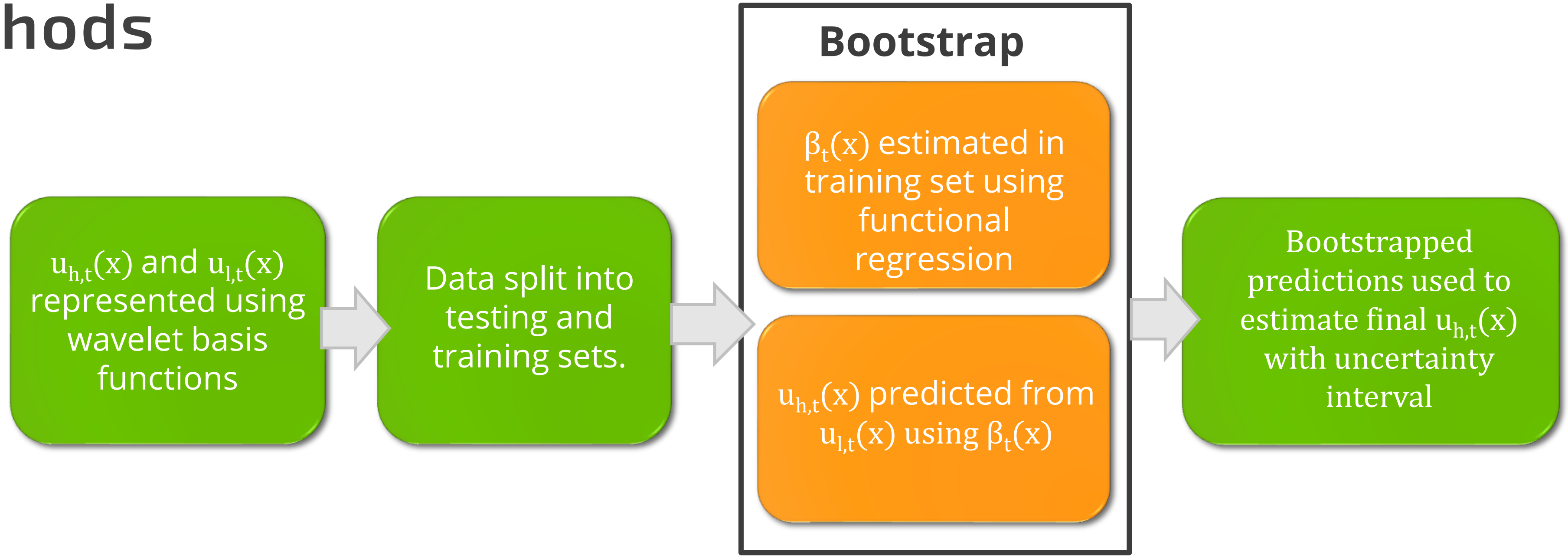
- $\beta_t(x)$ estimated in training set using functional regression
- $u_{h,t}(x)$ predicted from $u_{l,t}(x)$ using $\beta_t(x)$

High fidelity solution — Low fidelity solution — Error function

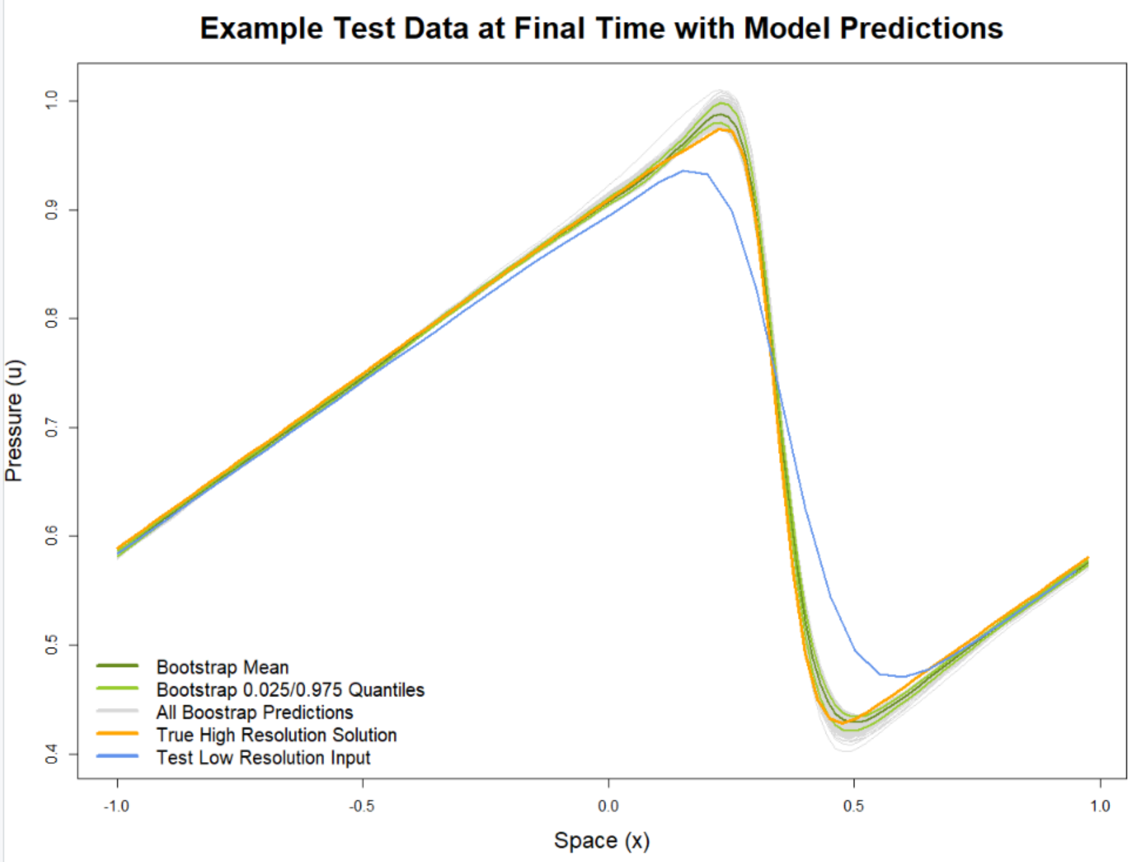
$$u_{h,t}(x) = \alpha_t(x) + \beta_t(x) u_{l,t}(x) + \epsilon_t(x)$$

