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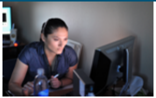
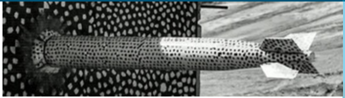
National Laboratories

Non-invasive Regional Multigrid for Semi-structured Grids





Peter Ohm, M. Mayr, L. Berger-Vergiat, R.S. Tuminaro

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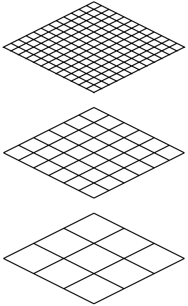
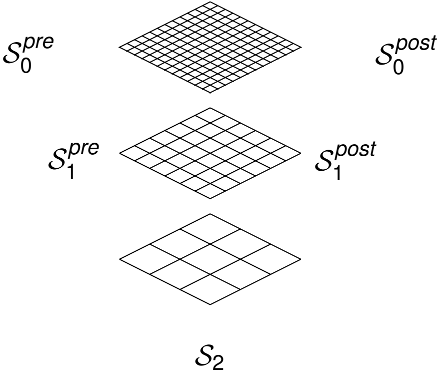
Basic multigrid algorithm



Basic ideas

- Reconstruct the fine level solution from information of coarse representations of the fine problem.

Basic multigrid algorithm



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- Reconstruct the fine level solution from information of coarse representations of the fine problem.
- Apply cheap **smoothers** on each multigrid level.

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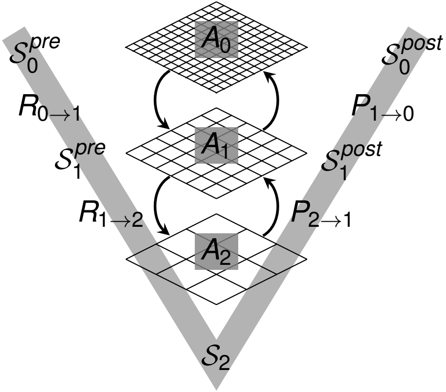

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- **Restriction** and **prolongation operators** transfer information between different multigrid levels.

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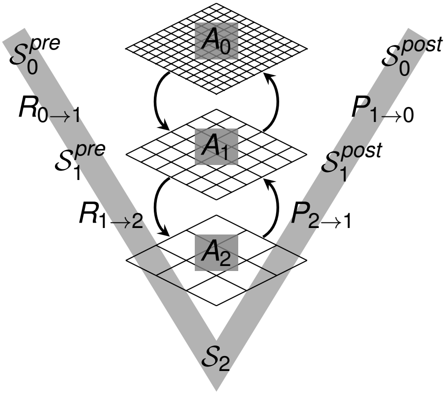

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- Apply cheap **smoothers** on each multigrid level.
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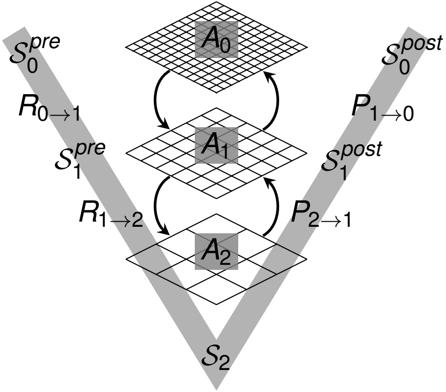
The multigrid method is fully defined by the **level smoothers** and **transfer operators**!

