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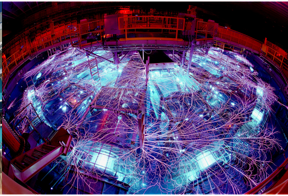

Randomized Spectral Graph Partitioning

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July 11, 2022

U.S. DEPARTMENT OF ENERGY
NNSA National Nuclear Security Administration

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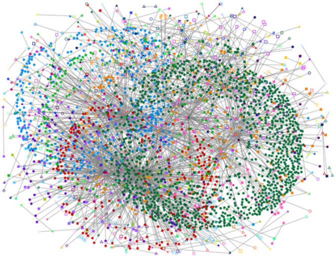
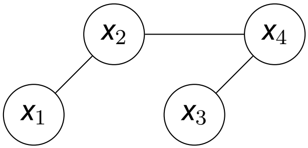
Introduction

- A graph is a mathematical representation of a network.
- Large/complicated graphs are difficult to analyze.
- The partitioning of a graph allows for easier analysis & gives better load balancing in parallel computing [1].



https://homepages.inf.ed.ac.uk/hsun4/talks.html

Graph Laplacian

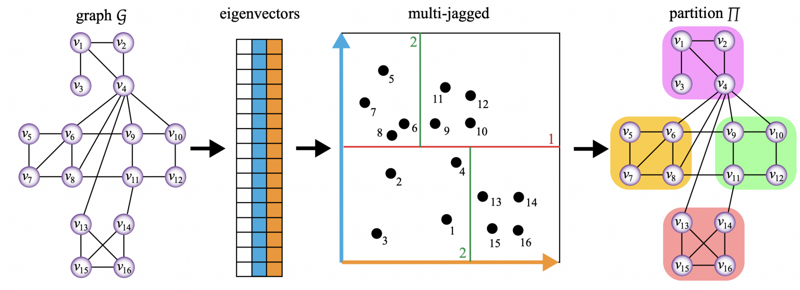


$$D - A = L$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

Spectral Partitioning Big Idea

Spectral graph partitioning was proposed by Pothen, Simon, and Liu (1990). We use the Sphynx package in Zoltan2 [2].



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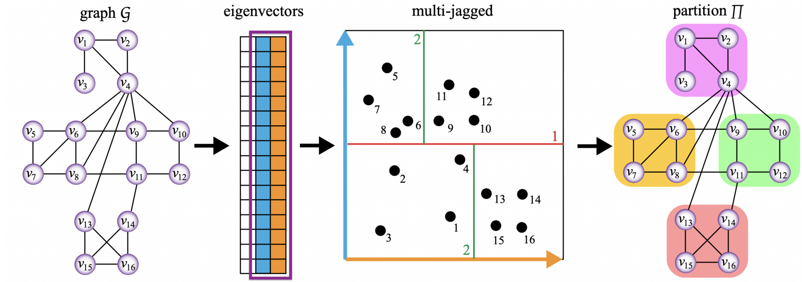

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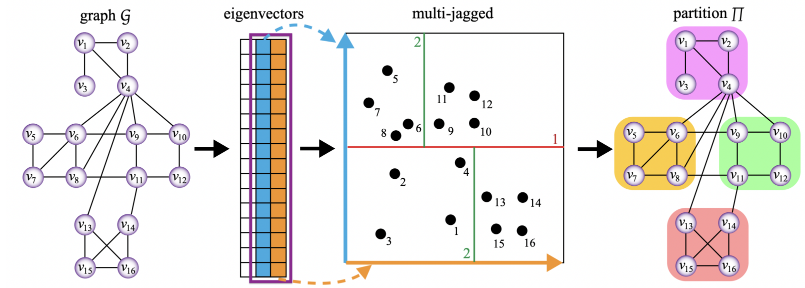

Eigenvector Computation

Computing eigenvectors for Sphynx spectral partitioning:

1. Take adjacency matrix of the graph.
2. Compute graph Laplacian $L = D - A$.
3. Compute eigenvectors ($Lx = \lambda x$).

We test two different methods with randomization for 3:

1. LOBPCG [3][4] with randomized Cholesky preconditioning.
2. A randomized eigenvector approximation method.

Method 1:

Part 1: LOBPCG with Randomized Preconditioning

- Jacobi Preconditioning
- Incomplete Cholesky (ichol) Preconditioning
- Randomized Cholesky (rchol) Preconditioning

(For this method, we only tested the convergence of LOBPCG. Did not plug eigenvector results into Sphynx partitioner.)