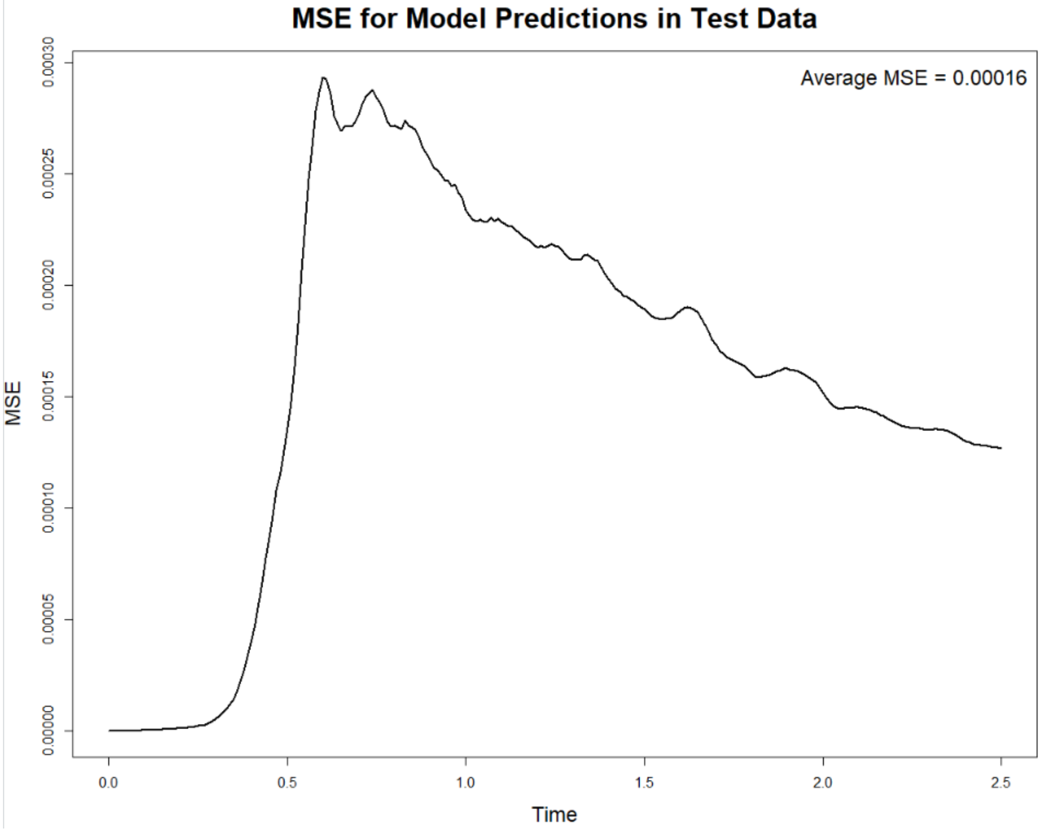
Y-intercept function — Regression coefficient

Methods


$u_{h,t}(x)$ and $u_{l,t}(x)$ represented using wavelet basis functions
Data split into testing and training sets.
Bootstrap
$\beta_t(x)$ estimated in training set using functional regression
$u_{h,t}(x)$ predicted from $u_{l,t}(x)$ using $\beta_t(x)$
Bootstrapped predictions used to estimate final $u_{h,t}(x)$ with uncertainty interval