Basic multigrid algorithm

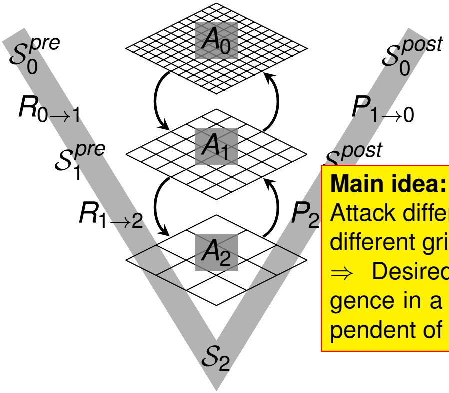


Recursive algorithm

Multigrid(x^k, b^k):

1. If $n < 1000$, use direct solver and return.
2. Presmoothing: Apply $\mathcal{S}_k^{pre}$ on x^k.
3. Transfer residual r^k to next coarser level:
$$r^{k+1} = R_{k\to k+1} r^k$$
4. Call Multigrid(x^{k+1}, r^{k+1}).
5. Transfer correction x^{k+1} to fine grid and add to x^k:
$$x^k = x^k + P_{k+1\to k} x^{k+1}$$
6. Postsmoothing: Apply $\mathcal{S}_k^{post}$ on x^k.

Basic multigrid algorithm



Main idea:
Attack different components of the error on different grids/levels!
$\Rightarrow$ Desired optimal behaviour: convergence in a fixed number of iterations independent of problem size *n*.

Recursive algorithm

Multigrid(x^k, b^k):

1. If $n < 1000$, use direct solver and return.
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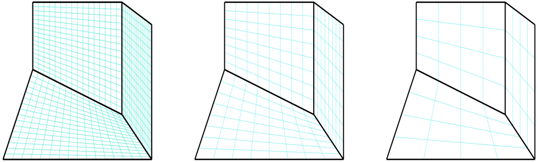
What Is MG-Region?

- Some advances have focused on MG for hybrid hierarchical grids (HHGs).



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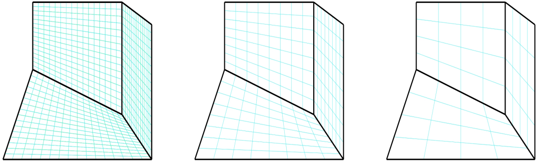
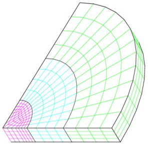
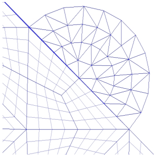
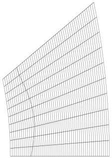
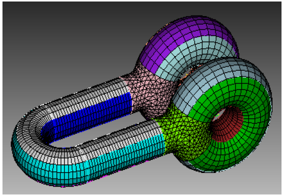


- `MG-Region` extends these ideas to the broader class of semi-structured meshes.



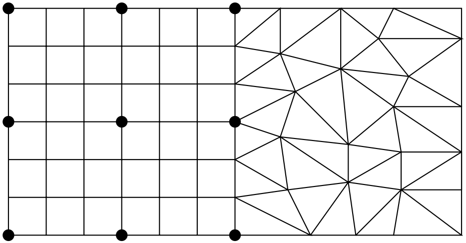






HHG with an unstructured region

Modify the HHG scheme to allow for unstructured regions



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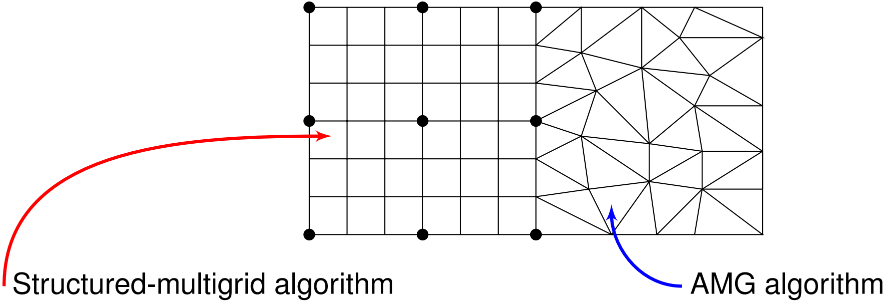