Randomized Cholesky (rchol) Method

LOBPCG is a known eigensolver.

Preconditioners replace original problem with a related one.

Cholesky Factorization decomposes a matrix

$$A = GG^T = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{12} & g_{22} & 0 \\ g_{13} & g_{21} & g_{33} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ 0 & g_{22} & g_{21} \\ 0 & 0 & g_{33} \end{bmatrix}$$

Randomized Cholesky Factorization was proposed by Chen, Liang, and Biros (2020). [5].

Test Matrices

Brick 3D 100 Matrix

- Synthetic discretization of the Laplacian on 27-point stencil
- Known Pattern and Structured Matrix (1,000,000 vertices)

Hollywood-2009 Matrix

- Unstructured Matrix with unknown properties ($\sim$1,070,000 vertices)
- Hollywood movie actor network from SuiteSparse Database

Rchol Results - Brick 3D 100 Matrix

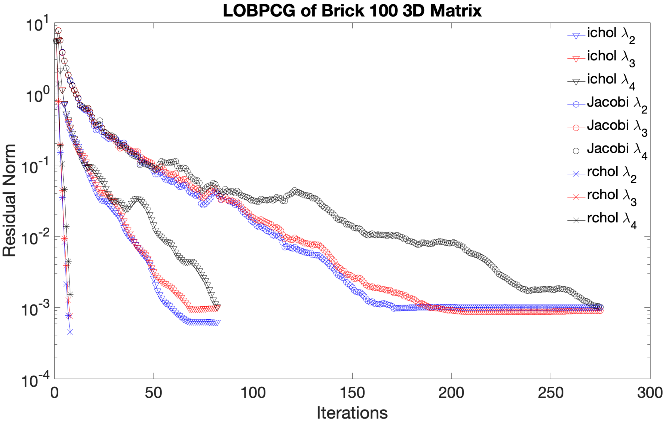

Figure: 100 3D Brick Matrix LOBPCG for 3 eigenvalues with varying preconditioners

Rchol Results - Brick 3D 100 Matrix

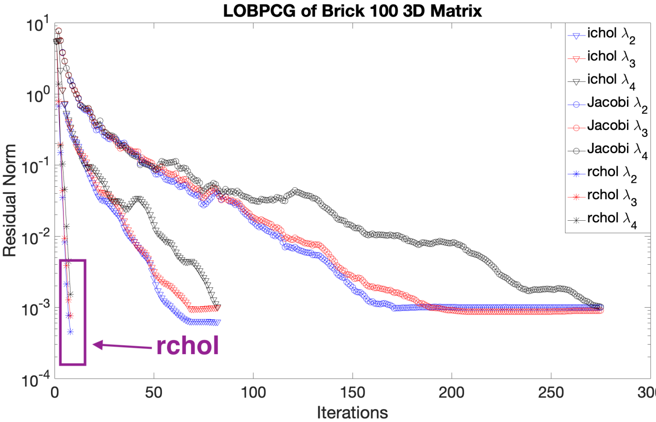

Figure: 100 3D Brick Matrix LOBPCG for 3 eigenvalues with varying preconditioners

