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Taking PDE Solutions from Low-Fidelity to High-Fidelity Using Function-on-Function Regression

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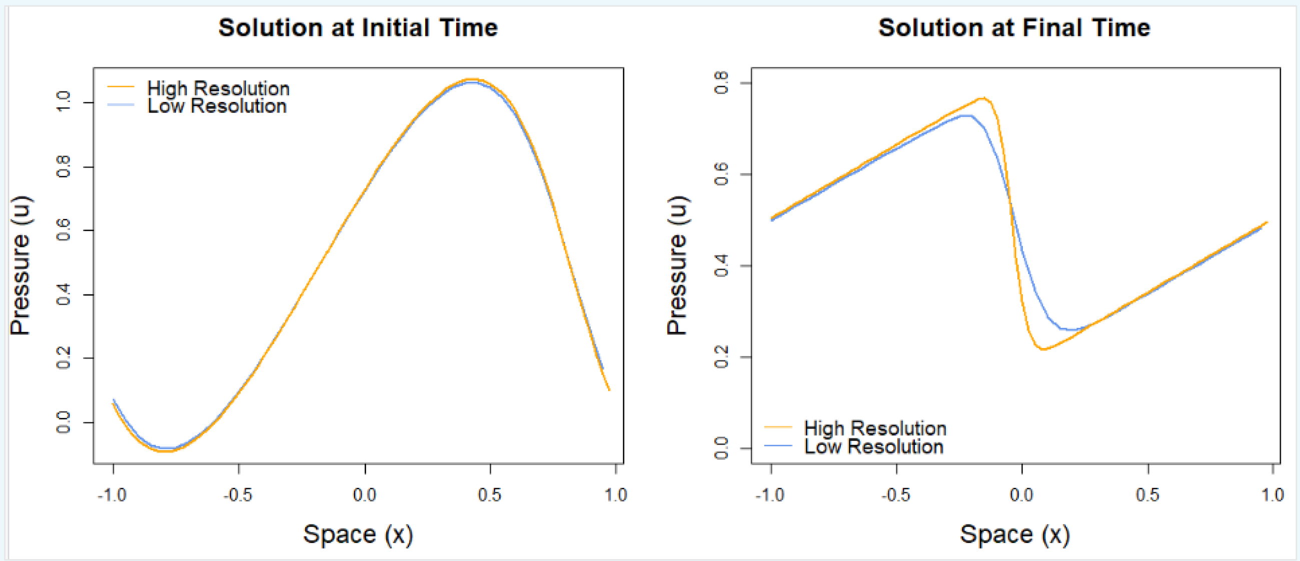
Introduction

Partial differential equations (PDEs) are often used to model physical processes. High fidelity solutions provide accuracy at a large computation cost. Low fidelity solutions have lower computational burden but can misrepresent key features.

1D Inviscid Burger's Equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$



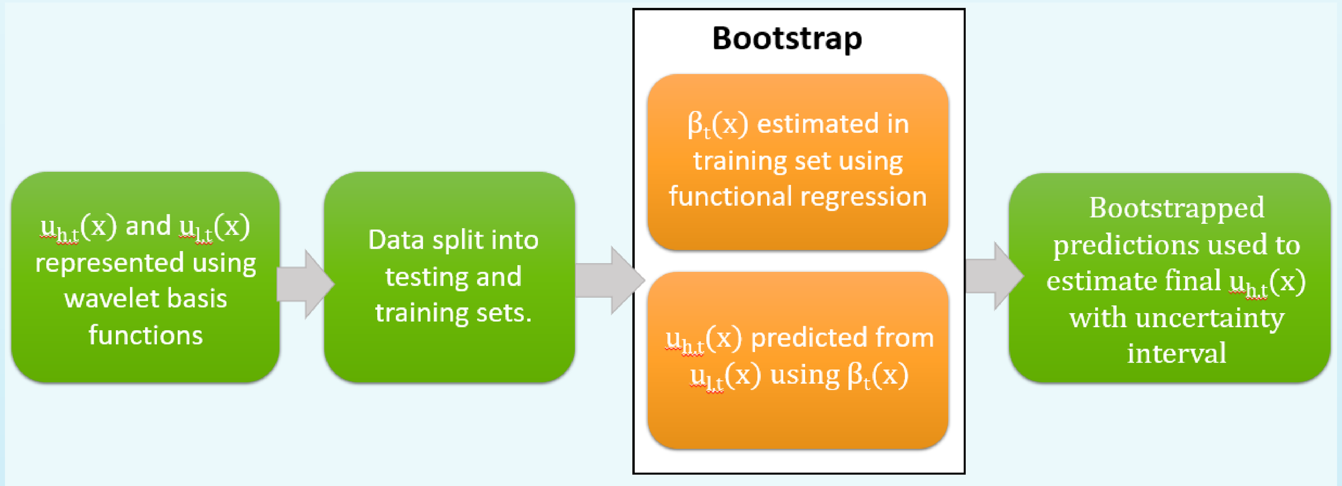
By treating high and low fidelity solutions as functionals we can apply functional regression to predict high fidelity solutions from low fidelity inputs at reduced computational burden.