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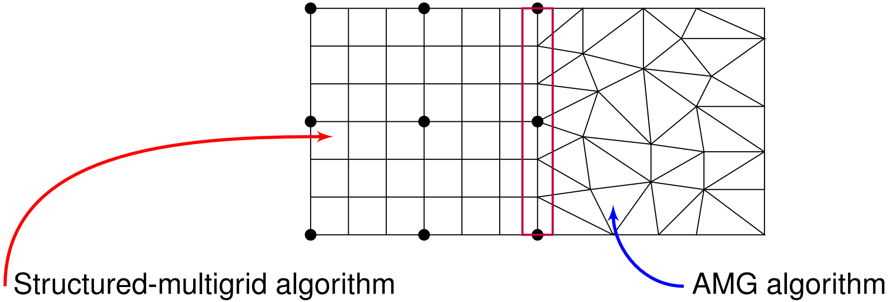

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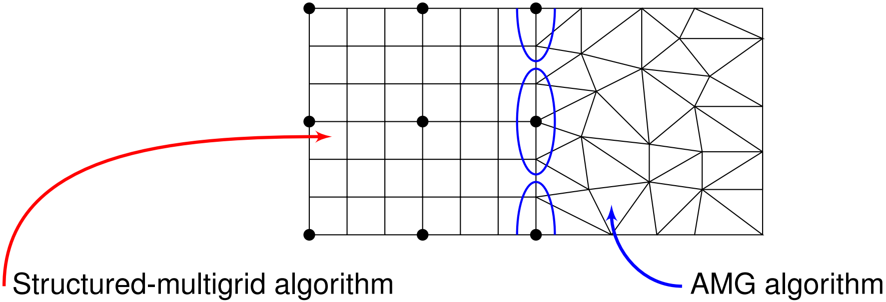

- Need coarse points on interface to agree between regions

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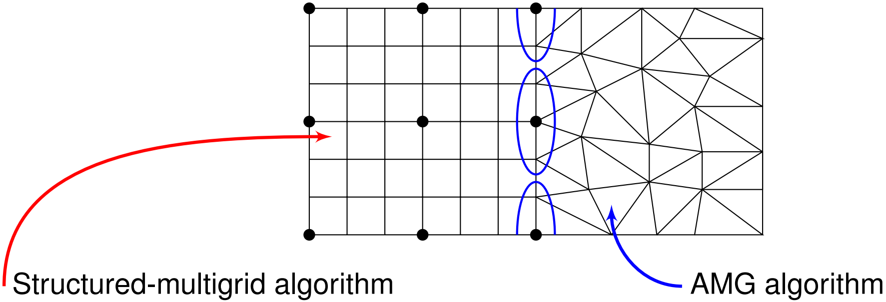
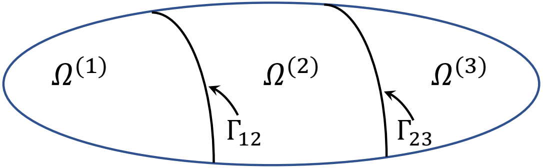


- Need coarse points on interface to agree between regions
- Want sparsity pattern of operators to agree on the interface

A domain decomposition framework

We would like to solve $Au = f$ on Ω.



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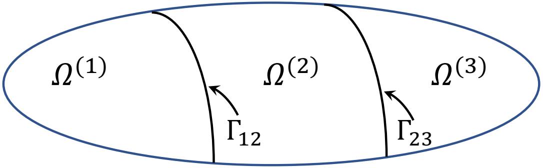

- Write $\Omega = \bigcup_{1 \leq k \leq n} \Omega^{(k)}$, $A = \sum_{1 \leq k \leq n} A^{(k)}$.

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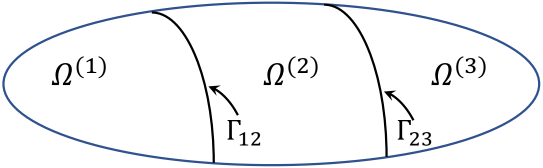
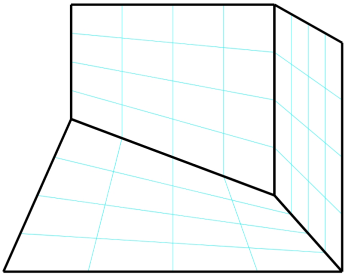


- Write $\Omega = \bigcup_{1 \leq k \leq n} \Omega^{(k)}$, $A = \sum_{1 \leq k \leq n} A^{(k)}$.
- $A^{(k)}$ is 0 for all DOFs outside of domain k.