Rchol Results - Hollywood Matrix

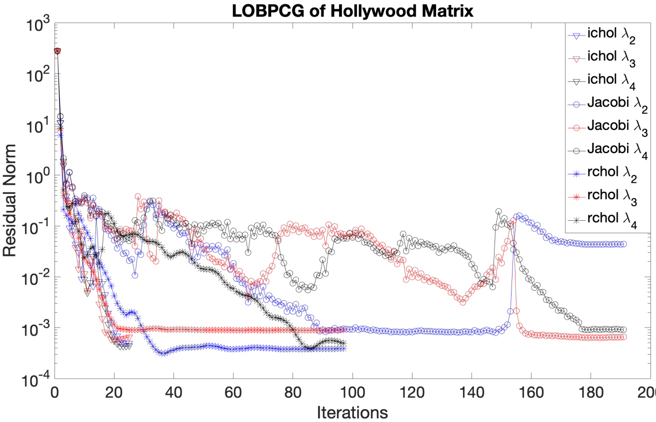

Figure: Hollywood LOBPCG for 3 eigenvalues with varying preconditioners

Rchol Results - Hollywood Matrix

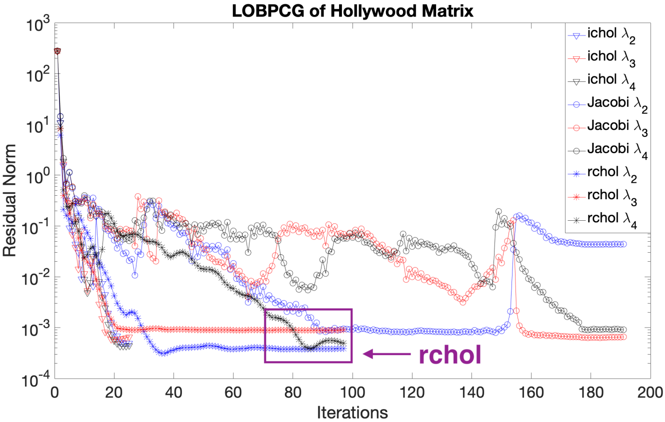

Figure: Hollywood LOBPCG for 3 eigenvalues with varying preconditioners

Method 2:

Part 2: Randomized Eigenvector Approximation

- Randomized Eigenvector approximation method
- LOBPCG & Jacobi Preconditioning

Randomized Eigenvector Approximation Method

Recall

$$L = D - A$$

Define

$$\hat{A} = D^{-1/2} A D^{1/2}$$

The randomized eigensolver has two main parts:

- **Part A:** Compute orthogonal basis Q of the range of $\hat{A}$.
- **Part B:** Solve projected eigenvalue system, and project back.

Randomized Eigenvector Approximation Method

Part A: Compute orthogonal basis Q of the range of $\hat{A}$.

1. Draw a random Gaussian matrix, $\Omega \in \mathbb{R}^{nxl}$
2. Form $Y = \hat{A}^q \Omega$.
3. Compute the skinny QR factors: $QR = Y$.

Randomized Eigenvector Approximation Method

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Part B: Solve the same projected eigenvalue system, and project back.

1. Compute projection $B = Q^T \hat{A} Q$.
2. Solve eigenproblem $B = V \Lambda V^T$.
3. Project back to large system $U = QV$.

Partitioning Results - Brick 3D100 Matrix

Recall $Y = \hat{A}^q \Omega$

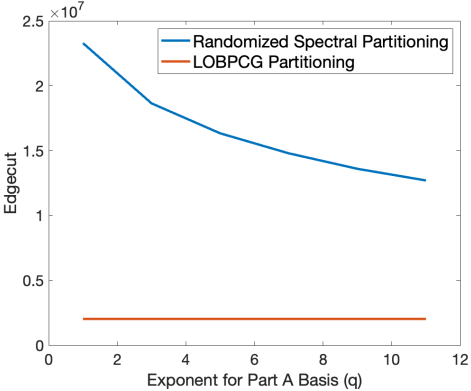
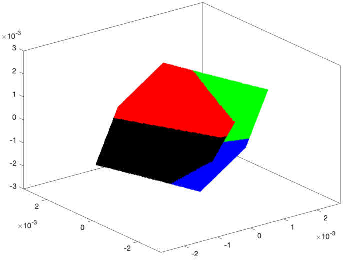
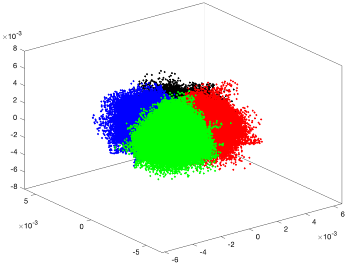


Figure: Brick3D100 Matrix Results

Partitioning Results - Brick 3D100 Matrix



(a) LOBPCG



(b) Randomized Spectral

Partitioning Results - Hollywood Matrix

Recall $Y = \hat{A}^q \Omega$

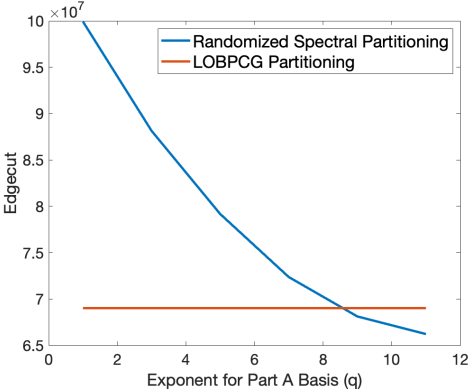


Figure: Hollywood Matrix Results

Future Work

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- Coding Randomized Methods into Trilinos
- Perform additional runs
- Adjust different Trilinos variables

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Thank you All!