Methodology

$$u_{h,t}(x) = \alpha_t(x) + \beta_t(x) u_{l,t}(x) + \epsilon_t(x)$$

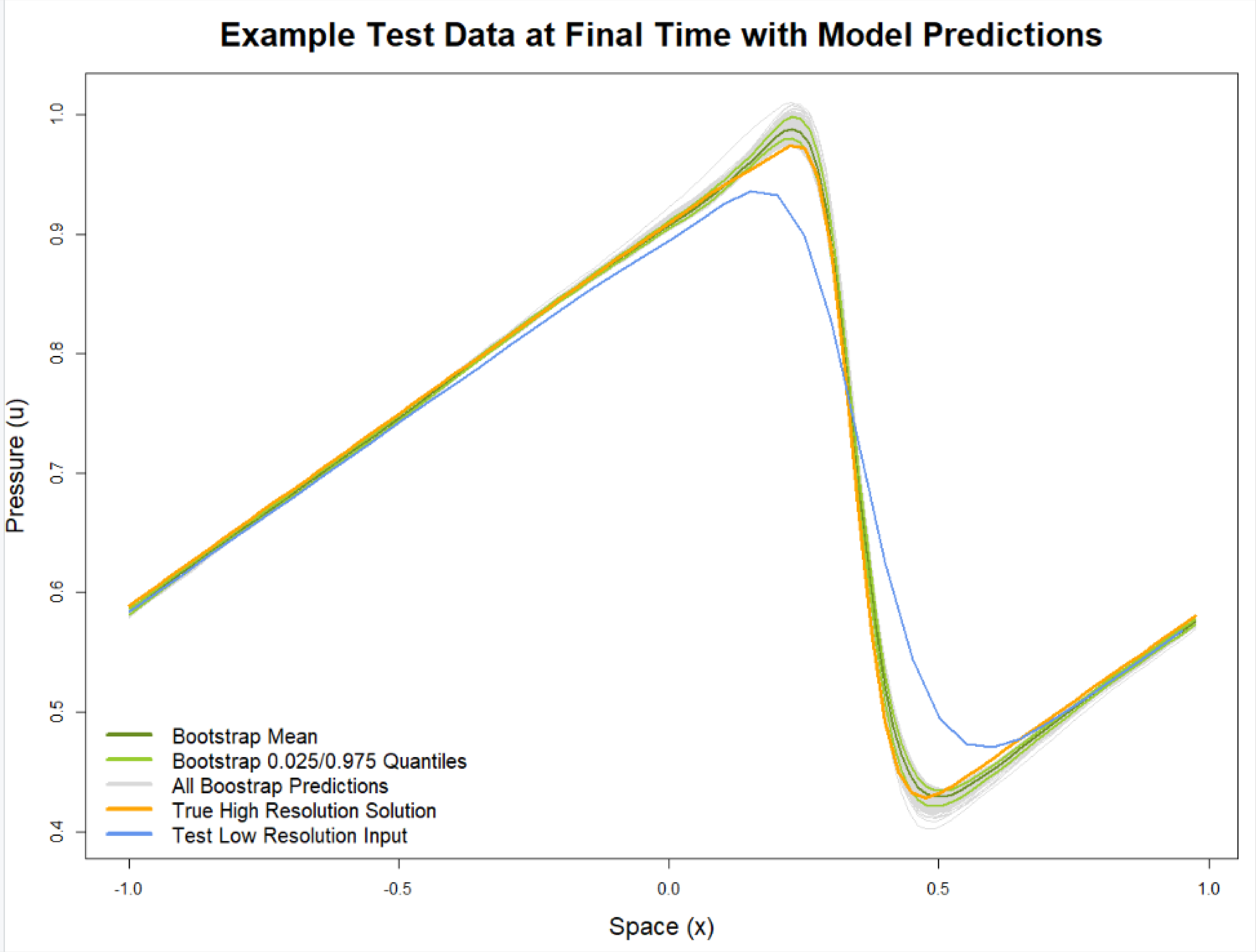
High fidelity solution: $u_{h,t}(x)$; Y-intercept function: $\alpha_t(x)$; Regression coefficient: $\beta_t(x)$; Low fidelity solution: $u_{l,t}(x)$; Error function: $\epsilon_t(x)$