The region format, $[\![v]\!]$

Convert traditional representations (composite) to region representations.

- $[\![v]\!] = \Psi^T v$.
  Ψ is an $n \times n_r$ boolean transform.



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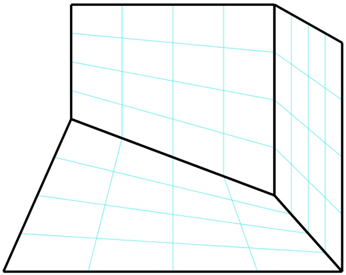

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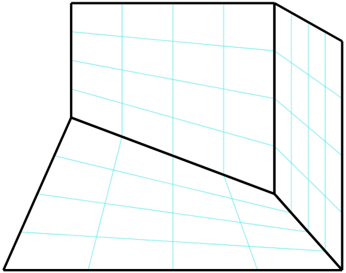
- $\Psi\Psi^T$ counts the number of region DOFs associated with a composite DOF.



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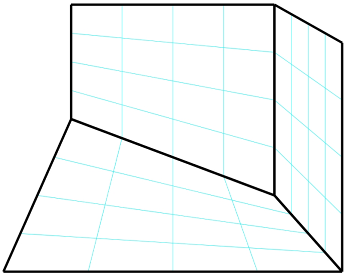
- $\Psi = [\Psi_1, \ldots, \Psi_m]$.



Non-Invasive

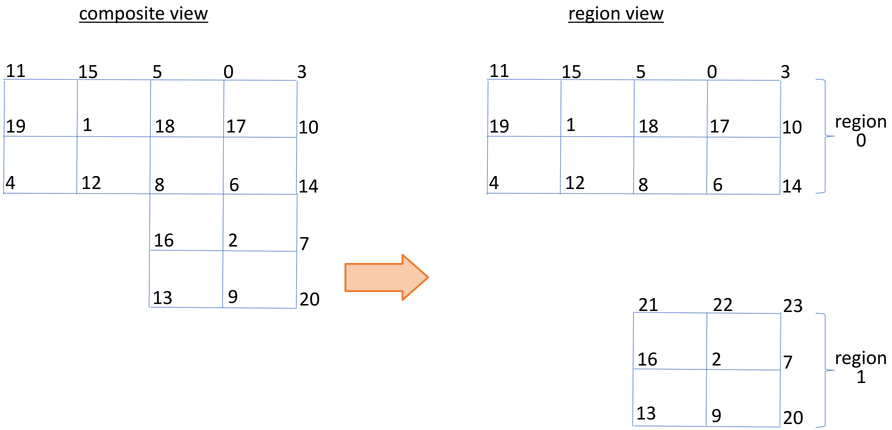


Figure: Sample user-provided mapping of mesh nodes to regions.

Utilizing the region format

Similarly, this may be applied to matrices.

$$[A] = \begin{pmatrix} \Psi_1^T A^{(1)} \Psi_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \Psi_m^T A^{(m)} \Psi_m \end{pmatrix}. \qquad (1)$$

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Lemma 1. Using $[A]$ and Ψ as defined previously, we have

$$\Psi [A] \Psi^T = A. \qquad (2)$$

Utilizing the region format

Let $\bar{\Psi}$ be the coarse version of Ψ. Define

$$[\![P]\!] = \begin{pmatrix} \Psi_1^T P \bar{\Psi}_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \Psi_m^T P \bar{\Psi}_m \end{pmatrix}, \tag{3}$$

$$[\![R]\!] = \begin{pmatrix} \bar{\Psi}_1^T R \Psi_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \bar{\Psi}_m^T R \Psi_m \end{pmatrix}. \tag{4}$$

Grid transfers must match on interfaces.