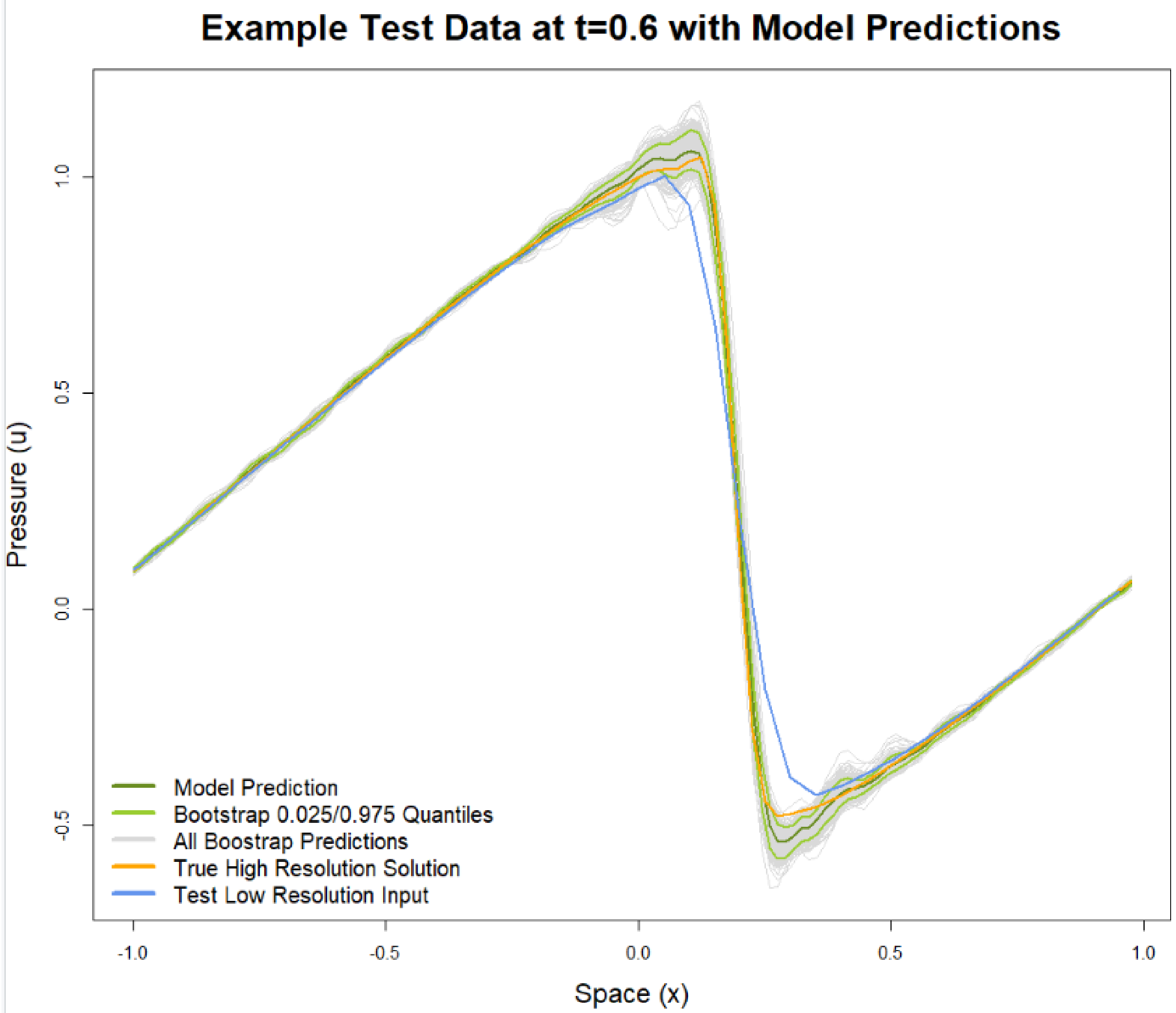
Results


Example Test Data at Final Time with Model Predictions
Pressure (u)
Space (x)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
-1.0
-0.5
0.0
0.5
1.0
Bootstrap Mean
Bootstrap 0.025/0.975 Quantiles
All Boostrap Predictions
True High Resolution Solution
Test Low Resolution Input


MSE for Model Predictions in Test Data
Average MSE = 0.00016
MSE
Time
0.00030
0.00025
0.00020
0.00015
0.00010
0.00005
0.00000
0.0
0.5
1.0
1.5
2.0
2.5

Conclusion

- Function on function regression can be used to predict high fidelity PDE solutions from low fidelity solutions
- Wavelet basis functions can capture abrupt, localized dynamics such as shocks
- Bootstrapping provides uncertainty quantification in recovered high fidelity solutions
- Future work:
  - Additional spatial dimensions
  - Modeling time-dependent dynamics



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