- Burger's equation was solved for 1000 initial pressure profiles $u_0(x)$ in both high and low fidelity training and test data
- The high fidelity solutions had 2x the spatial grid points as the low fidelity solutions
- Daubechies wavelets were used as the basis in the functional representation due to their ability to capture shock features better than traditional splines
- Bootstrap intervals were used to provide uncertainty estimates

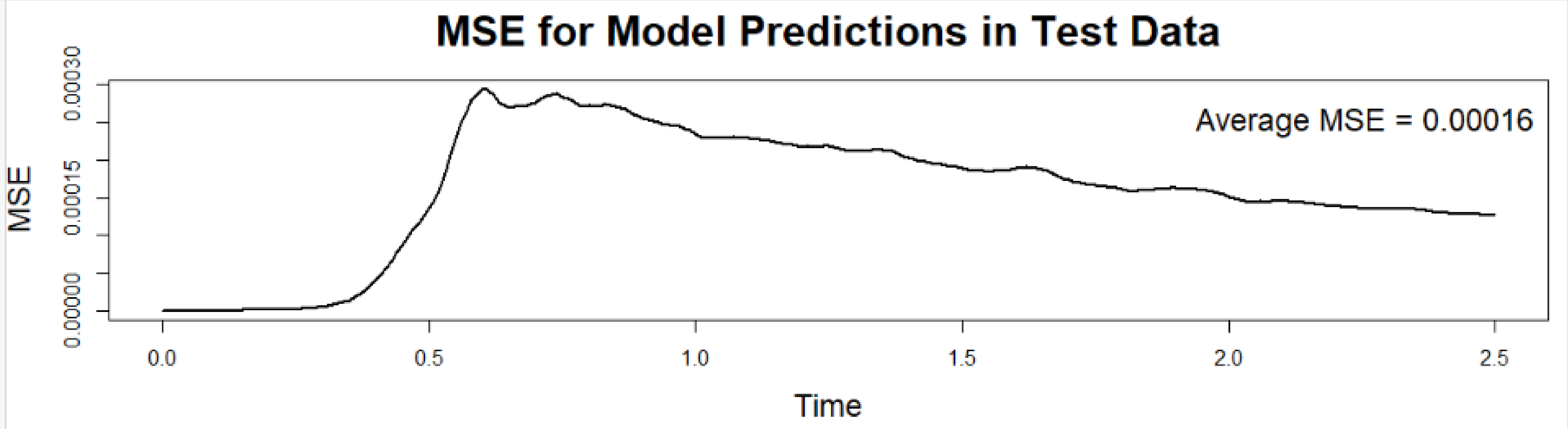


Results

Bootstrap mean and 0.025/0.975 quantiles along with true high fidelity solution and low fidelity test input are plotted below for one initial condition.



Mean squared error (MSE) was used to compare model prediction in the test data to true value and is plotted below at each time point. Model error peaks at t=0.60 where the shock front is the steepest and boundary effects the strongest, then tapers off.



Conclusion

Function-on-function linear regression provides a method to recover high fidelity PDE solutions from low fidelity solutions with uncertainty quantification. This method reduces the computational burden required to produce high fidelity PDE solutions and is generalizable to any 1D PDE.

For PDEs modeling shocks, such as Burger's equation, the wavelet basis functions can be used to capture nearly discontinuous features. Bootstrapping can be used to provide uncertainty quantification. This allows for better risk informed decision making and interpretability.

Future work includes extensions to higher spatial dimensions and modeling time dependence.

References

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