Utilizing the region format

Lemma 2. Let $[\![P]\!]$ and $[\![R]\!]$ be defined by (3), (4). Then if each row of P does not have nonzeros in multiple region interiors, and if each column of R does not have nonzeros in multiple region interiors,

$$[\![P]\!]\bar{\Psi}^T = \Psi P, \tag{5}$$

$$\bar{\Psi}[\![R]\!] = R\Psi. \tag{6}$$

Main result

Theorem 1. Let $[\![R]\!]$, $[A]$, $[\![P]\!]$ be as defined previously. Then

$$\bar{\Psi}[\![R]\!][A][\![P]\!]\bar{\Psi}^T = RAP. \tag{7}$$

Algorithm

Residual updates:

$$[\![r]\!] = [\![b]\!] - \Psi^T \Psi [A] [\![u]\!], \tag{8}$$

Jacobi smoothing:

$$[\![u]\!] \leftarrow [\![u]\!] + \omega \, [\![\tilde{D}^{-1}]\!] [\![r]\!] \tag{9}$$

Coarsen the regions using a structured algorithm, then either coarsen additionally with AMG or direct solve.

Algorithm

function mgSetup($[A]$) :

$[\![D]\!] \leftarrow \mathrm{diag}(\Psi^T \Psi\ \mathrm{diag}([A]))$

$[\![P]\!] \leftarrow \mathrm{construct}P([A])$

$[\![R]\!] \leftarrow [\![P]\!]^T$

$[\bar{A}] \leftarrow [\![R]\!][A][\![P]\!]$

$\bar{\Psi} \leftarrow \mathrm{inject}(\Psi)$

function mgCycle($[A], [\![u]\!], [\![b]\!]$) :

$[\![u]\!] \leftarrow \mathrm{applySmoother}([\![u]\!], [\![b]\!], [A])$

$[\![r]\!] \leftarrow [\![b]\!] - \Psi^T \Psi [A][\![u]\!]$

$[\![\bar{u}]\!] \leftarrow 0$

$[\![\bar{u}]\!] \leftarrow \mathrm{solve}([\bar{A}], [\![\bar{u}]\!], \bar{\Psi}^T \bar{\Psi} [\![R]\!][\![\Psi\Psi^T]\!]^{-1}[\![r]\!])$

$[\![u]\!] \leftarrow [\![u]\!] + [\![P]\!][\![\bar{u}]\!]$

Results - structured hierarchy match

Structured vs. `MG-Region` (structured), 730 $\times$ 730 **square mesh**
2D Laplace problem with a 7-point stencil, 1pre, 1post

	Jacobi		Gauss–Seidel		Chebyshev	
#its.	Structured	9 Regions	Structured	9 Region	Structured	9 Regions
0	1.00000000e+00	1.00000000e+00	1.00000000e+00	1.00000000e+00	1.00000000e+00	1.00000000e+00
1	1.77885821e-02	1.77885821e-02	1.34144214e-02	1.34395087e-02	1.42870540e-02	1.42868592e-02
2	3.09066249e-03	3.09066249e-03	1.22727384e-03	1.23709339e-03	9.93752447e-04	9.93713870e-04
3	6.17432509e-04	6.17432509e-04	1.27481334e-04	1.29627870e-04	1.21921975e-04	1.21914771e-04
4	1.29973612e-04	1.29973612e-04	1.41133381e-05	1.45165400e-05	1.58413729e-05	1.58401012e-05
5	2.81812370e-05	2.81812370e-05	1.61878817e-06	1.69088891e-06	2.11105538e-06	2.11083642e-06
6	6.22574415e-06	6.22574415e-06	1.89847271e-07	2.02561731e-07	2.86037857e-07	2.86000509e-07
7	1.39312700e-06	1.39312700e-06	2.26276959e-08	2.48757453e-08	3.92564304e-08	3.92500462e-08
8	3.14666393e-07	3.14666393e-07	2.73250326e-09	3.13452182e-09	5.44989750e-09	5.44879379e-09
9	7.15836477e-08	7.15836477e-08	3.33798476e-10	4.06768456e-10	7.65555357e-10	7.65361045e-10
10	1.63770972e-08	1.63770972e-08	4.12201997e-11	5.46524944e-11	1.08974518e-10	1.08939546e-10
11	3.76413472e-09	3.76413472e-09	5.14512205e-12	7.64221900e-12	1.57581213e-11	1.57516868e-11
12	8.68493274e-10	8.68493274e-10	6.49387222e-13	1.11538919e-12	2.32197807e-12	2.32077246e-12
13	2.01044350e-10	2.01044350e-10		1.69735837e-13	3.49742848e-13	3.49514354e-13
14	4.66714466e-11	4.66714466e-11				
15	1.08616953e-11	1.08616953e-11				
16	2.53347464e-12	2.53347464e-12				
17	5.92132868e-13	5.92132868e-13				

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3	6.17432509e-04	6.17432509e-04	1.27481334e-04	1.29627870e-04	1.21921975e-04	1.21914771e-04
4	1.29973612e-04	1.29973612e-04	1.41133381e-05	1.45165400e-05	1.58413729e-05	1.58401012e-05
5	2.81812370e-05	2.81812370e-05	1.61878817e-06	1.69088891e-06	2.11105538e-06	2.11083642e-06
6	6.22574415e-06	6.22574415e-06	1.89847271e-07	2.02561731e-07	2.86037857e-07	2.86000509e-07
7	1.39312700e-06	1.39312700e-06	2.26276959e-08	2.48757453e-08	3.92564304e-08	3.92500462e-08
8	3.14666393e-07	3.14666393e-07	2.73250326e-09	3.13452182e-09	5.44989750e-09	5.44879379e-09
9	7.15836477e-08	7.15836477e-08	3.33798476e-10	4.06768456e-10	7.65555357e-10	7.65361045e-10
10	1.63770972e-08	1.63770972e-08	4.12201997e-11	5.46524944e-11	1.08974518e-10	1.08939546e-10
11	3.76413472e-09	3.76413472e-09	5.14512205e-12	7.64221900e-12	1.57581213e-11	1.57516868e-11
12	8.68493274e-10	8.68493274e-10	6.49387222e-13	1.11538919e-12	2.32197807e-12	2.32077246e-12
13	2.01044350e-10	2.01044350e-10		1.69735837e-13	3.49742848e-13	3.49514354e-13
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`MG-Region` and composite structured hierarchies are identical, residuals match*

*GS and Cheby have slight implementation differences for composite/`MG-Region`

Results - hierarchy mismatch

Structured vs. `MG-Region` (structured), 700 × 720 **rectangular mesh**
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2	3.09747239e-03	3.08750444e-03	1.22692052e-03	1.22958855e-03	7.39937462e-04	7.40632616e-04
3	6.17958674e-04	6.15974350e-04	1.27486109e-04	1.28178073e-04	7.93385189e-05	7.96677401e-05
4	1.29899263e-04	1.29526261e-04	1.41232878e-05	1.42759476e-05	9.07488160e-06	9.15976761e-06
5	2.81258416e-05	2.80574257e-05	1.62135195e-06	1.65159920e-06	1.06848944e-06	1.08744092e-06
6	6.20516379e-06	6.19293768e-06	1.90317494e-07	1.95946605e-07	1.28512584e-07	1.32547397e-07
7	1.38672740e-06	1.38463243e-06	2.27023402e-08	2.37209815e-08	1.57501731e-08	1.65970557e-08
8	3.12830389e-07	3.12499063e-07	2.74346365e-09	2.92685712e-09	1.96757719e-09	2.14457022e-09
9	7.10802795e-08	7.10369390e-08	3.35333590e-10	3.68648440e-10	2.51098105e-10	2.87885059e-10
10	1.62430334e-08	1.62406131e-08	4.14287275e-11	4.75775741e-11	3.28456275e-11	4.04047421e-11
11	3.72913854e-09	3.73041913e-09	5.17289942e-12	6.32676763e-12	4.41956012e-12	5.94633832e-12
12	8.59490959e-10	8.60260047e-10	6.53051744e-13	8.72272622e-13	6.13278643e-13	9.15663966e-13
13	1.98754318e-10	1.99060540e-10				
14	4.60939586e-11	4.62011981e-11				
15	1.07170764e-11	1.07525542e-11				
16	2.49746150e-12	2.50886608e-12				
17	5.83206191e-13	5.86815903e-13				

Results - hierarchy mismatch

Structured vs. `MG-Region` (structured), 700 $\times$ 720 **rectangular mesh**
2D Laplace problem with a 7-point stencil, 1pre, 1post

	Jacobi		Gauss–Seidel		Chebyshev	
#its.	Structured	9 Regions	Structured	9 Region	Structured	9 Regions
0	1.00000000e+00	1.00000000e+00	1.00000000e+00	1.00000000e+00	1.00000000e+00	1.00000000e+00
1	1.78374178e-02	1.77971728e-02	1.34028366e-02	1.34057178e-02	1.26092241e-02	1.25980465e-02
2	3.09747239e-03	3.08750444e-03	1.22692052e-03	1.22958855e-03	7.39937462e-04	7.40632616e-04
3	6.17958674e-04	6.15974350e-04	1.27486109e-04	1.28178073e-04	7.93385189e-05	7.96677401e-05
4	1.29899263e-04	1.29526261e-04	1.41232878e-05	1.42759476e-05	9.07488160e-06	9.15976761e-06
5	2.81258416e-05	2.80574257e-05	1.62135195e-06	1.65159920e-06	1.06848944e-06	1.08744092e-06
6	6.20516379e-06	6.19293768e-06	1.90317494e-07	1.95946605e-07	1.28512584e-07	1.32547397e-07
7	1.38672740e-06	1.38463243e-06	2.27023402e-08	2.37209815e-08	1.57501731e-08	1.65970557e-08
8	3.12830389e-07	3.12499063e-07	2.74346365e-09	2.92685712e-09	1.96757719e-09	2.14457022e-09
9	7.10802795e-08	7.10369390e-08	3.35333590e-10	3.68648440e-10	2.51098105e-10	2.87885059e-10
10	1.62430334e-08	1.62406131e-08	4.14287275e-11	4.75775741e-11	3.28456275e-11	4.04047421e-11
11	3.72913854e-09	3.73041913e-09	5.17289942e-12	6.32676763e-12	4.41956012e-12	5.94633832e-12
12	8.59490959e-10	8.60260047e-10	6.53051744e-13	8.72272622e-13	6.13278643e-13	9.15663966e-13
13	1.98754318e-10	1.99060540e-10				
14	4.60939586e-11	4.62011981e-11				
15	1.07170764e-11	1.07525542e-11				
16	2.49746150e-12	2.50886608e-12				
17	5.83206191e-13	5.86815903e-13				

`MG-Region` and composite structured hierarchies are **not** identical due to region interfaces not matching refinement. Residuals still close.

Results - timing comparison

Mesh nodes	n^{proc}	L L_r/L_c (L)	Region MG #its	Region MG Setup	Region MG V-cycle	Pure AMG #its	Pure AMG Setup	Pure AMG V-cycle
82^3	27	3/2 (3)	13	0.0728 s	0.193 s	13	0.117 s	0.242 s
163^3	216	3/2 (4)	13	0.104 s	0.241 s	13	0.176 s	0.273 s
325^3	1728	3/3 (5)	13	0.352 s	0.428 s	13	0.581 s	0.400 s
622^3	12167	3/3 (6)	13	0.386 s	0.425 s	13	0.711 s	0.423 s

Table: Example 2: `MG-Region` vs AMG for three-dimensional Poisson

Mesh nodes	n^{proc}	#levels L_r/L_c (L)	Region MG #its	Region MG Setup	Region MG V-cycle	Pure AMG #its	Pure AMG Setup	Pure AMG V-cycle
82^3	27	3/2 (4)	22	0.333 s	1.94 s	35	2.46 s	4.23 s
163^3	216	3/3 (5)	21	0.423 s	1.97 s	33	2.78 s	4.34 s
325^3	1728	3/3 (5)	21	0.697 s	2.38 s	32	3.54 s	4.92 s
622^3	12167	3/4 (6)	20	1.199 s	2.63 s	32	3.92 s	5.06 s

Table: Example 2: `MG-Region` vs AMG for three-dimensional elasticity

Semi-structured aggregate visualization

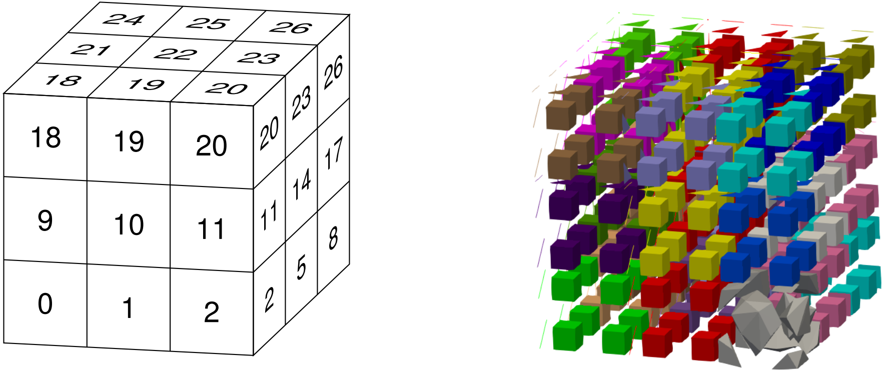
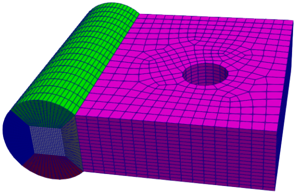


Figure: Visualization of a 3 $\times$ 3 $\times$ 3 region layout on a cube and an example of the region aggregates, with region 2 unstructured.

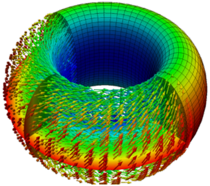
Semi-structured Plate with Hole



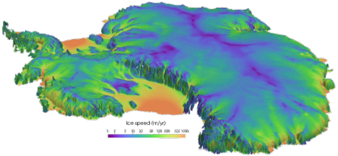
- Structured cylinder mesh (5 regions)
- Unstructured plate with hole mesh (1 region)
- Iteration counts (*its*) for solving a 3D Laplace equation
- conjugate gradient preconditioned with `MG-Region` V-cycle
- Two sweeps Summetric Gauss–Seidel relaxation

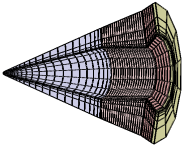
	MESH SIZE		
# levels	35910 (approx. 18^3 per region)	244290 (approx. 34^3 per region)	1777644 (approx. 66^3 per reigon)
2	16 its	18 its	22 its
3	21 its	24 its	27 its
4	25 its	33 its	43 its

Conclusion

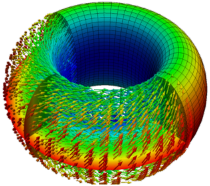


- `MG-Region` for semi-structured meshes
- Extension of HHG to unstructured regions
- Mayr, Berger-Vergiat, O., Tuminaro, *Non-Invasive Multigrid for Semi-structured Grids*, SISC, Accepted.





Conclusion



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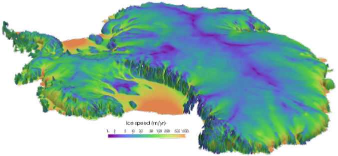

Future Work

- Integration with mesh refinement packages
- Further implementation optimization
- Apply to real-world applications